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A small-gain theorem for almost global convergence of monotone systems☆

D. Angeli[a,*], P. De Leenheer[b], E.D. Sontag[c,1]

[a] Dip. di Sistemi e Informatica, Universitá di Firenze, Via di S. Marta 3, 50139 Firenze, Italy
[b] DIMACS, Rutgers University, New Brunswick, NJ 08903, USA
[c] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

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Abstract

A small-gain theorem is presented for almost global stability of monotone control systems which are open-loop almost globally stable, when constant inputs are applied. The theorem assumes "negative feedback" interconnections. This typically destroys the monotonicity of the original flow and potentially destabilizes the resulting closed-loop system.

Keywords: Almost global convergence; Small-gain theorems; Monotone systems

1. Introduction and basic definitions

This paper deals with interconnected *monotone control systems*. Traditional monotonicity notions [10,16,17], however, only apply to autonomous dynamical systems. This, in some sense, prevents the possibility of studying interconnected structures or at least shifts the focus from a high-level and top-down description of a process to a bottom-up analytical approach. Motivated by the need for decomposing highly complex systems in smaller modules which individually are "nice" in some sense (for instance monotone), an extension of the classical notion of monotone system was proposed in [3], which allows to include inputs and outputs in the theory. Roughly speaking, a system is monotone whenever its flow enjoys some monotonicity property with respect to initial conditions and exogenous signals entering the system. Models satisfying such requirements are often encountered in many different fields, such as molecular biology, ecology, economics or chemistry, just to name a few. Motivations and examples for the use of the theory in this new set-up are emerging and can be found in several recent papers [2–5,7,8] (note that this usage of the word monotone is different from that which appears in the theory of "monotone operators"). One of the main results in [3] was the formulation of a small-gain theorem for systems which, together with

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* Corresponding author. Tel.: +39-055-4796359; fax: +39-055-479-6363.
E-mail addresses: angeli@dsi.unifi.it (D. Angeli), leenheer@math.rutgers.edu (P. De Leenheer), sontag@control.rutgers.edu (E.D. Sontag).
URL: http://www.math.rutgers.edu/∼sontag
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monotonicity, enjoy *global* asymptotic stability at some point of the state-space (not necessarily the origin) for each constant value of the input signal. Here we provide a non-trivial extension of that result, which applies to systems whose convergence to the unique asymptotically stable equilibrium is only *almost* global, i.e., zero-measure sets of initial conditions which are not attracted to the equilibrium are allowed. This is, to the best of our knowledge, one of the first successful attempts to generalize a small-gain theorem to an *almost-global* setting. (For other nonlinear small-gain theorems, see e.g. [6,11–13,18,20], as well as the more classical versions in [9,14,15,22].)

The usual setting for monotone dynamical systems is a metric space $\mathscr{X}$ with a partial order induced by a cone; see [17]. For our purposes however, $\mathscr{X}$ will be assumed to be a real Euclidean space. The partial order $\preceq$ is induced by a nonempty closed cone $K \subset \mathscr{X}$ (a cone K satisfies: (i) $aK \subset K$ whenever $a \in \mathbb{R}_+$, (ii) $K + K \subset K$, (iii) $K \cap (-K) = \{0\}$) as follows:

$$\text{for } x, y \in \mathscr{X}: \quad x \preceq y \ \text{ iff } \ y - x \in K.$$

For $x, y \in \mathscr{X}$ we sometimes use the notation $x \prec y$, meaning that $x \preceq y$ but $x \neq y$; also the notation $x \succeq y$ and $x \succ y$ will be used with obvious interpretations.

We will assume throughout that a subset X of $\mathscr{X}$ is given and will refer to X as the *state space*. Note that X is not necessarily a linear space; in fact in many applications it is not linear, for instance when $\mathscr{X} = \mathbb{R}^n$ and $X = \mathbb{R}^n_{\geqslant 0}$. We assume that X is the closure of its interior.

We also assume that a set U is given which will serve as the input set—this is the set of values for input signals of the system. This set U is assumed to be a partially ordered subset of a Euclidean space $\mathscr{U}$ where the partial order is also induced by a nonempty closed cone in $\mathscr{U}$ in the way described before. Since there is no risk of confusion, the partial order on $\mathscr{U}$ is denoted by $\preceq$ as well.

The input signals of the system are assumed to be Lebesgue measurable functions $u(\cdot): \mathbb{R}_{\geqslant 0} \to U$ which are locally essentially bounded, meaning that for each time interval $[0, T]$ there is a compact set $C_T \subset U$ such that $u(t) \in C_T$ for almost all $t \in [0, T]$. [2]

[2] More precisely, this and other definitions should be interpreted in an “almost everywhere” sense, since inputs are Lebesgue-measurable functions.

