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# Optimality for underwater vehicles

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**Abstract**

*In this paper, we study time-optimal trajectories for fully actuated planar underwater vehicles, with constraints on input forces. Using the Maximum Principle, we focus on the structure of singular extremals and their possible optimality.*

## 1 Introduction

We consider in this paper the time-optimal problem for fully actuated underwater vehicles. We view our work as a preliminary step towards understanding optimal paths for a special class of underwater vehicles called *underwater gliders*. An underwater glider has no propellers and typically no control surfaces; it operates by means of a buoyancy-driven engine. Advantages of this design include low noise and vibration, high reliability, and the potential for lower reliance on battery power as compared to thruster-driven vehicles. Moreover, the propulsion and steering mechanisms for a fixed-wing, underwater glider are totally contained inside the vehicle. Thus, vulnerability to the harsh effects of seawater is significantly reduced.

The equations of motion describing an underwater glider include a rigid body dynamic model with fluid dynamic forces associated with buoyancy and viscous effects and extra degrees of freedom corresponding to actuated mass redistribution for attitude control [5]. Further, there are constraints on the control inputs depending on the state.

To gain some insight into the time-optimality problem we first consider, in this work, a simplified model which uses control thrusters with magnitude limits. Such vehicles, described in Section 2, can be adequately modelled as conservative controlled mechanical systems. Our study is based on the results obtained in [1]. Here, we restrict to the fully actuated situation in the vertical plane. A next step towards the eventual goal of understanding underwater gliders, is to consider the underactuated case, also in the vertical plane, in which the two inputs (for a three degree-of-freedom system) correspond to a force in the inertial vertical direction (like a variable buoyancy) and a torque (such as that produced by a shifting mass).

While we focus on time-optimality, one would ideally want to minimize a combination of time and energy consumption. We conclude this paper with a brief discussion of energy minimization in the context of what we have done for time-optimality.

## 2 Statement of the problem

Dynamics of underwater vehicles are described in [4]. In this paper, we consider a neutrally buoyant, ellipsoidal vehicle restricted to the vertical plane. We assume vehicle mass is uniformly distributed and we neglect viscous effects so that Kirchhoff's equations describe the vehicle dynamics. The configuration space of the vehicle is SE(2). Denote by $(x, z)$ the absolute position of the vehicle, where $x$ is the horizontal position and $z$ the vertical position. The angle $\theta$ describes the vehicle's orientation in this plane so that vehicle configuration is given by $q = (x, z, \theta)$. Let $\Omega$ be the scalar angular rate in the plane and $v_1, v_3$ the horizontal and vertical components of vehicle velocity in body frame coordinates. Following Kirchhoff's potential flow model of a rigid body in a fluid, the kinetic energy for our vehicle restricted to the plane is given by $T = \frac{1}{2}(I\Omega^2 + m_1 v_1^2 + m_3 v_3^2)$ where $I$ is the body-fluid moment of inertia in the plane and $m_1, m_3$ are body-fluid mass terms in the body horizontal and vertical directions, respectively. We assume the planar vehicle is not a circle: $m_1 \neq m_3$.

Let the state vector be $w = (x, z, \theta, v_1, v_3, \Omega)$. The

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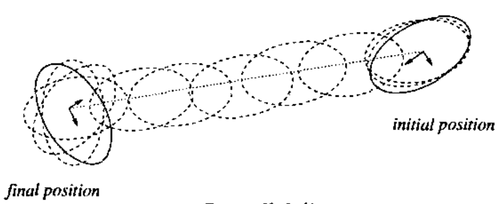
equations of motion are

$$\dot{w} = f(w) = \begin{pmatrix} \cos\theta v_1 + \sin\theta v_3 \\ \cos\theta v_3 - \sin\theta v_1 \\ \Omega \\ -v_3\Omega\frac{m_3}{m_1} \\ v_1\Omega\frac{m_1}{m_3} \\ v_1 v_3 \frac{m_3-m_1}{I} \end{pmatrix}$$

