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DISCRETE-TIME TRANSITIVITY AND ACCESSIBILITY: ANALYTIC SYSTEMS*

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Abstract. A basic open question for discrete-time nonlinear systems is that of determining when, in analogy with the classical continuous-time "positive form of Chow's Lemma," accessibility follows from transitivity of a natural group action.

This paper studies the problem and establishes the desired implication for analytic systems in several cases: (i) compact state space, (ii) under a Poisson stability condition, and (iii) in a generic sense. In addition, the paper studies accessibility properties of the "controllability sets" recently introduced in the context of dynamical systems studies. Finally, various examples and counterexamples are provided relating the various Lie algebras introduced in past work.

Key words. discrete-time, nonlinear systems, transitivity, accessibility

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1. Introduction.

This paper continues the study, initiated in [7], of systems of the type

$$x(t+1) = f(x(t), u(t)) , \quad t = 0, 1, 2, \dots , \tag{1}$$

where x and u take values in manifolds. The smooth mapping f is assumed to be invertible on x for each fixed u, a restriction that models systems that arise when dealing with continuous-time plants under digital control. See [7] for further motivation for the study of such systems, and [12] for general definitions of systems.

Given the system (1), we may introduce the *reachable* or *forward-accessible set* from a state x^0, which we will denote by $R(x^0)$. This is the set of states to which we may steer x^0 using arbitrary controls. Clearly, reachable sets are one of the central concepts in control theory.

A mathematically far easier object to deal with is the *orbit* or *forward-backward accessible set* from x^0, which we will denote by $O(x^0)$. This is defined as the set consisting of all states to which x^0 can be steered using both motions of the system and negative time motions: a state z is in the orbit of x^0 if there exists a sequence of states

$$x_0 = x^0, x_1, \dots , x_k = z$$

such that, for each $i = 1, \dots , k$, either x_i is reachable from x_{i-1} or x_{i-1} is reachable from x_i.

Of course, $R(x^0)$ is always included in $O(x^0)$, but these two sets are in general different. Observe that $O(x^0)$ is the orbit of x^0 under the group action induced by all the diffeomorphisms $f(\cdot, u)$, while the main interest in control theory—since negative time motions are in general not physically realizable—is in $R(x^0)$, the orbit under the corresponding semigroup. One reason that orbits are easier to study is that they have

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a natural structure of submanifold of the state space; this induces a decomposition of the state space into invariant submanifolds that integrate a natural distribution of vector fields (see, for instance, [13] and [11]).

One of the central facts in continuous-time controllability is the following property, valid for analytic systems and arbitrary states x^0:

(C)	$R(x^0)$ *has nonempty interior in* $O(x^0)$.

This property follows directly from the orbit theorem, but it can also be established for general smooth systems, under appropriate Lie-algebraic assumptions; it is often known as the "positive form of Chow's Lemma." Thus, for continuous-time, the state space can be partitioned into invariant submanifolds, and inside each submanifold we can reach an open set from each state. In particular, the interior of the reachable set from x^0 is nonempty—we then say that there is *forward accessibility from* x^0—if and only if the orbit is open—i.e., there is *transitivity from* x^0.

In contrast, property (C) may fail in discrete-time, even for systems obtained through the time-sampling of one-dimensional analytic continuous-time systems; see the examples in [7]. There are two known cases where (C) does hold:

(a) When x^0 is an *equilibrium point* (and the system is analytic and the control-value set is connected); this is one of the main results in [7].

(b) If the map f is *rational* on states and controls; see [9].

Both of these properties are quite restrictive; equilibria are in general few, and the rationality assumption is too strong in discrete-time (note that even when sampling very simple—for instance, polynomial— continuous-time systems we do not in general obtain rational equations.)

In this paper we extend the validity of property (C). For analytic systems, we prove that property (C) does hold if the orbit from x^0 is compact (see Remark 4.1), or under certain stability hypotheses related to Hamiltonian dynamics. Another result shows that if there is only one orbit (the system is transitive), then forward accessibility holds from an *open dense* set of states, assuming the state space to have at most finitely many connected components. Building upon the results in this paper, [2] provides further conditions under which property (C) holds.

Low-dimensional cases are of interest because certain special implications hold in those cases, and as sources of examples and counterexamples. For instance, we show that in dimension one transitivity from a given state x^0 implies either forward accessibility from x^0 or backward accessibility (controllability from some open set to x^0), but that this result fails in dimension two.

Recently, Colonius and Kliemann introduced the notion of *controllability subsets* of the state space of continuous-time systems. These are essentially sets where "almost reachability" holds. Controllability sets have proved to be an extremely useful concept; in particular, in [4] these authors established an interesting relationship between such sets and chaotic behavior in subsets of an associated dynamical system. The extension to discrete-time of the results of Colonius and Kliemann depends critically on the better understanding of the forward accessibility properties of controllability sets, so we devote the last part of this paper to that goal. The reader is referred to the conference paper [1] for a detailed explanation of how the results in [4] can indeed be extended when applying the techniques developed here.

2. Basic definitions. In this paper we will deal with discrete-time nonlinear systems Σ of the type (1) where $x(t) \in X$ and $u(t) \in U$. We assume that the state space X is a connected, second countable, Hausdorff, differentiable manifold of dimension n, except in §5.1, where we wish to study what happens if the connectedness assumption is dropped.

The control-value space U is always assumed to be a subset of $\mathbb{R}^m$ that satisfies the assumptions

$$U \subseteq \operatorname{clos} \operatorname{int} U$$

and $0 \in U$. We always assume that U is a connected set, except in §§3.1 and 6 where this assumption can be dropped.

The system is *of class* C^k, with $k = \infty$ or ω, if the manifold X is of class C^k and the function

$$f: \; X \times U \to X$$

is of class C^k (i.e., there exists a C^k extension of f to an open neighbourhood of $X \times U$ in $X \times \mathbb{R}^m$). We call systems of class C^∞ *smooth systems* and those of class C^ω *analytic systems.*

The most restrictive technical assumption to be made is that the system is *invertible*; this means that for each $u \in U$ the map $f_u = f(\cdot, u): \; X \to X$ is a global diffeomorphism of X. Invertibility allows the application of the techniques in [7]; the assumption is satisfied when dealing with systems obtained by sampling a continuous-time one. We will use f_u^{-1} to denote the inverse of the map f_u.

Unless otherwise stated, from now on we assume that a fixed smooth system Σ is given.

2.1. Some notation. If there exists an integer $k \geq 0$ and a k-tuple $(u_k, \dots, u_1) \in U^k$ such that $f_{u_k,\dots,u_1}(x) = z$, we will write

$$x \underset{k}{\leadsto} z \, .$$

As usual, $f_{u_k,\dots,u_1}$ denotes $f_{u_k} \circ \cdots \circ f_{u_1}$. For any fixed state x and any nonnegative integer k define:

$$\psi_{k,x}(\mathbf{u}) \;:=\; f_{u_k,\dots,u_1}(x),$$

where $\mathbf{u} = (u_k, \dots, u_1) \in U^k$. For each $\mathbf{u}$, let $\rho_{k,x}(\mathbf{u})$ be the rank of $\partial / \partial \mathbf{u} \psi_{k,x}[\mathbf{u}]$, and denote

$$\bar{\rho}_{k,x} := \max_{\mathbf{u} \in U^k} \rho_{k,x}(\mathbf{u}).$$

For each x, let also

$$\bar{\rho}_x := \max_{k \geq 0} \bar{\rho}_{k,x};$$

roughly, this is the largest possible dimension of a manifold reachable from x. Observe that $k' \geq k$ implies

$$\bar{\rho}_{k',x} \geq \bar{\rho}_{k,x} \tag{2}$$

because if $\mathbf{u} \in U^k$ achieves $\rho_{k,x}(\mathbf{u}) = \bar{\rho}_{k,x}$ then also $\rho_{k',x}(\tilde{\mathbf{u}}) \geq \rho_{k,x}(\mathbf{u})$ for any $\tilde{\mathbf{u}} \in U^{k'}$ that extends $\mathbf{u}$. We define the following sets:

$$R^k(x) := \; \{z \mid x \underset{k}{\leadsto} z\}$$

is the set of states *reachable from x in* (exactly) *k steps*,

$$\tilde{R}^k(x) := \{\psi_{k,x}(\mathbf{u}) \mid \mathbf{u} \in U^k,\ \ \rho_{k,x}(\mathbf{u}) = \bar{\rho}_x\}$$

is the set of states that are *maximal-rank reachable* from x in (exactly) k steps,

$$\bar{R}^k(x) := \{\psi_{k,x}(\mathbf{u}) \mid \mathbf{u} \in U^k,\ \ \rho_{k,x}(\mathbf{u}) = n\}$$

is the set of states that are *nonsingularly reachable from x in k steps.* Observe that, clearly,

$$\bar{R}^k(x) \subseteq \tilde{R}^k(x) \subseteq R^k(x).$$

We let

$$R(x) := \bigcup_{k \geq 0} R^k(x)$$

and analogously for $\tilde{R}(x)$ and $\bar{R}(x)$. Recall that Σ is said to be *forward accessible from x* if and only if $\operatorname{int} R(x) \neq \emptyset$.