The set of all input signals is denoted by $\mathscr{U}_\infty$. The partial order on U extends to a partial order on $\mathscr{U}_\infty$ in the following natural way:

$$\text{for } u_1(\cdot), u_2(\cdot) \in \mathscr{U}_\infty: u_1(\cdot) \preceq u_2(\cdot) \Leftrightarrow u_1(t) \preceq u_2(t)$$

$$\text{for almost all } \ t \geqslant 0.$$

A *controlled dynamical system* consists of a state space X and an input set U as above, and a mapping $x: E \to X$ with

$$E := \{(t, \xi, u(\cdot)) \mid \xi \in X, u(\cdot) \in \mathscr{U}_\infty, \ t \in [0, T_{\xi, u(\cdot)})\},$$

where $0 < T_{\xi,u(\cdot)} \leqslant +\infty$ possibly depends on ξ and $u(\cdot)$, such that the usual semigroup properties hold. (Namely, $x(0, \xi, u(\cdot)) = \xi$ and $x(t, x(s, \xi, u_1(\cdot)), u_2(\cdot)) = x(s + t, \xi, v(\cdot))$, where $v(\cdot)$ is the restriction of $u_1(\cdot)$ to the interval $[0, s]$ concatenated with the restriction of $u_2(\cdot)$ to $[s, +\infty)$.) Moreover, for technical reasons, we specialize to flows induced by the solutions of controlled differential equations with $\mathscr{C}^1$ right-hand side, so that for every $(\xi, u(\cdot)) \in X \times \mathscr{U}_\infty$ and every associated $t \in [0, T_{\xi,u(\cdot)})$, there exists an open neighborhood N_ξ of ξ such that the map $x_{t,\xi,u(\cdot)}: N_\xi \to X$ defined as $x_{t,\xi,u(\cdot)}(\xi) = x(t, \xi, u(\cdot))$ is a diffeomorphism onto its image.

We interpret $x(t, \xi, u(\cdot))$ as the state at time t for given initial state ξ and input signal $u(\cdot)$. Instead of $x(t, \xi, u)$ we will sometimes just write “$x(t)$” when we intend to emphasize time-dependence. We use “x” to denote both states (i.e., elements of X) and trajectories in case there is no risk of confusion, but we reserve Greek letters—possibly subscripted—such as ξ to denote states. Similarly, “u” may refer to an input value (element of U) or to an input signal (element of $\mathscr{U}_\infty$).

Definition 1.1. A controlled dynamical system $x: E \to X$ is *monotone* if the implication below holds:

$$u_1 \preceq u_2, \ x_1 \preceq x_2 \ \Rightarrow \ x(t, x_1, u_1) \preceq x(t, x_2, u_2),$$

for all t for which both states $x(t, x_1, u_1)$ and $x(t, x_2, u_2)$ are defined.

We will also consider monotone systems *with outputs* $y = h(x)$. These are specified by a monotone controlled dynamical system together with a continuous monotone ($x_1 \preceq x_2 \Rightarrow h(x_1) \preceq h(x_2)$) map $h: X \to Y$, where Y, the set of measurement or output values, is a subset of some partially ordered Euclidean space $\mathscr{Y}$ with a partial order induced by some nonempty closed cone in $\mathscr{Y}$ as before. We often use the shorthand $y(t,x,u)$ instead of $h(x(t,x,u))$, to denote the output at time t corresponding to the state obtained from initial state x and input signal u.

The central idea in the paper is that knowledge of the behavior of a monotone system when constant inputs are applied may provide a direct way of estimating the input-output gain of the system. In this context the following notion is crucial to the following developments.

Definition 1.2. We say that a system *is endowed with an input-state quasi-characteristic* $k_x : U \to X$, if for each constant input $u(t) \equiv q \in U$ and each initial state $\xi \in X$, the solution $x(t,\xi,q)$ is well defined for all $t \geqslant 0$, and there exists a *zero-measure* set $\mathscr{B}_q$ with the following property:

$$\forall \xi \in X \setminus \mathscr{B}_q \qquad \lim_{t\to+\infty} x(t,\xi,q) = k_x(q). \tag{1}$$

Accordingly, we may define an input–output quasi-characteristic $k_y : U \to Y$ by composition with the output map $h(x)$, i.e., $k_y(u) := h(k_x(u))$.

2. Asymptotic behavior of monotone systems

In this section we make more precise the statement that knowledge of I/S and I/O quasi-characteristics allows to quantify the asymptotic input–output behavior of the system. The next proposition is in fact the main technical tool used in the proof of the small gain theorem.

From now on, we consider only input (and output) spaces $U \subset \mathbb{R}^m$ ($Y \subset \mathbb{R}^p$) *which are ordered with respect to the standard order induced by the closed positive orthant*. In other words we let $K_u = \mathbb{R}^m_{\geqslant 0}$ and $K_y = \mathbb{R}^p_{\geqslant 0}$ so that, for instance, $u_1 \succeq u_2$ holds if and only if $[u_1]_i \geqslant [u_2]_i$ is true for all $i = 1, 2, \ldots, m$. Moreover we denote by $\limsup_{t\to+\infty} u(t)$ (respectively, $\liminf_{t\to+\infty} u(t)$) the componentwise limits, viz:

$$\limsup_{t\to+\infty} u(t) = \begin{bmatrix} \limsup\limits_{t\to+\infty} [u(t)]_1 \\ \vdots \\ \limsup\limits_{t\to+\infty} [u(t)]_m \end{bmatrix}.$$

It will be useful in order to guarantee that componentwise limits still make sense as input signals, to introduce the following notion:

Definition 2.1. We say that a set U is closed with respect to componentwise maximization (minimization) if the following implication holds:

$$u_1, u_2 \in U \quad \Rightarrow \quad \max(\min)\{u_1, u_2\} \in U \tag{2}$$

(where the max and min in (2) is interpreted componentwise).