For the fully actuated case considered here, the control vector is $u = (u_1, u_2, u_3)$ where $u_1$ is a force in the body 1-axis, $u_2$ is a force in the body 3-axis and $u_3$ is a pure torque in the plane. Accordingly, the input vector fields $g_i$ are given by $g_1 = (0,0,0,\frac{1}{m_1},0,0)^t$, $g_2 = (0,0,0,0,\frac{1}{m_3},0)^t$ and $g_3 = (0,0,0,0,0,\frac{1}{I})^t$. The following constraints will be assumed on the domain of control: $\{\alpha_i \leq u_i \leq \beta_i;\ \alpha_i, \beta_i \in I\!R, \alpha_i < 0, \beta_i > 0\}$, $i = 1,2,3$.

Clearly, the fully actuated system is controllable if we assume the initial and final configurations to be at rest. Indeed, such a pair of configurations can be joined by a motion formed of pure rotations ($u_1 = u_2 \equiv 0$ and $u_3$ takes its value everywhere in $\{\alpha_3, \beta_3\}$) and pure translations ($u_2 = u_3 \equiv 0$ and $u_1$ takes its value everywhere in $\{\alpha_1, \beta_1\}$) as depicted in Figure 1 (in this paper, we will prove, however, that such a motion is not time optimal unless there is only a single piece formed by a pure rotation or a pure translation). This result can be generalized to complete controllability by showing that any configuration $w_0$ can be steered to a given configuration with zero velocities using a control satisfying $|\ u_i(t)\ | \leq \min(|\ \alpha_i\ |, \beta_i)$ (we take the minimum in order to have a symmetry property). We also note that from [1], any fully actuated controlled mechanical system is flat, here the flat outputs are the position and orientation variables $(x, z, \theta)$.




*Controllability*

Figure 1

In [1], we used the Maximum Principle (see [6] for a general reference) to obtain necessary conditions for a trajectory of a control system to be time optimal. Let us recall what these conditions are in our case. If $w(.)$ is a time-optimal trajectory defined on $[0,T]$, and $u(.)$ is the corresponding time-optimal control, then there exists an absolutely continuous vector function, called the *adjoint vector*, $\lambda : [0,T] \to I\!R^6$, such that the following conditions are satisfied:

1. $\lambda(t) \neq 0$ for all $t \in [0,T]$;
2. $H(w(t), \lambda(t), u(t)) = \max_v H(w(t), \lambda(t), v) \geq 0$ where the Hamiltonian function is given by

$$\begin{aligned} H(w,\lambda,u) &= \lambda_1(\cos\theta v_1 + \sin\theta v_3) \\ &+ \lambda_2(\cos\theta v_3 - \sin\theta v_1) + \lambda_3\Omega - \lambda_4 v_3\Omega\frac{m_3}{m_1} \\ &+ \lambda_5 v_1\Omega\frac{m_1}{m_3} + \lambda_6 v_1 v_3\frac{m_3-m_1}{I} + \lambda_4 u_1\frac{1}{m_1} \\ &+ \lambda_5 u_2\frac{1}{m_3} + \lambda_6 u_3\frac{1}{I}; \end{aligned}$$

3. the adjoint vector $\lambda(.)$ satisfies the following equations:

$$\dot{\lambda}_1 = 0 \quad (1)$$

$$\dot{\lambda}_2 = 0 \quad (2)$$

$$\dot{\lambda}_3 = \lambda_1(v_1\sin\theta - v_3\cos\theta) + \lambda_2(v_1\cos\theta + v_3\sin\theta) \quad (3)$$

$$\dot{\lambda}_4 = -\lambda_1\cos\theta + \lambda_2\sin\theta - \lambda_5\Omega\frac{m_1}{m_3} - \lambda_6 v_3\alpha \quad (4)$$

$$\dot{\lambda}_5 = -\lambda_1\sin\theta - \lambda_2\cos\theta + \lambda_4\Omega\frac{m_3}{m_1} - \lambda_6 v_1\alpha \quad (5)$$

$$\dot{\lambda}_6 = -\lambda_3 + \lambda_4 v_3\frac{m_3}{m_1} - \lambda_5 v_1\frac{m_1}{m_3} \quad (6)$$

where $\alpha = \frac{m_3-m_1}{I}$ is a nonzero constant. Note that as the two variables $x$ and $z$ do not appear explicitly in the expression of the Hamiltonian the corresponding adjoint variables $\lambda_1, \lambda_2$ are constant.