We also define

$$\begin{array}{rcl} O^0(x) & = & x, \\ O^k(x) & = & \{\ z \mid \exists z_1 \in O^{k-1} \ \text{and}\ \ z_1 \underset{1}{\rightsquigarrow} z \ \ \text{or}\ \ z \underset{1}{\rightsquigarrow} z_1\ \}, \end{array}$$

and

$$O(x) = \bigcup_{k \geq 0} O^k(x).$$

Thus $O(x)$ is the orbit from x; Σ is said to be *transitive from x* if and only if $\operatorname{int} O(x) \neq \emptyset$.

Note that, given any state x, there is a well-defined restriction of the system to the orbit $O(x)$. Hence all results can be, in principle, applied in each orbit. The only difficulty is that orbits are often *not* connected, while most results hold only under the blanket assumption that the state space must be connected. In §5.1 we make some further comments about this issue.

Certain Lie algebras of vector fields L, L^+, Γ, Γ^+ were introduced in [7] (see also [5] and [6] for previous work) we repeat their definitions here for the convenience of the reader.

First we let X_u^+ and X_u^- be the following vector fields:

$$X_{u,i}^+(x) = \left.\frac{\partial}{\partial v_i}\right|_{v=0} f_u^{-1} \circ f_{u+v}(x),$$

$$X_{u,i}^-(x) = \left.\frac{\partial}{\partial v_i}\right|_{v=0} f_u \circ f_{u+v}^{-1}(x),$$

one for each $i = 1, \dots, m$ (for computational aspects associated to these vector fields see [3]). Given a vector field Y and a control value u, we can define another vector field from Y by applying a change of coordinates given by the diffeomorphism f_u,

$$(\mathrm{Ad}_u Y)(x) = (\mathrm{d}f_u(x))^{-1} Y(f_u(x)).$$

Here $\mathrm{d}f_u$ stands for the differential of f_u with respect to x. In the same way, but now using the diffeomorphism f_u^{-1}, we also define Ad_u^{-1}. We let

$$\mathrm{Ad}^{\epsilon_k\cdots\epsilon_1}_{u_k\cdots u_1} Y = \mathrm{Ad}^{\epsilon_1}_{u_1}\cdots\mathrm{Ad}^{\epsilon_k}_{u_k} Y. \tag{3}$$

We will use the abbreviated notation $\mathrm{Ad}_0^k Y$ for $\mathrm{Ad}_{0\cdots0}Y$ with $u=0$ repeated k-times, if $k>0$, and for $\mathrm{Ad}_{0\,\cdots\,0}^{-1\cdots-1}Y$, if $k<0$. Additionally, $\mathrm{Ad}_0^0 Y = Y$. The Lie algebras Γ^+ and Γ are now defined as

$$\Gamma^+ = \{\mathrm{Ad}_{u_k\cdots u_1} X^+_{u_0,i} | k \geq 0,\ 1 \leq i \leq m,\ u_0,\dots,u_k \in U\},$$

$$\Gamma = \{\mathrm{Ad}^{\epsilon_k\cdots\epsilon_1}_{u_k\cdots u_1} X^\sigma_{u_0,i} | k \geq 0,\ 1 \leq i \leq m,\ u_0,\dots,u_k \in U, \epsilon_1,\dots,\epsilon_k = \pm1, \sigma = \pm\}.$$

Finally the Lie algebras L^+ and L are as follows:

$$L^+ = \mathrm{Lie}\,\{\mathrm{Ad}_0^k X^+_{u,i} \mid k \geq 0,\ 1 \leq i \leq m,\ u \in U\},$$

$$L = \mathrm{Lie}\,\{\ \mathrm{Ad}_0^k X^+_{u,i} \mid k \in \mathbb{Z},\ 1 \leq i \leq m,\ u \in U, \}.$$

We look at the sets of states in which various rank conditions fail, or forward accessibility fails:

$$\begin{aligned} B^+ &:= \{x \mid \mathrm{int}\, R(x) \ = \ \emptyset\} \\ B^+_L &:= \{x \mid \dim L^+(x) \ < \ n\} \\ B^+_\Gamma &:= \{x \mid \dim \Gamma^+(x) \ < \ n\}\,. \end{aligned}$$

Although well-defined always, the set B^+_L will be of interest only when the system is analytic.

2.2. Review of main known facts. With this notation, many of the results obtained in [7] can be visualized by the following diagram, where an arrow "$A \to B$" indicates inclusion $A \subseteq B$, and the inclusions involving B^+_L are only valid in the analytic case:

$$\begin{array}{ccccc} \tilde{R}(B^+) & \longrightarrow & B^+_\Gamma & \longrightarrow & B^+_L \\ \downarrow & & \downarrow & \swarrow & \\ R(B^+) & \longrightarrow & B^+ & & \end{array}$$

Note. The inclusion

$$\tilde{R}(B^+) \subseteq B^+_\Gamma \tag{4}$$

rephrases the result obtained in Corollary 4.4 of [7]. The inclusion $B^+_\Gamma \subseteq B^+$ expresses the result in Theorem 6 part (a) of [7]. The inclusion $B^+_L \subseteq B^+$ represents the result in Theorem 6 part (b) of [7].

3. Some new general properties. In this section we prove a number of general facts that can be conveniently expressed in terms of the sets just defined.

Remark 3.1. If there exists any k_0 such that $\bar{R}^{k_0}(x)$ is nonempty, then for all $k \geq k_0$ we have $\bar{R}^k(x) = \tilde{R}^k(x)$. Indeed, the assumption implies that $\bar{\rho}_x = n$.

PROPOSITION 3.1. *For each $x \in X$, the following properties are equivalent:*

(a) $\operatorname{int} \bar{R}(x) \neq \emptyset$;
(b) $\operatorname{int} \tilde{R}(x) \neq \emptyset$;
(c) $\operatorname{int} R(x) \neq \emptyset$.

Proof. Since $\bar{R}(x) \subseteq \tilde{R}(x) \subseteq R(x)$, it is only necessary to show that (c) implies (a).

We will show the following two properties:

1. for each $k \geq 0$ if $\operatorname{int} \bar{R}^k(x) = \emptyset$ then $\bar{R}^k(x) = \emptyset$;
2. if $\bar{R}^k(x) = \emptyset$ for all $k \geq 0$ then $\operatorname{int} R(x) = \emptyset$.

Combining (1) and (2) we have that if $\operatorname{int} \bar{R}(x) = \emptyset$ then all $\operatorname{int} \bar{R}^k(x) = \emptyset$ too, so $\operatorname{int} R(x) = \emptyset$, as desired.

We first prove (1). Suppose that $\bar{R}^k(x) \neq \emptyset$, so that there exists some sequence $\bar{\mathbf{u}}$ for which the rank $\rho_{k,x}(\cdot)$ is equal to n at $\bar{\mathbf{u}}$. Since we assume $U \subset \operatorname{clos} \operatorname{int} U$, there exists also some $\tilde{\mathbf{u}} \in \operatorname{int} U^k$ so that $\rho_{k,x}(\mathbf{u}) = n$ for each $\mathbf{u}$ in some neighbourhood of $\tilde{\mathbf{u}}$. By the implicit mapping theorem, $\tilde{z} = \psi_{k,x}(\tilde{\mathbf{u}})$ belongs to $\operatorname{int} \bar{R}^k(x)$.

We now prove (2). If $\bar{R}^k(x) = \emptyset$ for all $k \geq 0$ then each $\mathbf{u} \in U^k$ is a singular point of the map $\psi_{k,x}$, for each k. Thus by Sard's theorem $\psi_{k,x}(U^k)$ has measure zero for all $k \geq 0$. It follows that also

$$R(x) = \bigcup_{k \geq 0} R^k(x) = \bigcup_{k \geq 0} \psi_{k,x}(U^k)$$

has measure zero, and hence $\operatorname{int} R(x) = \emptyset$, as desired. □

PROPOSITION 3.2. *If the system Σ is analytic then, for any $x \in X$:*

$$\operatorname{clos} R^k(x) \;=\; \operatorname{clos} \tilde{R}^k(x)$$

for all k sufficiently large.

Proof. Fix $x \in X$, and let k_0 be so that $\bar{\rho}_{k_0,x} = \bar{\rho}_x$. For all $k \geq k_0$, let

$$A_k(x) \;=\; \{\mathbf{u} \mid \rho_{k,x}(\mathbf{u}) = \bar{\rho}_x\} \,.$$

We claim that $A_k(x)$ is an open dense set of U^k. This is because $A_k(x) \neq \emptyset$ by (2) and the complement of $A_k(x)$ is a set defined by the vanishing of certain analytic functions (suitable determinants) of $\mathbf{u}$.

