Comments on "Structure and Stability of Certain Chemical Networks and Applications to the Kinetic Proofreading Model of T-Cell Receptor Signal Transduction"

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Abstract— **A lemma in [1] was incorrect. A replacement is provided here.**

I. The error

There is an error in the statement and proof of Lemma VIII.2 in the paper [1]. This was pointed out by Madalena Chaves, whom the author also wishes to thank for very useful discussions.

The estimate (55) is incorrect, and it should be replaced by the following one:

$$\langle \vec{\rho}(x)-\vec{\rho}(z), f(x)\rangle \;\leq\; -\frac{c(x)\delta(x,z)}{4+\delta(x,z)} + \langle v(\vec{\rho}(x)-\vec{\rho}(z)), f(z)\rangle$$

(the first term in (55) was $-c(z)\delta(x,z)$). The mistake was made when passing from (57) to the next line, because the function f_a is not always negative. (As a side remark, note that the relevant equation numbers in the published version are inconsistent with the discussion, due to a typesetting error.)

We will explain here how this new estimate is proved, and why the main results are not affected by the change. All notations are as in [1].

II. The fix

We first note that (56) can also be written like this:

$$g(x,z) = \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} e^{\langle b_j,\vec{\rho}(x)\rangle}\left(e^{q_i-q_j}-1\right).$$

The main derivation is now as follows:

$$\begin{aligned}
\langle \vec{\rho}(x)-\vec{\rho}(z), f(x)\rangle &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} e^{\langle b_j,\vec{\rho}(x)\rangle}(q_i-q_j)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} e^{\langle b_j,\vec{\rho}(x)\rangle}(q_i-q_j-e^{q_i-q_j}+1) + g(x,z) \qquad (1)\\
&\leq -\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} e^{\langle b_j,\vec{\rho}(x)\rangle}\frac{(q_i-q_j)^2}{4+\delta(x,z)} + g(x,z) \qquad (2)\\
&\leq -\frac{c_0(x)}{4+\delta(x,z)}\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(q_i-q_j)^2 + g(x,z)\\
&= -\frac{c_0(x)}{4+\delta(x,z)}\, Q(q_1,\ldots,q_m) + g(x,z).
\end{aligned}$$

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As earlier, Equality (1) follows by adding and subtracting $g(x,z)$. To justify (2), we note first that

$$1+h-e^h \;\leq\; -\frac{h^2}{4+h^2}$$

for all $h\in\mathbb{R}$. We apply this inequality with $h=q_i-q_j$, and use that $(q_i-q_j)^2 \leq \sum_\ell\sum_k (q_\ell-q_k)^2 = \delta(x,z)$.

Lemma VIII.1 gives that $Q(q_1,\ldots,q_m) \;\geq\; \kappa\delta(x,z)$. Thus, we may take $c(x) := \kappa c_0(x)$. This completes the proof of the revised version of Lemma VIII.2.

III. The Proofs of the Main Results

Because of the new statement, a couple of small changes must be made in the proofs of the main results of the paper. First of all, in the proof of Theorem 4, instead of $\delta_S(x) = \frac{1}{4}\,c(\bar{x})^2\,\delta(x,\bar{x})$, one should define:

$$\delta_S(x) \;:=\; \frac{1}{4}\,\frac{c(x)^2\,\delta(x,\bar{x})}{(4+\delta(x,\bar{x}))^2}$$

which then leads to an upper bound as follows in (71):

$$-(1/2)\frac{c(x)\,\delta(x,\bar{x})}{4+\delta(x,\bar{x})}.$$

Finally, Remark VIII.C on exponential stability is still valid with the modified formulas, because

$$-(1/2)\frac{c(x)\,\delta(x,\bar{x})}{4+\delta(x,\bar{x})} \leq -\kappa\delta(x,\bar{x})$$

for any constant $\kappa > 0$ which lower-bounds the values $(1/2)\frac{c(x)}{4+\delta(x,\bar{x})}$ on a neighborhood of $\bar{x}$.

References

[1] E.D. Sontag, "Structure and stability of certain chemical networks and applications to the kinetic proofreading model of T-cell receptor signal transduction," *IEEE Trans. Autom. Control*, vol. 46, pp. 1028-1047, 2001.