A triple $(w, \lambda, u)$ solution of conditions 1, 2 and 3 is called an *extremal*. From the maximization condition of the Hamiltonian:

$$H(w(t), \lambda(t), u(t)) = \max_v H(w(t), \lambda(t), v),$$

we have that $u_i(t) = \begin{cases} \beta_i & if\ \lambda^t(t)g_i(w(t)) > 0 \\ \alpha_i & if\ \lambda^t(t)g_i(x(t)) < 0 \end{cases}$ for $i = 1,2,3$. The functions $\phi_i(t) = \lambda^t(t)g_i(w(t))$, defined along an extremal $(w, \lambda, u)$, are called the *switching functions* associated to that extremal. Clearly, the zeroes of these functions are crucial for the study of optimal synthesis. If there exists a nonempty interval such that a given $\phi_i(.)$ is identically zero, we say that the extremal is $u_i$-*singular* on that interval. If an extremal is $u_i$-singular for all $i = 1,2,3$, it is called *totally singular*. Assume now the extremal to be *bang-bang* for the component $u_i$ of the control, i.e. $u_i$ takes its value in $\{\alpha_i, \beta_i\}$ for almost all $t$. A time $t_s$ such that $u_i$ is not almost everywhere constant on any interval of the form $]t_s - \varepsilon, t_s + \varepsilon[$, $\varepsilon > 0$ is called a $u_i$-*switching time* for $u_i$ and the corresponding state a $u_i$-*switching state* (or *point*).

From Propositions 3.2 and 3.3 in [1] we deduce the following result for the fully actuated underwater vehicle considered here:

**Proposition 1** *Along an extremal, there cannot exist any common accumulation point of zeroes for all switching functions. In particular, there is no totally singular extremal.*

This means, in particular, that if the underwater vehicle follows the equations of motion of the conservative mechanical system (this corresponds to the control being identically equal to 0), such a motion is not time-optimal. Hence we have to study extremals with at most two components of the control singular at the same time.

In our case the switching functions are given by the last three variables of the adjoint vector. Indeed, we have $\phi_1(t) = \frac{\lambda_4(t)}{m_1}$, $\phi_2(t) = \frac{\lambda_5(t)}{m_3}$ and $\phi_3(t) = \frac{\lambda_6(t)}{I}$. It is well known that the key tool in the study of the zeroes of these functions is the Lie algebra generated by the Lie bracket of the vector fields $f, g_i$. The Lie brackets of length 2 and 3 are computed in [2]. We know also from [1] that in the case of one nonsingular control, the nonsingular control has a finite number of switches along an extremal defined on an interval $[0, T]$ with $T > 0$. We prove here that there is a uniform bound on this number and that this bound is in fact 1. Notice that the existence of a bound is not a consequence of the Maximum Principle. Finding bounds on number of switches is a well-known problem in optimal control; see for instance in [7], where the author provides examples of optimal trajectories with an infinite number of switchings.

## 3 2-singular extremals

### 3.1 $u_1, u_2$-singular extremals

Along a $u_1, u_2$-singular extremal, both corresponding switching functions $\phi_1$ and $\phi_2$ are identically zero. This is equivalent to $\lambda_4 \equiv 0$ and $\lambda_5 \equiv 0$.