We claim that

$$R^k(x) \subseteq \operatorname{clos} \tilde{R}^k(x),$$

which implies

$$\operatorname{clos} R^k(x) \;\subseteq\; \operatorname{clos} \tilde{R}^k(x) \;\text{ for each such } k. \tag{5}$$

This will establish the result, the other inclusion being obvious.

Indeed, pick $k \geq k_0$ and take any $z \in R^k(x)$. Then $z = \psi_{k,x}(\mathbf{u})$ for some $\mathbf{u} = (u_k, \ldots, u_1)$. Since $A_k(x)$ is dense, we can find a sequence $\{\mathbf{u}_l\}$ such that

$$\mathbf{u}_l = (u_k^{(l)}, \ldots, u_1^{(l)}) \;\rightarrow\; \mathbf{u} = (u_k, \ldots, u_1) \quad \text{as} \;\; n \rightarrow \infty$$

and $\mathbf{u}_l \in A_k(x)$ for each l.

Let $z_l = \psi_{k,x}(\mathbf{u}_l) \in \tilde{R}^k(x)$. By continuity, $z_l \rightarrow z$, which proves (5). □

Remark 3.2. Assume that the system Σ is analytic, and that there exists an $x_0 \in X$ and a $k_0 \geq 0$ for which $\bar{R}^{k_0}(x_0) \neq \emptyset$. Then the proof of the previous result together with Remark (3.1) imply that

$$\operatorname{clos} R^k(x_0) \;=\; \operatorname{clos} \bar{R}^k(x_0)$$

for all $k \geq k_0$.

Moreover, since $\partial/\partial \mathbf{u}\psi_{k,x}[\mathbf{u}]$ is analytic also with respect to the x-variable, this particular k_0 works also for an open dense set of states $x \in X$. Thus, under these assumptions, we have that

$$\operatorname{clos} R^k(x) \;=\; \operatorname{clos} \tilde{R}^k(x) \;=\; \operatorname{clos} \bar{R}^k(x)$$

for all $k \geq k_0$ and for almost all $x \in X$.

3.1. Regular points. We call x a *regular point* if $\bar{\rho}_x$ is constant in a neighbourhood of x. The following fact will be useful later; it is of course a well-known general fact about smooth mappings.

LEMMA 3.3. *The regular points form an open dense subset of* X.

Proof. Let

$$\bar{\rho} \;=\; \max_{x \in X} \bar{\rho}_x \,.$$

We have $\bar{\rho} \in \{0, \ldots, n\}$. We will prove our thesis by induction on $\bar{\rho}$.

If $\bar{\rho} = 0$, then each $x \in X$ is a regular point, thus the statement is true.

Let $\bar{\rho} > 0$. Define

$$\begin{aligned} X_1 &:= \{x \in X \mid x \ \text{ is a regular point and } \ \bar{\rho}_x = \bar{\rho}\} \,, \\ Y_1 &:= \operatorname{int}\{X \setminus X_1\} \,. \end{aligned}$$

Then X_1 and Y_1 are open. Moreover $X_1 \bigcup Y_1$ is dense in X, since its complement is the boundary of X_1 which is a nowhere dense set. If we call

$$\bar{\rho}_1 \;=\; \max_{x \in Y_1} \bar{\rho}_x$$

we have $\bar{\rho}_1 < \bar{\rho}$.

Thus, applying the inductive assumption to $(\bar{\rho}_1,\ Y_1)$, we have that the set of regular points in Y_1, denote it by Y_r, is dense in Y_1. But since the set of regular points of X is given by $X_1 \bigcup Y_r$ and $X_1 \bigcup Y_1$ is dense in X, then $X_1 \bigcup Y_r$ is also dense in X. □

Note that, in the particular case in which the system is analytic, then in the above proof the set X_1 is already dense, because the rank is less then $\bar{\rho}$ if and only if certain determinants, which are analytic functions of x, vanish and this can happen only in a nowhere dense set.

4. More results for analytic systems. In this section we always assume the system Σ to be *analytic*.

LEMMA 4.1. *Suppose that for a fixed $x \in X$ there exists a sequence of elements $\{x_{n_k}\}$ and some $y \in X$ so that $\dim L^+(y) = n$, such that*

1. *$x_{n_k} \in R^{n_k}(x)$, with $n_k \to \infty$;*
2. *$x_{n_k} \to y$.*

Then the system is forward accessible from x (i.e. $x \notin B^+$).

Proof. Since $x_{n_k} \to y$ and $\dim L^+(y) = n$ there is some integer $k_0 \geq 0$ such that $\dim L^+(x_{n_k}) = n$ for all $k \geq k_0$. But for k sufficiently large we know (by Proposition (3.2)) that $x_{n_k} \in \operatorname{clos} \tilde{R}^{n_k}(x)$. Thus there exists some $z \in X$ such that $z \in \tilde{R}^{n_k}(x)$ and $\dim L^+(z) = n$. So we can conclude forward accessibility from x by (4). □

Remarks 4.1.

1. The result is also true if the weaker assumption $\dim \Gamma^+(y) = n$ is made, but we will apply it in the above form.

2. If x and y are as in the previous lemma, and U is any open neighbourhood of y, then, in particular, we have that $R(x) \cap U$ is also open.

3. If for a fixed $x \in X$ there exists a sequence of elements $\{x_{n_k}\}$ such that $x_{n_k} \in R^{n_k}(x)$, with $n_k \to \infty$ and $x_{n_k} \to x$ then, by the previous lemma, we can conclude that forward accessibility from x is equivalent to $\dim L^+(x) = n$. We will see later that in dimension 1 this equivalence is always true, but it can fail in higher dimensions.

For each $x \in X$, we will denote by $y^k_{0,x}$ the image under $\psi_{k,x}(\cdot)$ of the zero control; i.e.,

$$y^k_{0,x} = \psi_{k,x}(\underbrace{0,\dots,0}_{k-times}).$$

LEMMA 4.2. *Suppose that x, $y \in X$ are so that*

1. *the system is transitive from y, (or equivalently, $\dim L(y) = n$,)*
2. *there exists a sequence $\{y^{n_k}_{0,x}\}$ with $n_k \to \infty$ such that $y^{n_k}_{0,x} \to y$.*

Then $\dim L^+(x) = n$.

Proof. Choose n vector fields $v_1, \dots, v_n$ in L such that

$$\{v_1(y), \dots, v_n(y)\}$$

is a basis for $L(y)$.

As in the proof of Proposition 4.2 in [7], we can assume that the v_i's involve Lie brackets of a finite numbers of vector fields of the form $Ad_0^{k_j} X^+_{u_j}$, with $k_j \in \mathbb{Z}$. Choose a positive integer k_0 so that $k_j + k_0 \geq 0$ for all such j.

Since the v_i's are linearly independent at y, they are still linearly independent in some neighbourhood U_y of y. By assumption (2), there is some n_k so that $y^{n_k}_{0,x} \in U_y$ and $n_k \geq k_0$.

Applying the operator $Ad_0^{n_k}$ to the v_i's, there result n linearly independent vectors in $L^+(x)$, as desired. □

4.1. Poisson stability. Recall that if Y is a vector field on a manifold M, one says that $x \in M$ is a *positively Poisson stable point* for Y if and only if for each neighbourhood V of x and each $T \geq 0$ there exists some $t > T$ such that $e^{tY}(x) \in V$, where $e^{tY}(\cdot)$ represents the flow of Y.

Analoguosly, we can define positive Poisson stability in discrete time, as follows.

DEFINITION 4.1. Let $f : X \to X$ be a global diffeomorphism. The point $x \in X$ is *positively Poisson stable* if and only if for each neighbourhood V of x and each integer $N \geq 0$ there exists some integer $k > N$ such that $f^k(x) \in V$.

THEOREM 4.3. *Let $x \in X$ be a positively Poisson stable point for $f_0 = f(\cdot, 0)$. Then transitivity from x implies forward accessibility from x.*

Proof. Positive Poisson stability from x implies the existence of a sequence $\{y^{n_k}_{0,x}\}$, with $n_k \to \infty$, convergent to x. Thus the result follows immediately combining Lemmas (4.1), (4.2) (applied with $y = x$). □

4.2. Compact state space. For each $k \geq 0$ we define the following sets:

$$C^k(x) := \{\, y \mid y \underset{k}{\rightsquigarrow} x\},$$

i.e., the set of states *controllable to* x in (exactly) k steps, and

$$C(x) = \bigcup_{k \geq 0} C^k(x).$$

A system is *backward accessible from* x if and only if $\operatorname{int} C(x) \neq \emptyset$.