**Proposition 2** *Along a $u_1, u_2$-singular extremal, there is at most one $u_3$-switching and a necessary condition for there to be one switching is that $v_1 \equiv v_3 \equiv 0$ along the extremal (in which case $u_1, u_2$ are identically zero). Conversely if along an extremal we have $v_1$ and $v_3$ identically zero, then the corresponding trajectory is a $u_1, u_2$-singular extremal with at most one $u_3$-switching and $u_1 \equiv u_2 \equiv 0$.*

Proof: Along a $u_1, u_2$-singular extremal, we have $\phi_1 \equiv 0$ and $\phi_2 \equiv 0$. As the first derivative of the switching functions are absolutely continuous functions, this implies 4 relations on the adjoint vector: $\phi_i(t) = \lambda^t(t) g_i(w(t)) = 0$ and $\dot\phi_i(t) = \lambda^t(t)[f, g_i](w(t)) = 0$ for all $t$, $i = 1, 2$. Since $\dot\phi_1(t) = \frac{\dot\lambda_4(t)}{m_1}$ and $\dot\phi_2(t) = \frac{\dot\lambda_5(t)}{m_3}$, from equations (4),(5) we have:

$$\lambda_4 \equiv 0, \lambda_5 \equiv 0, \tag{7}$$

$$-\lambda_1 \cos\theta + \lambda_2 \sin\theta - \lambda_6 v_3 \alpha \equiv 0, \tag{8}$$

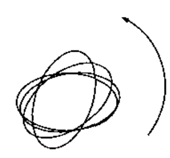
$$-\lambda_1 \sin\theta - \lambda_2 \cos\theta - \lambda_6 v_1 \alpha \equiv 0. \tag{9}$$

Applying Proposition 3.3 from [1], the $u_3$-switching states are contained in the set of points $S_3^c = A_1 \cap A_2$ where $A_i = \{w;\ det(g_1, g_2, g_3, [f, g_1], [f, g_2], ad_f^2 g_i) = 0\}$, $i = 1, 2$. Computing we find $A_1 = \{w;\ \frac{\alpha}{m_1^3 m_3^2} v_3 = 0\}$, $A_2 = \{w;\ \frac{\alpha}{m_1^2 m_3^3} v_1 = 0\}$. Hence: $S_3^c = \{w;\ v_1 = v_3 = 0\}$. Assume $t_0$ to be a $u_3$-switching time: $\phi_3(t_0) = \frac{1}{I}\lambda_6(t_0) = 0$. We must have $v_1(t_0) = v_3(t_0) = 0$ and equations (8), (9) evaluated at $t_0$ imply $\lambda_1 = 0, \lambda_2 = 0$. From equations (3) and (6), we deduce $\lambda_3 = constant$ and $\dot\lambda_6(t) = -\lambda_3$. Hence, the nonsingular switching function $\phi_3$ is linear in $t$ and given by $\phi_3(t) = -\frac{\lambda_3}{I} t + \frac{\lambda_6(0)}{I}$ (this function cannot be identically zero because $u_3$ is nonsingular). It follows that there is at most one $u_3$-switching along a $u_1, u_2$-singular extremal. Finally, replacing $\lambda_1 = \lambda_2 = 0$ in equations (8),(9) we conclude that a necessary condition for a $u_1, u_2$-singular extremal to have a $u_3$-switching is that the linear velocities $v_1$ and $v_3$ are identically zero. Conversely if $v_1 \equiv 0$ and $v_3 \equiv 0$ we have from the equations of motion $\dot v_1(t) = \frac{u_1(t)}{m_1} = 0$, $\dot v_3(t) = \frac{u_2(t)}{m_3} = 0$ for all $t$. Hence, the components of the control $u_1, u_2$ are identically zero.

Remark. From the equation for $\phi_3$: $\phi_3(t) = -\frac{\lambda_3}{I} t + \frac{\lambda_6(0)}{I}$. If there is a $u_3$-switching time $t_s$ it is given by $t_s = \frac{\lambda_6(0)}{\lambda_3}$.

Physical interpretation. Along a $u_1, u_2$-singular extremal with $v_1, v_3$ identically zero, the corresponding motion for the underwater vehicle is a pure rotation with constant angular acceleration (the angular velocity is a linear function of time), see Figure 2. In the event of one switching, the angular acceleration changes sign, i.e., after an acceleration in one direction the vehicle will first slow down and then accelerate in the other direction.