THEOREM 4.4. *Let* Σ *be a discrete time, analytic, invertible system, and assume that the state space* X *is compact.*

Then, Σ *is transitive if and only if it is forward accessible.*

Proof. By [7], Theorem 3, it will be enough to show that $\dim L^+(x) = n$ for all $x \in X$. Fix any $x \in X$, and consider the sequence

$$y^l_{0,x} = \psi_{l,x}(0, \ldots, 0).$$

Then since X is compact (and second countable) there exists a subsequence $\{y^{l_k}_{0,x}\}$ which converges; let y be so that $y^{l_k}_{0,x} \to y$. Since Σ is transitive, $\dim L(y) = n$, so, by Lemma (4.2), $\dim L^+(x) = n$ as wanted. □

Remark 4.1. Notice that, in the previous theorem, the blanket assumption of connectedness of the state space X is not needed. In particular, the result holds if the orbit from a state x is compact.

Remark 4.2. Clearly, using the same arguments as in Theorem 4.4, we also have that, if the state space is compact, then transitivity from all $x \in X$ is equivalent to backward accessibility from all $x \in X$. We will not use this fact, however.

Recall that for a space Z with a σ-algebra F and a finite measure μ, we say that a measurable transformation $T : Z \to Z$ is *measure-preserving* if for every $A \in F$ we have $\mu(T^{-1}A) = \mu(A)$.

The following controllability result is an analogue for discrete-time systems of the result in [8]. The proof is very similar, but it uses the facts just established.

PROPOSITION 4.5. *Assume that the state space* X *is a compact Riemannian analytic manifold, and that for all* $u \in U$ *the map* f_u *is a measure preserving transformation (for the natural measure in* X*). Then* Σ *is transitive if and only if* Σ *is controllable.*

Proof. We need only to prove that transitivity implies controllability.

For each u, since f_u is a measure preserving map, by the Poincaré recurrence theorem the set of positively Poisson stable points for f_u is known to be dense in X.

Let $x, y \in X$; we need $y \in R(x)$. By Theorem 4.4, we know that Σ is both forward and backward accessible from x and y. Choose $\bar{x} \in \operatorname{int} R(x)$ and $\bar{y} \in \operatorname{int} C(y)$; since Σ is transitive there exist k, $(u_k, \ldots, u_1)$, and $(\epsilon_k, \ldots, \epsilon_1)$, with each $u_i \in U$ and $\epsilon_i = 1$ or -1, such that

$$f^{\epsilon_k}_{u_k} \circ \ldots \circ f^{\epsilon_1}_{u_1}(\bar{x}) = \bar{y}.$$

Let l = number of $\epsilon_i = -1$. We will show by induction on l the following fact:

there exist $\tilde{x} \in \operatorname{int} R(x)$ and $\tilde{y} \in \operatorname{int} C(y)$ such that $\tilde{y} \in R(\tilde{x})$.

Clearly the previous statement implies our thesis.

If $l = 0$ then the statement holds with $\tilde{x} = \bar{x}$ and $\tilde{y} = \bar{y}$. So let $l > 0$ and let i be the first index such that $\epsilon_i = -1$. Define

$$x_i = f_{u_{i-1}} \circ \ldots \circ f_{u_1}(\bar{x})$$

and

$$y_i = f^{-1}_{u_i}(x_i)\ .$$

Since $\bar{y} \in \operatorname{int} C(y)$, there exists a neighbourhood V of y_i such that

$$f_{u_k}^{\epsilon_k} \circ \ldots \circ f_{u_{i+1}}^{\epsilon_{i+1}}(V) \subseteq C(y)\ ;$$

let $W = f_{u_i}(V)$. Since $\bar{x} \in \operatorname{int} R(x)$ we can assume (taking V smaller if necessary) that $W \subseteq R(x)$.

Choose $z_i \in W$ positively Poisson stable for f_{u_i}; then there exists some $n > 1$ such that $f_{u_i}^n(z_i) \in W$ and the following properties hold:

- $f_{u_i}^{n-1}(z_i) = f_{u_i}^{-1} \circ f_{u_i}^n(z_i) \in V$,
- $\hat{y} = f_{u_k}^{\epsilon_k} \circ \ldots \circ f_{u_{i+1}}^{\epsilon_{i+1}}(z_i) \in \operatorname{int} C(y)$.

So we have constructed a trajectory joining $z_i \in \operatorname{int} R(x)$ to $\hat{y} \in \operatorname{int} C(y)$ with a number of negative steps strictly less than l; the statement follows by induction. □

Remark 4.3. The result obtained in the previous proposition can be applied to any discrete-time system Σ that arises through the time-sampling of a continuous-time system, if the vector fields in the right-hand side of the differential equation are conservative. The latter happens for Hamiltonian systems; see for instance [10] for many examples of such Hamiltonian control systems, and the last section of [11] for conditions under which transitivity is preserved under sampling.

5. Accessibility almost everywhere. For analytic systems, we say here that a property holds for "almost all" $x \in X$ if it holds on a set which is the complement of the set of zeros of a nonzero analytic function; note that such a set is open dense and its complement has zero measure.

LEMMA 5.1. *Let Σ be an n-dimensional, discrete-time, invertible, and analytic system. Then the following are equivalent:*

(1) *Σ is transitive from almost all $x \in X$;*
(2) $\dim L(x) = n$ *for almost all $x \in X$;*
(3) *Σ is forward accessible from almost all $x \in X$;*
(4) $\dim L^+(x) = n$ *for almost all $x \in X$.*

Proof. We will show (1) $\to$ (2) $\to$ (4) $\to$ (3) $\to$ (1).

(1) $\to$ (2) This is a consequence of Theorem 4 in [7].

(2) $\to$ (4) Since the system is analytic, and X is connected it will be enough to show that there is at least one x with $\dim L^+(x) = n$, because the set where this property holds is either empty or open and dense. To show that there exists such an x we will use the same procedure used in proving Lemma 4.2.

Fix any $y \in X$ for which $\dim L(y) = n$, and let $v_1, \ldots, v_n \in L$ be so that

$$\{v_1(y), \ldots, v_n(y)\}$$

is a basis for $L(y)$. Assume that the v_i's involve vector fields of the form

$$Ad_0^{k_j} X_{u_j}^+,$$

with $k_j \in \mathbb{Z}$, and choose a positive integer k_0 so that $k_j + k_0 \geq 0$ for all such j. Applying the operator $Ad_0^{k_0}$ to the v_i's, there result n linearly independent vectors in $L^+(x)$, where $x := f_0^{-k_0}(y)$. Thus $\dim L^+(x) = n$.

(4) $\to$ (3) Again by analyticity, it will be sufficient to find at least one x form which Σ is forward accessible. Choose $\bar{x}$ regular and let k, $\mathbf{u} = (u_k, \ldots, u_1)$, and $\bar{z}$ be such that

$$\psi_{k,\bar{x}}(\mathbf{u}) = \bar{z} \quad \text{and} \quad \rho_{k,\bar{x}}(\mathbf{u}) = \bar{\rho}_{\bar{x}}.$$

Let W be some neighbourhood of $\bar{x}$ so that

$$\rho_{k,x}(\mathbf{u}) \geq \rho_{k,\bar{x}}(\mathbf{u}) = \bar{\rho}_{\bar{x}}$$

for each $x \in W$. As $\bar{x}$ is regular

$$\bar{\rho}_{\bar{x}} = \bar{\rho}_x \geq \rho_{k,x}(\mathbf{u}),$$

so there is equality, $\rho_{k,x}(\mathbf{u}) = \rho_{k,\bar{x}}(\mathbf{u})$. Define

$$U = f_{\mathbf{u}}(W);$$

since $f_{\mathbf{u}}$ is a diffeomorphism, U is open. Moreover, by maximality of the rank, we have

$$U \subseteq \tilde{R}^k(W).$$

Since $\dim L^+(x) = n$ for almost all x, we can choose some $z \in U$ for which $\dim L^+(z) = n$. Let

$$y := f_{\mathbf{u}}^{-1}(z) \in W.$$

Note that then $z \in \tilde{R}^k(y)$ and $\dim L^+(z) = n$.

We can conclude forward accessibility from y by (4).

(3) $\rightarrow$ (1) This is clear. □

Remarks 5.1. (1) Since Σ is analytic, in each of the previous statements we can substitute "there exists $x \in X$" instead of "for almost all $x \in X$."

(2) Note that, in general, the open dense sets in which the previous statements hold are *not* the same, except for those in parts (1) and (2). In particular, if we denote

$$B := \{x \mid \dim L(x) < n\},$$

we have

- $B = \{x \mid x \text{ is not transitive }\}$;
- $B \subseteq B_L^+ \subseteq B^+$;

and the previous inclusions can be proper. For example, for the system described in Example 6.1 below we have

$$\begin{aligned} B &= \emptyset \\ B_L^+ &= \{\ (k,y) \mid k \geq 1,\ k \in \mathbb{Z},\ -k \leq y \leq k\ \} \\ B^+ &= \{\ (k,y) \mid k \geq 0,\ k \in \mathbb{Z},\ -k \leq y \leq k\ \}\ =\ B_L^+ \bigcup \{(0,0)\}. \end{aligned}$$

(3) Let L^- be the Lie algebras defined in the same way as L^+, but using the vector fields $X_{u,i}^-$ instead of $X_{u,i}^+$, and $k \leq 0$ instead of $k \geq 0$. Given this definition, the conclusions of Lemma 5.1 hold substituting (3) and (4) with the following properties:

(3') Σ is backward accessible from almost all $x \in X$.