From the proof of Proposition 2, if $v_1$ and $v_3$ have a common zero then they are identically zero along the extremal. Consider now the case in which $v_1$ and $v_3$ never vanish at the same time. In that case the switching function $\phi_3$ does not vanish and the nonsingular control $u_3$ is constant. To compute $u_1$ and $u_2$ we use Proposition 3.3 from [1]. These computations are done in [2] with an example of such motion.



*pure rotation*

Figure 2

### 3.2 $u_1, u_3$-singular extremals

In this case $\phi_1$ and $\phi_3$ are identically zero along the extremal: $\lambda_4 \equiv 0$, $\lambda_6 \equiv 0$.

**Lemma 1** *Along a $u_1, u_3$-singular extremal, the component $u_2$ of the control is nonsingular with a finite number of switchings, and if $t_0$ is a $u_2$-switching time:*

$\phi_2(t_0) = 0$ *and* $v_1(t_0) = \Omega(t_0) = 0$, *but* $\dot{\phi}_2(t_0)$ *does not vanish. Moreover, along a* $u_1, u_3$*-singular extremal the following differential equation is satisfied:* $\ddot{\phi}_2(t) = \frac{m_1}{m_3}\Omega^2(t)\phi_2(t)$.

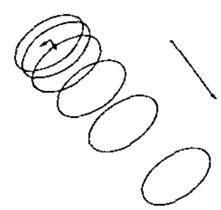
Proof: From [1], as discussed above, for an extremal with two components of the control singular, the non-singular control has a finite number of switchings. Assume $t_0$ to be such that $\phi_2(t_0) = \frac{1}{m_3}\lambda_5(t_0) = 0$. Following [1] we can prove that the $u_2$-switching states belong to the set $S_2^c = \{w;\ v_1 = \Omega = 0\}$. Hence, $v_1(t_0) = \Omega(t_0) = 0$. Using $\dot{\phi}_1(t) = \frac{\dot{\lambda}_4(t)}{m_1}$, $\dot{\phi}_3(t) = \frac{\dot{\lambda}_6(t)}{I}$ and equations (4) and (6), we obtain $-\lambda_1 \cos\theta(t_0) + \lambda_2 \sin\theta(t_0) = 0$ and $\lambda_3(t_0) = 0$. As $\dot{\phi}_2(t) = \frac{\dot{\lambda}_5(t)}{m_3}$, if $\dot{\phi}_2(t_0) = 0$ it would imply by equation (5) that $\lambda_1 \sin\theta(t_0) + \lambda_2 \cos\theta(t_0) = 0$. Thus, $\lambda_1 = \lambda_2 = 0$. As we have also $\lambda_4 \equiv 0$ and $\lambda_6 \equiv 0$ we deduce $\lambda(t_0) = 0$. This contradicts the fact that the adjoint vector cannot vanish along an extremal. Moreover, computing the second derivative of $\lambda_5(.)$ we have $\ddot{\lambda}_5(t) = (-\lambda_1\cos(\theta(t)) + \lambda_2\sin(\theta(t)))\Omega(t)$ and using $\dot{\lambda}_4 \equiv 0$, $\phi_2(t) = \frac{1}{m_3}\lambda_5(t)$ we obtain $\ddot{\phi}_2(t) = \frac{m_1}{m_3}\Omega^2(t)\phi_2(t)$.

**Proposition 3** *Along a* $u_1, u_3$*-singular extremal there is at most one* $u_2$*-switching. Moreover, if there is one, then* $\Omega \equiv 0$, $v_1 \equiv 0$ *and both singular components of the control are identically zero. Conversely, if* $\Omega \equiv 0$ *and* $v_1 \equiv 0$, *the corresponding trajectory is a* $u_1, u_3$*-singular extremal with at most one* $u_2$*-switching and* $u_1 \equiv u_3 \equiv 0$.

Proof: From Lemma 1 the switching function $\phi_2(.)$ and its second derivative are absolutely continuous functions with the same sign. Assume the extremal is defined on $[0, T]$ and $t_1, t_2$ such that $0 \leq t_1 < t_2 \leq T$. If $\phi_2(t_1) = \phi_2(t_2) = 0$, there exists $t_1 < t_m < t_2$ such that $\phi(t_m)$ is a maximum or a minimum. This contradicts the fact that $\text{sign}\phi_2(t) = \text{sign}\ddot{\phi}_2(t)$. Assume there is one $u_2$-switching time $t_s$ along the extremal: $\phi_2(t_s) = 0$. Then, using the differential equation for $\phi_2(.)$ and the fact that at a switching point $\Omega$ has to be zero (see proof of Lemma 1), we can easily verify that $\phi_2^{(n)}(t_s) = 0$ for $n \geq 2$. So, $\phi_2(.)$ is a linear function and has at most one zero. This implies $\ddot{\phi}_2(t) = 0$ for all $t$, hence $\Omega \equiv 0$ along the extremal. To prove that $v_1$ must also be identically 0 we use the algorithm described in [1] to compute singular controls. We find that $u_1(t) = -\frac{v_1(t)\gamma_1\dot{\phi}_2}{\phi_2(t)}$. Note that $\dot{\phi}_2$ is a constant and $\gamma_1$ is given by $\frac{m_1}{m_3} - 1$ (see [2]). From the equations of motion, we have $v_1(t) = \phi_2(t)^{-\gamma_1}C$ where $C$ is a constant. We proved in Lemma 1 that at a $u_2$-switching $v_1$ must vanish, i.e., $v_1(t_s) = 0$. Since $\phi_2(t_s) = 0$, then if $\gamma_1 > 0$ the only solution is $v_1 \equiv 0$. If $\gamma_1 < 0$, then $v_1(t_s) = 0$ but $u_1(t) \to \infty$ when $t \to t_s$. This contradicts the assumption on the domain of controls. Finally, if $\Omega \equiv 0$ and $v_1 \equiv 0$, using the equations of motion we conclude that $u_1$ and $u_3$ must be identically zero. Conversely, if $\Omega \equiv 0$ and $v_1 \equiv 0$, then from the equations of motion we have $\dot{v}_1(t) = \frac{u_1(t)}{m_1}$, $\dot{\Omega}(t) = \frac{u_3(t)}{I}$. This implies $u_1 \equiv 0$ and $u_3 \equiv 0$.

Remark. From the equation for $\phi_2$, if there is a $u_2$-switching time it occurs at $t_s = \frac{\lambda_5(0)}{\lambda_1 \sin\theta + \lambda_2 \cos\theta}$ (remember that if there is one switching, then $\theta$ is constant along the motion).

Physical interpretation. Along a $u_1, u_3$-singular extremal with $\Omega$ and $v_1$ identically zero, the corresponding motion for the underwater vehicle is a pure translation with constant linear acceleration in the direction of the vertical axis of the body frame ($v_3$ is a linear function of time), see Figure 3. Both singular components of the control $u_1, u_3$ are identically zero along such a motion. In the event of one switching, the linear acceleration changes sign, i.e., after an acceleration in one direction the vehicle will first slow down and then accelerate in the other direction. If $\Omega$ and $v_1$ are not identically zero, then the nonsingular component of the control $u_2$ is constant and the corresponding trajectory can be computed using Proposition 3.3 from [1].



*vertical translation in the body frame coordinate*

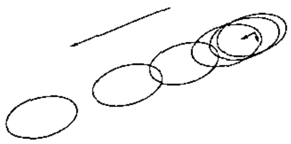
Figure 3

### 3.3 $u_2, u_3$-singular extremals

In this case $\phi_2$ and $\phi_3$ are identically zero along the extremal: $\lambda_5 \equiv 0$, $\lambda_6 \equiv 0$. The results and proofs are similar to the ones of Section 3.2 (simply invert the roles of $u_1$ and $u_2$).

**Proposition 4** *Along a* $u_2, u_3$*-singular extremal there is at most one* $u_1$*-switching. Moreover, if there is one, then* $\Omega \equiv 0$, $v_3 \equiv 0$ *and both singular components of the control are identically zero. Conversely, if* $\Omega \equiv 0$ *and* $v_3 \equiv 0$, *the corresponding trajectory is a* $u_2, u_3$*-singular extremal with at most one* $u_1$*-switching and* $u_2 \equiv u_3 \equiv 0$.