(4') $\dim L^-(x) = n$ for almost all $x \in X$.

5.1. Nonconnected orbits. Given any system Σ, its state space can be partitioned into invariant submanifolds, the orbits. Since the system restricted to each orbit is transitive, we would like to conclude that relative to each orbit there is forward accessibility from almost every state. Unfortunately, this conclusion is false in general (see Example 5.1 below), because orbits are in general not connected. We can prove this fact, however, in the particular case of orbits with at most finitely many connected components, as follows from the next result.

PROPOSITION 5.2. *Let Σ be an n-dimensional, discrete-time, invertible and analytic system, and assume that the state space X has finitely many connected components. If Σ is transitive then it is forward accessible from almost all $x \in X$.*

Proof. Partition $X = \bigcup_{i=1}^{l} X_i$ into disjoint nonempty open connected subsets. Note that, if $x \in X_i$ and $f(x,u) \in X_j$, then since $X_i \times U$ is connected we have that

$$f(X_i \times U) \subseteq X_j, \tag{6}$$

by continuity of f. Then for each i there is some $j(i)$ so that

$$f_u(X_i) \subseteq X_{j(i)} \quad i = 1, \dots, l,$$

for every $u \in U$.

Fix now any $u \in U$. Since $f_u(X) = X$, necessarily $\bigcup_{i=1}^{l} X_{j(i)} = X$. As f_u is a diffeomorphism of X, the $X_{j(i)}$ are all distinct and $f_u(X_i) = X_{j(i)}$. Since Σ is transitive, we can conclude that for any $p = 1, \dots, l-1$, denoting by

$$f_u^p = f\underbrace{u, \dots, u}_{p-times},$$

the following holds:

$$f_u^p(X_i) \neq X_i \quad \forall\, i = 1, \dots, l. \tag{7}$$

If this were not the case and there exists such p and i, then applying (6) p-times we would have

$$f_{u_1,\dots,u_p}(X_i) = X_i$$

for *all* $(u_1, \dots, u_p) \in U^p$. Thus the set

$$\bigcup_{j=0}^{p-1} f_u^j(X_i)$$

will be an invariant set different from X, which contradicts the assumption that Σ is an orbit. Moreover, from (7), since l is finite, we can conclude that

$$f_{u_1,\dots,u_l}(X_i) = X_i \qquad \forall\, i. \tag{8}$$

By repeating the arguments used in the proof of the Lemma 5.1 ((2) $\to$ (4)) we conclude that there exists $x \in X$ such that $\dim L^+(x) = n$. Assume that $x \in X_i$. Since X_i is connected we have

$$\dim L^+(y) = n \quad \text{from almost all} \quad y \in X_i.$$

Choose $\bar{x} \in X_i$, $\bar{x}$ regular and let k, $\mathbf{u} = (u_k, \ldots, u_1)$, and $\bar{z}$ be such that

$$\psi_{k,\bar{x}}(\mathbf{u}) = \bar{z} \quad \text{and} \quad \rho_{k,\bar{x}}(\mathbf{u}) = \bar{\rho}_{\bar{x}}.$$

By inequality (2) we can assume that k is a multiple of l. Thus, by (7), we get that $\bar{z} \in X_i$. Now, we can repeat the arguments used in the proof of the Lemma 5.1 ((4) $\rightarrow$ (3)) and conclude that Σ is forward accessible from almost all $x \in X_i$.

To conclude that Σ is forward accessible from almost all $x \in X$ it is enough to note that, for any $j \neq i$, (7) implies that there exists p such that

$$f_{u_1,\ldots,u_p}(X_j) = X_i. \qquad \square$$

Example 5.1. Consider the following analytic system, with $X = \mathbb{R}^2$, $U = \mathbb{R}$, and equations:

$$\begin{aligned} x^+ &= x + 1, \\ y^+ &= y + uh(x) , \end{aligned}$$

where $h(x)$ is any analytic function whose zeros are exactly at the positive integers $\{1, 2, 3, \ldots\}$. This system is easily seen to be invertible. Let $z_0 = (0, 0)$. Then it is easy to verify that the orbit $O(z_0)$ is as follows:

$$O(z_0) = \bigcup_{i \in \mathbb{Z}} R_i,$$

$$R_i = \{ \ (i, y) \mid y \in \mathbb{R} \ \}.$$

If we restrict the system to this orbit, the restricted system is not forward accessible from any the points in R_i, for each $i = 1, 2, 3, \ldots$. This is because there it holds that $h(x) = 0$, so z^+ and z must have the same y-coordinate.

6. Low-dimensional cases.

In this section we make some remarks about one- and two-dimensional systems.

6.1. Dimension one.

There we consider systems for which the state space X is of dimension one. The pointwise versions of [7, Thm. 3] hold for these systems as follows.

LEMMA 6.1. *Let Σ be as above, and pick $x \in X$. Then*

1. *if Σ is smooth then*
Σ is forward accessible from x if and only if $\dim \Gamma^+(x) = 1$;
2. *if Σ is analytic and U is connected then*
Σ is forward accessible from x if and only if $\dim L^+(x) = 1$.

Proof. (1) The necessary part follows from part (a) of Theorem 6 in [7], so we will prove sufficiency. If Σ is not forward accessible from x then $f(x, u)$ must be independent of u. Moreover if $y = f_{u_k,\ldots,u_1}(x)$, since Σ is also not forward accessible from y, also

$$f(y, u) = f(f_{u_k,\ldots,u_1}(x), u)$$

must be independent of u. Thus

$$Ad_{u_k,\ldots,u_1} X^+_{u_0}(x) = \left.\frac{\partial}{\partial v}\right|_{v=0} f^{-1}_{u_k,\ldots,u_1} \circ f^{-1}_{u_0} \circ f_{u_0+v} \circ f_{u_k,\ldots,u_1}(x) = 0,$$

which implies $\dim \Gamma^+(x) = 0$.

(2) The necessary part follows from part (b) of Theorem 6 in [7]. Sufficiency is a consequence of (1), since $L^+(x) \subseteq \Gamma^+(x)$. □

LEMMA 6.2. *Let Σ be a one-dimensional, discrete-time, invertible system, and pick any $x \in X$ so that Σ is transitive from x. Then, either Σ is forward accessible from x or Σ is backward accessible from x.*

Proof. Suppose that neither conclusion holds.

We claim that, for each $u \in U$, Σ is not forward nor backward accessible from $y = f_u(x)$. Since x is not forward accessible, $f(x,u)$ is independent of u. Thus $y = f_u(x)$ for *all* $u \in U$, so also

$$f_u^{-1}(y) = x \quad \text{for all} \quad u \in U.$$

It follows that $C^1(y) = x$, which implies that

$$C^k(y) = C^{k-1}(x) \quad \text{for all} \quad k \geq 1.$$

Thus if Σ would be backward accessible from y also Σ would be backward accessible from x. Clearly, forward accessibility from y would imply forward accessibility from x (in any dimension). So the claim is proved.

With the same arguments we can prove that Σ is not forward nor backward accessible from $z = f_u^{-1}(x)$ for all $u \in U$.

Now we want to prove that $\dim \Gamma(x) = 0$, which implies that Σ in not transitive from x. In order to do that, we will show that

$$Ad_{u_k,\ldots,u_1}^{\epsilon_k,\ldots,\epsilon_1} X_{u_0}^{\sigma}(x) = 0$$

for all $k \geq 0$, $(u_k, \ldots, u_1)$, $\epsilon_i = 1$ or -1, $\sigma = 1$ or -1, and for all x which are neither forward nor backward accessible.

We will use induction on k. Take first $k = 0$.

- If $\sigma = 1$

$$X_{u_0}^{+}(x) = \left.\frac{\partial}{\partial v}\right|_{v=0} f_{u_0}^{-1} \circ f_{u_0+v}(x) = 0$$

since $f(x,\cdot)$ is independent of u (Σ is not forward accessible from x).

- If $\sigma = -1$

$$X_{u_0}^{-}(x) = \left.\frac{\partial}{\partial v}\right|_{v=0} f_{u_0} \circ f_{u_0+v}^{-1}(x) = 0$$

since $f^{-1}(x,\cdot)$ is independent of u (Σ is not backward accessible from x).

Take now any $k > 0$ and note that

$$Ad_{u_k,\ldots,u_1}^{\epsilon_k,\ldots,\epsilon_1} X_{u_0}^{\sigma}(x) = (df_{u_k}^{\epsilon_k}(x))^{-1} Ad_{u_{k-1},\ldots,u_1}^{\epsilon_{k-1},\ldots,\epsilon_1} X_{u_0}^{\sigma}(f_{u_k}^{\epsilon_k}(x)).$$

From the first part of the proof, we have that Σ is also neither forward nor backward accessible from $f_{u_k}^{\epsilon_k}(x)$, so, by inductive assumption, this last vector is zero. □

Remark 6.1. A consequence of the two previous lemmas is that, for each x:

1. $\dim L(x) = 1$ if and only if $\dim L^+(x) = 1$ or $\dim L^-(x) = 1$.
2. $\dim \Gamma(x) = 1$ if and only if $\dim \Gamma^+(x) = 1$ or $\dim \Gamma^-(x) = 1$.

The result in Lemma 6.2 is true only pointwise. In fact we can find a one-dimensional, analytic system Σ that is transitive but is neither forward nor backward accessible. One example of such a system is as follows.

Consider the following system:

$$x^+ = 1 + x + \frac{u}{2}[g(x) + g(x-1)] \tag{9}$$

with $X = \mathbb{R}$, $U = [-1, 1]$, and where $g(x)$ is the following function:

$$g(x) = \frac{\sin(\pi x)}{\pi x}. \tag{10}$$

It is easy to verify that $|g'(x)| \leq 1$ for all $x \in \mathbb{R}$. Moreover, $g(x) = 0$ if and only if $x \in \mathbb{Z} \setminus \{0\}$. Since $|g'(x)| \leq 1$, this system is invertible. Moreover the following properties hold and are easily verified:

1. Σ is transitive;
2. if $x = 2, 3, \ldots$ then Σ is backward accessible but not forward accessible from x;
3. if $x = -1, -2, -3, \ldots$ then Σ is forward accessible but not backward accessible from x.

6.2. Dimension two. We now show that both the results in Lemmas 6.1 and 6.2 are false if the dimension of the state space X is greater than one, even if the system is invertible, analytic and with a connected control space U.

The following example illustrates these facts.

Example 6.1. Consider the discrete-time, analytic system with $X = \mathbb{R}^2$, $U = [-1, 1]^2$, and equations:

$$x^+ = x + 1 + \frac{u}{2}\sin(y)g(x),$$
$$y^+ = y + v,$$

where $g(x)$ is the function in (10).

This system is invertible. In fact, the determinant of the Jacobian matrix of the map $f_{u,v}(x, y)$ is given by

$$1 + \frac{u}{2}\sin(y)g'(x).$$

Since $u \in [-1, 1]$, $|\sin(y)| \leq 1$ and $|g'(x)| \leq 1$,

$$|\frac{u}{2}\sin(y)g'(x)| < 1$$

so the determinant is nonzero for all x, y. Moreover it is easy to verify that for each $(u, v) \in U$, the map $f_{u,v}(\cdot, \cdot)$ is bijective.

We wish to study the behavior of this system when starting from $x = 0$, $y = 0$. Let $z_0 = (0, 0)$. We prove the following properties:

(1) the system is not forward accessible from z_0;
(2) the system is not backward accessible from z_0;
(3) $\dim L^+(z_0) = 2$;
(4) the system is transitive from z_0.

Proof.

(1) This follows from the equality

$$R^k(z_0) \;=\; \{\ (k,y) \mid -k \le y \le k\ \},$$

which holds for each $k \ge 1$ and it is clear from the equations.

(2) It will be sufficient to show that

$$C^k(z_0) \;=\; \{\ (-k,y) \mid -k \le y \le k\ \}. \tag{11}$$

First note that if $(x_k, y_k) \in C^k(z_0)$ then we can write $y_k + v_1 + \ldots + v_k = 0$ with all $|v_i| \le 1$, so $|y_k| \le k$.

To prove (11), it is now sufficient to note the following. For any fixed $u \in [-1,1]$ and any $y \in \mathbb{R}$, the function

$$x \overset{h}{\longmapsto} x + 1 + \frac{u}{2}\sin(y)g(x)$$

is invertible. Moreover

$$h^{-1}(-k+1) = -k \quad \text{for all} \quad k \ge 1,$$

independently of u and y. Thus $x = -k$ is the only solution of $h(x) = -k+1$, for all u, y, and we have proved

$$C^k(z_0) \;\subseteq\; \{\ (-k,y) \mid -k \le y \le k\ \}.$$

The other inclusion is obvious.

(3) Consider the vector fields

$$X^+_{(u,v),1}(z) = (df_{(u,v)}(z))^{-1} \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} f_{(u+\epsilon,v)}(z)$$

and

$$X^+_{(u,v),2}(z) = (df_{(u,v)}(z))^{-1} \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} f_{(u,v+\epsilon)}(z)\,.$$

Fix $(u,v) = (0,0)$. Then

$$df_{(0,0)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all $z \in \mathbb{R}^2$,

$$X^+_{(0,0),1}(z) = \begin{pmatrix} \frac{\sin(y)g(x)}{2} \\ 0 \end{pmatrix},$$

and

$$X^+_{(0,0),2}(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So

$$\left[X^+_{(0,0),1}, X^+_{(0,0),2}\right](z) \;=\; \begin{pmatrix} \frac{\cos(y)g(x)}{2} \\ 0 \end{pmatrix}.$$

In particular,

$$X^+_{(0,0),2}(z_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\left[X^+_{(0,0),1}, X^+_{(0,0),2}\right](z_0) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$

so $\dim L^+(z_0) = 2$ as desired.

(4) Transitivity at z_0 is a consequence of (3) since $\dim L^+(z_0) = 2$ implies $\dim L(z_0) = 2$. □

7. Controllability sets. The next definition is a discrete-time analogue of that in [4], except that we make the assumption of nonempty interior.

DEFINITION 7.1. A set $D \subseteq X$ is called a *precontrollability set* if

$$D \subseteq \operatorname{clos} R(x) \quad \text{for all} \quad x \in D$$

and $\operatorname{int} D \neq \emptyset$. A precontrollability set which is maximal with respect to set inclusion is called a *controllability set*.

Note that if D is a precontrollability set, then in D the system Σ is "almost" controllable, in the sense that if $x, y \in D$ then from x it is possible to reach any neighbourhood of y.

LEMMA 7.1. *Let $D \subseteq X$ be a controllability set. Pick any two elements $\bar{x}$, $\bar{y}$ in $\operatorname{int} D$. Then, for each sequence $(u_0, \dots, u_T) \in U^{T+1}$ such that*

$$f_{u_T} \circ \dots \circ f_{u_0}(\bar{x}) = \bar{y}$$

we have that, necessarily, also

$$f_{u_k} \circ \dots \circ f_{u_0}(\bar{x}) \in \operatorname{int} D \quad \text{for} \quad k = 0, \dots, T-1.$$

Proof. Let $\bar{x}$, $\bar{y}$, $u_0, \dots, u_T$ be as in the statement and let E be the following set:

$$E := \{ f_{u_k} \circ \dots \circ f_{u_0}(\bar{x}), \quad k = 0, \dots, T-1 \}.$$

We will first prove that $E \subseteq D$, by showing that $D' = D \bigcup E$ is again a precontrollability set and using that D is maximal. For this, we must prove that

$$D' \subseteq \operatorname{clos} R(x) \quad \text{for each} \quad x \in D'.$$

Observe that $E \subseteq R(\bar{x}) \subseteq \operatorname{clos} R(\bar{x})$ and $\bar{y} \in R(y) \subseteq \operatorname{clos} R(y)$ for all $y \in E$. Thus

- $E \subseteq \operatorname{clos} R(\bar{x}) \subseteq \operatorname{clos} R(\operatorname{clos} R(x)) = \operatorname{clos} R(x) \quad \forall x \in D$;
- $D \subseteq \operatorname{clos} R(x) \quad \forall x \in D$;
- If $y \in E$ then $D \subseteq \operatorname{clos} R(\bar{y}) \subseteq \operatorname{clos} R(\operatorname{clos} R(y)) = \operatorname{clos} R(y)$ and $E \subseteq \operatorname{clos} R(\bar{x}) \subseteq \operatorname{clos} R(\operatorname{clos} R(\bar{y})) \subseteq \operatorname{clos} R(y)$.

Thus $D \bigcup E = D' \subseteq \operatorname{clos} R(x)\ \forall x \in D'$.

So we have proved that, for any two points $\bar{x}$, $\bar{y}$ in $\operatorname{int} D$ and any trajectory joining them, all the intermediate states must be in D. We now prove that such intermediate points must be in $\operatorname{int} D$.