Physical interpretation. Along a $u_2, u_3$-singular extremal with $\Omega$ and $v_3$ identically zero, the corresponding motion for the underwater vehicle is a pure translation with constant linear acceleration in the direction of the horizontal axis of the body frame ($v_1$ is a linear function of time), see Figure 3. Both singular components of the control $u_2, u_3$ are identically zero along such a motion. In the event of one switching, the linear acceleration changes sign, i.e., after an acceleration in one direction the vehicle will first slow down and then accelerate in the other direction. If $\Omega$ and $v_3$ are not identically zero, then the nonsingular component of the control $u_1$ is constant and the corresponding trajectory can be computed using Proposition 3.3 from [1].



*horizontal translation in the body frame coordinates*

Figure 4

### 3.4 Concatenation of 2-singular extremals

In [1], we state necessary conditions on the Lie brackets of the vector fields $f, g_i$ under which some concatenations of singular extremals are not time-optimal. Here, we apply these conditions to fully actuated underwater vehicles.

**Proposition 5** *The concatenations of two different 2-singular extremals is not an extremal. In particular, such motions are not time-optimal.*

Proof: The result can be obtained by applying Proposition 3.5 in [1] or it can be also be proved directly. Indeed, assume $(w, \lambda, u)$ defined on $[0, T]$ to be the concatenation of a $u_1, u_2$-singular extremal and a $u_1, u_3$-singular extremal. Then, there exists $0 < t_0 < T$ such that $\phi_i(t_0)$ vanishes as well as its derivative $\dot{\phi}_i(t_0)$, for $i = 1, 2, 3$ ($w(t_0)$ is the state corresponding to the switch from one extremal to the other). From the differential equations satisfied by the adjoint vector (see Condition 3 in Section 2) we deduce that $\lambda(t_0) = 0$ which is a contradiction with the Maximum principle (see Condition 1). The proof is similar for the concatenation of a $u_1, u_2$-singular extremal (resp. $u_1, u_3$) with a $u_2, u_3$-singular extremal (resp. $u_2, u_3$).

Physical interpretation.

A consequence of Proposition 5 is that any motion formed by the concatenation of a pure rotation with a pure translation (in the direction of a body frame axis) or by the concatenation of a horizontal and a vertical translation (in the body frame) are not time-optimal. In particular, the motion described in Figure 1 is not time-optimal. A physical interpretation of this result is that in order to switch between two 2-singular extremals, the underwater vehicle has to be at rest, i.e., it has to spend time slowing down in the first direction before it can begin accelerating in the second direction.

### 3.5 Conclusions on 2-singular extremals

In the previous sections, we have proved that along a 2-singular extremal there is at most one switching for the nonsingular component of the control. Moreover, we showed that the basic motions: pure rotations and pure translations in the direction of a body frame axis, are 2-singular extremals. We conjecture the time-optimality of these basic motions; hence, they are likely to play a crucial role in time-optimal synthesis of motion. This means, in particular, we should not restrict ourselves to the study of bang-bang trajectories when dealing with time-optimality.

Remark. It is straightforward to verify that along an abnormal 2-singular extremal, a switching time can occur only when the vehicle is stoping: $v_1(t_s) = v_3(t_s) = \Omega(t_s) = 0$. We conjecture the general result that along a time-optimal trajectory the underwater vehicle never stops moving, hence such extremals would not have to be taken into account to determine time-optimal motions.

## 4 Optimality of some specific trajectories

Let us consider the initial and final positions illustrated in Figure 5 (same orientation and vertical position $z$) and assume the underwater vehicle is at rest at these positions. In the case that $\alpha_1 = -\beta_1$, we conjecture that the horizontal translation with one $u_1$-switching at half of the travel time is time-optimal, see Figure 6. As we saw previously this motion is a $u_2, u_3$-singular extremal.