Pick any $\bar{x}$, $\bar{y}$, $u_0, \ldots, u_T$ as above. Let $k \in \{0, \ldots, T-1\}$ and $\bar{z} = f_{u_k} \circ \ldots \circ f_{u_0}(\bar{x})$. By continuity of $f_{u_0}^{-1} \circ \ldots \circ f_{u_k}^{-1}$ and of $f_{u_{k+1}} \circ \ldots \circ f_{u_T}$, there exists some open neighbourhood V of $\bar{z}$ such that

$$f_{u_0}^{-1} \circ \ldots \circ f_{u_k}^{-1}(V) \subseteq \operatorname{int} D \quad \text{and} \quad f_{u_{k+1}} \circ \ldots \circ f_{u_T}(V) \subseteq \operatorname{int} D.$$

Pick any $z \in V$. For such a z,

$$z = f_{u_k} \circ \ldots \circ f_{u_0}(x)$$

for some $x \in \operatorname{int} D$ and

$$y = f_{u_{k+1}} \circ \ldots \circ f_{u_T}(z) \in \operatorname{int} D.$$

Thus, applying the first part of the proof to x and y (rather than to $\bar{x}$ and $\bar{y}$), it follows that $z \in D$. We conclude that $V \subseteq D$, so $\bar{z}$ is in $\operatorname{int} D$, as desired. □

LEMMA 7.2. *Let $D \subseteq X$ be a precontrollability set. Then we have*

$$D \subseteq \operatorname{clos} F_k(\operatorname{int} D) \quad \text{for all} \quad k = 0, 1, 2, \ldots,$$

where

$$F_k(\operatorname{int} D) = \bigcup_{l \geq k} R^l(\operatorname{int} D).$$

Proof. We proceed by induction on k. The case $k = 0$ follows directly from the defintion of controllability set. So let $k \geq 1$ and pick any $x \in D$.

Choose $y \in \operatorname{int} D$, $y \neq x$. By inductive assumption there exists a sequence $y_n \to y$ with

$$y_n \in F_k(\operatorname{int} D).$$

For $\bar{n}$ sufficiently large, $y_{\bar{n}} \in D$ (since $y \in \operatorname{int} D$) and $y_{\bar{n}} \neq x$ (since $y \neq x$), where each $y_{\bar{n}}$ is of the form

$$y_{\bar{n}} = \psi_{l,z}(\mathbf{u})$$

with $z \in \operatorname{int} D$, $l \geq k$, for some $\mathbf{u} \in U^l$. Pick one such $\bar{n}$. Since $x \in \operatorname{clos} R(y_{\bar{n}})$, there exist a sequence $\{t_n\}$ and a sequence $\{z_n\}$ so that

$$z_n \in R^{t_n}(y_{\bar{n}}) \quad \text{and} \quad z_n \to x.$$

Since $y_{\bar{n}} \neq x$ we can assume $t_n \geq 1$ for all n. Thus

$$z_n \in R^{l+t_n}(z) \subseteq F_{k+1}(\operatorname{int} D),$$

which implies $x \in \operatorname{clos} F_{k+1}(\operatorname{int} D)$. □

Remark 7.1. The conclusion of the previous lemma can be rephrased by saying that

$$D \subseteq \overline{\lim}_k R^k(\operatorname{int} D),$$

where for any family of sets E_k, $\overline{\lim}_k E_k = \bigcap_{k=0}^{\infty} \bigcup_{l \geq k} E_l$.

LEMMA 7.3. *Let $D \subseteq X$ be a controllability set. Then*

$$\operatorname{clos} D = \operatorname{clos} \operatorname{int} D.$$

Proof. Let $x \in D$. We only need to prove that for any neighbourhood W of x,

$$W \cap \operatorname{int} D \neq \emptyset.$$

Pick any such W and choose any $y \in \operatorname{int} D$. Since $y \in \operatorname{clos} R(x)$, we can find $z = \psi_{k,x}(\mathbf{u})$ for some $k \geq 0$ and some $\mathbf{u} \in U^k$, such that $z \in \operatorname{int} D$. Let U_z be a neighbourhood of z contained in D. Then, by continuity, there exists a neighbourhood U_x of x such that for all $y \in U_x$, $\psi_{k,y}(\mathbf{u})$ is in U_z and so, in particular, in $\operatorname{int} D$.

Let $W_x = U_x \cap W$. Choose $y' \in \operatorname{int} D$. Since $x \in \operatorname{clos} R(y')$, we can find k', $\mathbf{u}'$ such that

$$\bar{x} := \psi_{k',y'}(\mathbf{u}') \in W_x.$$

Let $\bar{\mathbf{u}}$ be the concatenation of $\mathbf{u}'$ and $\mathbf{u}$. Since $\bar{x} \in U_x$,

$$\psi_{k,\bar{x}}(\mathbf{u}) \in \operatorname{int} D.$$

Thus

$$\psi_{k+k',y'}(\bar{\mathbf{u}}) \in \operatorname{int} D,$$

so by Lemma (7.1), $\bar{x} \in \operatorname{int} D$. Hence

$$W_x \cap \operatorname{int} D \neq \emptyset,$$

so $W \cap \operatorname{int} D \neq \emptyset$ as wanted. □

DEFINITION 7.2. Let $x \in X$ and $S \subseteq X$. We say that x is *forward accessible in* S (respectively, *backward accessible in* S) if

$$\operatorname{int}(R(x) \cap S) \neq \emptyset$$

(respectively, $\operatorname{int}(C(x) \cap S) \neq \emptyset$).

If we simply say that x is forward (backward) accessible, we mean forward (backward) accessible in X.

LEMMA 7.4. *Let $S \subseteq X$ and define*

$$S_f = \{ \, x \in M \mid x \text{ is forward accessible in } S \, \},$$

then S_f is open.

Proof. If $S_f = \emptyset$, then it is trivially open; thus assume $S_f \neq \emptyset$. Pick any $x \in S_f$.

By assumption there exists $W \subseteq S$ open such that $W \subseteq R(x)$; therefore there exists k such that $W \cap R^k(x)$ has nonzero measure. Let

$$U_W^k = \{ \, \mathbf{u} \mid \mathbf{u} \in U^k, \text{ and } \psi_{k,x}(\mathbf{u}) \in W \, \},$$

then U_W^k is open and the image of

$$\psi_{k,x}|_{U_W^k}$$

has nonzero measure. It follows, by Sard's theorem, that there exists $\mathbf{u} \in U^k$ such that $\rho_{k,x}(\mathbf{u}) = n$. We may assume, without loss of generality, that $\mathbf{u} \in \operatorname{int} U^k$.

Now pick any neighbourhood V of x such that $\psi_{k,V}(\mathbf{u}) \subseteq W$ and still $\rho_{k,y}(\mathbf{u}) = n$ for all $y \in V$. By the implicit mapping theorem, $V \subseteq S_f$; therefore S_f is open. □

We assume from now on that the system Σ to be *analytic* and *transitive.* In this case, we can conclude the following important property of precontrollability sets.

THEOREM 7.5. *Let $D \subseteq X$ be a precontrollability set. Then every point of D is forward accessible in D.*

Proof. Since Σ is transitive and analytic, by Lemma 5.1 we have that there exists an open dense set of points for which $\dim L^+(x) = n$. If we intersect this set with $\operatorname{int} D$ then this intersection, which we denote by W, is open.

Pick any $x \in D$, and y in W, with $x \neq y$. Now, we will construct a sequence of elements y_n such that

$$y_n \to y,\ y_n \in R^{k_n}(x) \quad \text{and} \quad k_n \to \infty \ \text{ as } \ n \to \infty.$$

Then, using Lemma 4.1 and its succesive remarks, we can conclude that from x it is possible to reach an open set within any neighbourhood of y, i.e., since $y \in \operatorname{int} D$, x is forward accessible in D, as desired.

To construct the y_n's we proceed as follows. Let's denote by W_n a neighbourhood of y. Since $y \in \operatorname{clos} R(x)$, we can find $y_1 \in W_1$, $y_1 \neq y$, and $y_1 \in R^{k_1}(x)$ where (since $y \neq x$) we can assume $k_1 \geq 1$.

Now we proceed by induction. Suppose that we have found $y_1, \ldots, y_n$ such that

$$y_i \in W_i,\ y_i \neq y,\ y_i \in R^{k_i}(x) \ \text{ and } \ k_i \geq i \ \text{ for } \ i = 1, \ldots, n.$$

Since $y \in \operatorname{clos} R(y_n)$ we can find $y_{n+1} \in W_{n+1}$ such that

$$y_{n+1} \neq y,\ y_{n+1} \in R^l(y_n)$$

with $l \geq 1$. Since $y_n \in R^{k_n}(x)$,

$$y_{n+1} \in R^{k_n + l}(x)$$

and $k_{n+1} = k_n + l \geq n + 1$. Thus $k_n \to \infty$ as $n \to \infty$; moreover, we can choose the W_i's in such a way that $y_n \to y$. □

The definition of precontrollability set is not reversible in time, so we cannot conclude backward accessibility from every point. However, the next result provides backward accessibility from a dense subset.

PROPOSITION 7.6. *Let $D \subseteq X$ be a controllability set. Then there exists some (necessarily nonempty) open subset $E \subseteq D$ such that*

(1) $\operatorname{clos} E = \operatorname{clos} D$;

(2) *if $y \in E$ then y is backward accessible in D.*

Proof. Since Σ is transitive and analytic, by Lemma 5.1 applied to the "inverse" system

$$x(t+1) = f_u^-(x(t)),$$

we know that there exists an open dense set from which we have backward accessibility. Moreover, there exists some integer k_0 such that the set G of states $x \in X$ for which $\operatorname{int} C^{k_0}(x) \neq \emptyset$ and

$$\operatorname{clos} \operatorname{int} C^k(x) = \operatorname{clos} C^k(x)$$

for all $k \geq k_0$ is itself open dense (Remark 3.2). Consider first the open set

$$E' = \operatorname{int} D \cap G.$$

We claim that E' is open and

$$\operatorname{clos} E' = \operatorname{clos} \operatorname{int} D.$$

To show this, it is enough to establish that $\operatorname{int} D \subseteq \operatorname{clos} E'$. So take any $x \in \operatorname{int} D$. By density of G, there exists some sequence $\{y_n\}$ with $y_n \in G$ for all n, $y_n \to x$. Thus

$$y_n \in \operatorname{int} D \cap G$$

for all large enough n, and this shows that $x \in \operatorname{clos} E'$. Finally, let

$$E = E' \cap F_{k_0}(\operatorname{int} D),$$

where $F_{k_0}(\operatorname{int} D)$ is defined as in Lemma (7.2). Then E is also open, since $F_k(\operatorname{int} D)$ is open for any k. Moreover, using the result in Lemma (7.2) (i.e., $D \subseteq \operatorname{clos} F_{k_0}(\operatorname{int} D)$) and the same arguments used before we have

$$\operatorname{clos} E = \operatorname{clos} E' = \operatorname{clos} \operatorname{int} D.$$

Thus, by Lemma (7.3),

$$\operatorname{clos} E = \operatorname{clos} D.$$

So E satisfies property (1). We prove next that it also satisfies (2).

Pick $y \in E$. Since $y \in F_{k_0}(\operatorname{int} D)$ then there exists $x \in \operatorname{int} D$ so that $y \in R^k(x)$ for some $k \geq k_0$. This means that $x \in C^k(y)$. Since $y \in G$,

$$x \in \operatorname{clos} \operatorname{int} C^k(y).$$

Thus, since $x \in \operatorname{int} D$, we can find some $z \in \operatorname{int} D \cap \operatorname{int} C^k(y)$, which means that y is backward accessible in D. Thus (2) is proved. □

LEMMA 7.7. *Let $D \subseteq X$ be a controllability set and let E be any set as in the conclusion of the previous proposition. Then*

$$E \subseteq R(x) \quad \text{for each} \quad x \in D.$$

Proof. Take any $y \in E$ and $x \in D$. By the previous proposition, there exists some nonempty open set $W \subseteq D \cap C(y)$. Choose any $z \in W$. Since D is a controllability set, $z \in \operatorname{clos} R(x)$, so there exists also $\tilde{z} \in R(x) \cap W$. Thus $\tilde{z} \in R(x)$ and $y \in R(\tilde{z})$ (since $\tilde{z} \in C(y)$) imply $y \in R(x)$. □

DEFINITION 7.3. For any set $S \subseteq X$, define

$$\operatorname{Core}(S) := \{\ x \in \operatorname{int} S \mid x \ \text{ is forward and backward accessible in } \ S\ \}.$$

Using Lemma 7.4 twice (once for Σ and another time for the "inverse" system $x(t+1) = f_u^-(x(t))$), we can conclude the following.

LEMMA 7.8. *For any subset $S \subseteq X$,* Core (S) *is open.*

For a controllability set D, we proved (see results in Theorem 7.5 and Proposition (7.6)) that $\operatorname{Core}(D) \supseteq E$ for some $E \subseteq D$ such that $\operatorname{clos} E = \operatorname{clos} D$. Thus we have

$$\boxed{\operatorname{clos} \operatorname{Core}(D) = \operatorname{clos} D \quad \text{for a controllability set } \ D.} \tag{12}$$

Moreover, the result in Lemma (7.7) can be rephrased as follows.

PROPOSITION 7.9. *If D is a controllability set, and $E = \mathrm{Core}\,(D)$, then $E \subseteq R(x)$ for all $x \in D$.*

If D is a controllability set, then, by the previous results, $\mathrm{Core}\,(D)$ is a dense subset of D in which we have exact controllability. Note that if Σ was a continuous time system then $\mathrm{Core}\,(D)$ would have been equal to $\mathrm{int}\, D$. However for discrete-time systems there are controllability sets D for which $\mathrm{Core}\,(D)$ is strictly contained in $\mathrm{int}\, D$, as it is shown in the next example.

Example 7.1. Let us consider the discrete-time, analytic system with $X = \mathbb{R}^2$, $U = [-1,1]^2$, and equations

$$x^+ = x + 1 + uy,$$
$$y^+ = y + \frac{v}{2} g(x),$$

where $g(x)$ is the function in (10).

This system is invertible. In fact the determinant of the Jacobian matrix of the map $f_{u,v}(x,y)$ is given by

$$1 - \frac{uv}{2} g'(x).$$

Since $u, v \in [-1,1]$, and $|g'(x)| \leq 1$,

$$\left| \frac{uv}{2} g'(x) \right| \leq \frac{1}{2}$$

so the determinant is nonzero for all x, y. Moreover it is easy to verify that for each $(u,v) \in U$, the map $f_{u,v}(\cdot,\cdot)$ is bijective. It is also easy to prove that this system is transitive.

For this system we can see that for all $k \in \mathbb{N}$ with $k \geq 1$ the following hold:

1. the points of the type $(-k, 0)$ are not backward accessible;
2. the points of the type $(k, 0)$ are not forward accessible.

Let

$$B = \{\ (k,0) \mid k \in \mathbb{N},\quad k \geq 1\ \}.$$

Next we want to show that $D = \mathbb{R}^2 \setminus B$ is a controllability set.

Note that D is certainly maximal; in fact, no points in B could belong to a controllability set, since they are not forward accessible. To prove that D satisfies

$$D \subseteq \bar{R}(\xi) \quad \text{for all } \ \xi \in D \tag{13}$$

we will prove the following:

$$\mathbb{R}^2 \setminus \{\ (k,y) \mid k \in \mathbb{Z},\ y \in \mathbb{R}\ \} \subseteq R(\xi), \tag{14}$$

which, by taking the closure in both sides, implies (13). Let $F = \{\ (k,y) \mid k \in \mathbb{Z},\ y \in \mathbb{R}\ \}$.

First we note that, since $|\sin(\pi(x+1))| = |\sin(\pi x)|$, if we apply to any (x,y) a control sequence of the following form:

$$u_l = 0, \quad v_l = \ \mathrm{sign}\ (g(x + l - 1)), \tag{15}$$

then, after k steps, we will reach the following point:

$$x_k = x + k$$
$$y_k = y + \frac{|\sin(\pi x)|}{2\pi} \sum_{l=0}^{k-1} \frac{1}{|x+l|}.$$

Using this fact and the divergence of the series $\sum_n 1/n$ we will prove (14).

Fix $(\bar{x}, \bar{y}) \in D$ and $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \setminus F$. Note that, since $(\bar{x}, \bar{y}) \notin B$, it is not restrictive to assume

$$g(\bar{x}) \neq 0 \quad \text{and} \quad \bar{y} \neq 0.$$

First we choose u_l, v_l as in (15). Since $g(\bar{x}) \neq 0$ there exists k such that $y_k > 1$. Next we apply a control sequence with all $v_l = 0$ so as to reach a state (x', y') of the type

$$x' = \tilde{x} - n \quad \text{and} \quad y' = y_k,$$

where n is a positive integer that will be chosen later. Note that we can assume $\tilde{y} < y'$.

Now we want to find a sequence of controls $(0, v_l)$ such that we get the state $(\tilde{x}, \tilde{y})$ in exactly n steps. It is clear that this is possible if and only if

$$y' - \frac{|\sin(\pi \tilde{x})|}{2\pi} \sum_{l=0}^{n-1} \frac{1}{|\tilde{x} - n + l|} \leq \tilde{y}. \tag{16}$$

So we just have to choose n large enough such that (16) is satisfied. This is possible since $\sin(\pi\tilde{x}) \neq 0$ and

$$\sum_{l=0}^{n-1} \frac{1}{|\tilde{x} - n + l|} = \sum_{m=1}^{n} \frac{1}{|\tilde{x} - m|}$$

is divergent. Thus D is a controllability set.

Note that, for this controllability set D, Core (D) is strictly contained in $D =$ int D. In fact, none of the points of the type $(-k, 0)$ with k a strictly positive integer belongs to Core (D).

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