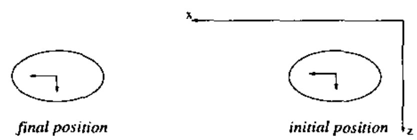
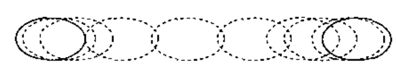



Figure 5



*motion = horizontal translation*

Figure 6

It is important to note further that the time used to travel between these two positions is not altered if to the horizontal motion we add some vertical motion as described by Figure 7. It can be verified that these motions are $u_3$-singular with one $u_1$-switching and a finite number of $u_2$-switchings. But as illustrated by Figure 7, there is no uniform bound on the number of $u_2$-switchings. We remark that in order for the singular component $u_3$ (determined using $\dot{\Omega} \equiv 0$ along such motion) to satisfy the constraint $\alpha_3 \leq u_3 \leq \beta_3$ along the trajectory, we have to assume that the initial and final positions are not too far away. The same kind of argument applies to motions represented on Figure 8 (along these motions $u_2$ is singular and $u_3$ is bang-bang with a finite number of switchings). The conclusion is that there exists an infinite number of trajectories joining these initial and final positions with the same elapsed time, and we conjecture that they are all optimal. This is true for more than just this particular choice of positions. Hence, this makes the use of numerical algorithms very sensitive for our problem.

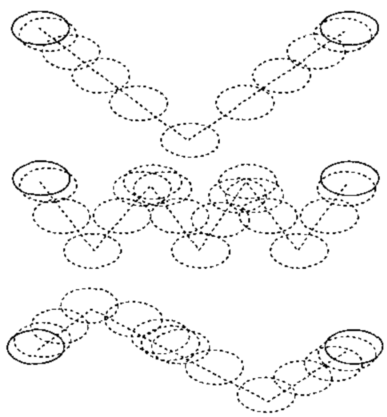


Figure 7

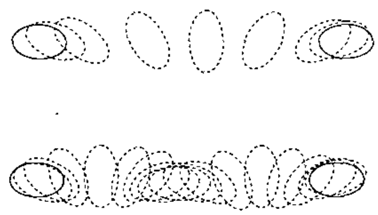


Figure 8

This leads us to consider the equally important problem of minimizing energy. When studying underwater vehicles, one is most often not only concerned with time of travel, but also with the energy spent in the process. Of course, a precise definition of "energy cost" has to be made. In [3], the author studies underwater vehicles with buoyancy-controlling mechanisms (there is no independent means of redistributing mass in the case studied). This buoyancy force is converted to horizontal motion by wing and fuselage lift (see also [8]). As for our gliders, a horizontal motion is not a feasible one (it would have been energy minimizing), and we have to look at motions such as those described in Figure 7 in order to find the time-optimal ones. In [3], it is shown that if we minimize the energy spent per meter of horizontal motion travel at a given horizontal speed, then the propulsion energy required for depths of 30 meters or less is more than 80 percent greater than that required at depths of 100 meters. Therefore, it costs less energy to let the vehicle glide deeply and make only one switch, than to make it oscillate around the straight line with a lot of switching. It is of interest to extend these results to our setting (with a glider that can change buoyancy and redistribute mass independently) and also to determine the optimal gliding angle.

It has been recently pointed out, see [9], on marine mammals that, as for underwater vehicles with gravity-buoyancy controlling mechanisms, travelling in the horizontal plane near the water surface demands different behaviors and energy cost as compared to travelling in the vertical plane. Using video cameras, biologists have observed that for depths exceeding 300 meters, nearly 80 percent of the descent is spent in a diving mode (which entails no locomotor activity for the mammal). This allows the animal to conserve energy and oxygen, in contrast to the swimming mode, and it helps explain the secret of marine mammals being able to diving deeply without breathing. This seems to be directly related to the previously mentioned results on energy expense for underwater vehicles driven with a buoyancy-engine.

In summary, to single out among the candidates for time-optimality those which minimize an energy cost, we should consider making our glider behave in this sense like a dolphin. Of course, in a realistic oceanic environment, there will in addition be constraints due to terrain and obstacles. We expect to extend our work in the future to include energy considerations as well as workspace constraints.

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