It is easy to check, for the standard orthant order, that "box" intervals $[u_1, u_2] := \{u \in \mathbb{R}^m : u_1 \preceq u \preceq u_2\}$ enjoy the property of closure with respect to component-wise maximization and minimization. Some useful facts about sets closed with respect to componentwise maximization are proved in the Appendix.

Before stating the main result of this section we recall the usual notion of ω-limit set, which will be used in the following proposition. Given a trajectory $x(\cdot,\xi,u)$ defined over $[0,+\infty)$ we denote by $\Omega(x(\cdot,\xi,u))$ the set of limit-points, viz:

$$\Omega(x(\cdot,\xi,u)) := \{\bar{\bar{\xi}} \in X : \exists t_n \to +\infty \text{ as } n \to \infty \text{ such that } x(t_n,\xi,u) \to \bar{\bar{\xi}}\}. \tag{3}$$

Proposition 2.2. *Consider a monotone system with* continuous *I/S and I/O quasi-characteristics, respectively* $k_x(\cdot)$ *and* $k_y(\cdot)$.

Assume that inputs as well as outputs are ordered according to the standard order induced by the positive orthant in U *and* Y. *Let the input set* U *be closed (as a set) and closed with respect to componentwise maximization and minimization. Then, the I/S quasi-characteristic is nondecreasing and there exists a zero measure set* $\mathscr{B} \subset X$ *so that for each initial condition* $\xi \in X \setminus \mathscr{B}$ *and each bounded input* $u(\cdot)$ *such that the solution* $x(\cdot,\xi,u)$ *is defined for all* $t \geqslant 0$, *the following holds:*

$$\Omega(x(\cdot,\xi,u)) \subseteq [k_x(u_{\inf}), k_x(u_{\sup})], \tag{4}$$

where $u_{\inf} := \liminf_{t\to+\infty} u(t)$ *and* $u_{\sup} := \limsup_{t\to+\infty} u(t)$. *Moreover, the I/O quasi-characteristic is also nondecreasing, and for each initial condition* $\xi \in X \setminus \mathscr{B}$ *and each bounded input* $u(\cdot)$ *such* *that the solution $x(\cdot,\xi,u)$ is defined for all $t \geqslant 0$, the following inequality holds:*

$$k_y(u_{\inf}) \leqslant \liminf_{t\to+\infty} y(t,\xi,u) \leqslant \limsup_{t\to+\infty} y(t,\xi,u) \leqslant k_y(u_{\sup}). \quad (5)$$

Remark 2.3. Before proving the result we remark that the set $\mathscr{B}$ of non convergent initial conditions can be taken to be of zero-measure and independent of the input signal $u(\cdot)$. Of course, for each given u, the set of non-convergent initial conditions will typically be a proper subset of $\mathscr{B}$, possibly depending on u. Dependence upon u is therefore allowed, but strongly constrained, as only a zero-measure set is spanned as $u(\cdot)$ varies. This is a very strong property in general, and only follows by virtue of monotonicity.

Proof. Monotonicity of the I/S static quasi/characteristic is straightforward. In fact, let $u_1 \preceq u_2$ be arbitrary constant input signals and ξ any initial condition in $X \setminus (\mathscr{B}_{u_1} \cup \mathscr{B}_{u_2})$ (this set is non-empty as both $\mathscr{B}_{u_1}$ and $\mathscr{B}_{u_2}$ are zero measures while X is the closure of its interior and hence has strictly positive Lebesgue measure). By definition of I/S quasi-characteristic and monotonicity we have

$$k(u_1) = \lim_{t\to+\infty} x(t,\xi,u_1) \preceq \lim_{t\to+\infty} x(t,\xi,u_2) = k(u_2).$$

Let the sequences $v_k^{\inf}$ and $v_k^{\sup}$ be as in Lemma A.3. For each constant input $u(t) \equiv q$, let B_q denote the set of initial conditions which give rise to non-converging solutions (this set has zero-measure by assumption). We define $\mathscr{B}$ as follows:

$$\mathscr{B} := \bigcup_{\substack{n,k\in\mathbb{N} \\ \sigma\in\{\inf,\sup\} \\ q\in\mathscr{U}_0}} x(-n,B_q,v_k^\sigma), \quad (6)$$

where $\mathscr{U}_0$ is any countable and dense subset of U (every subset of $\mathbb{R}^m$ is separable). The sets $x(-n,B_q,v_k^\sigma)$ are not defined here by taking preimages of B_q under the flows of constant controls, since we have not made the assumption that backward flows are everywhere defined. Instead, they consist by definition of all those states ξ for which $x(n,\xi,v_k^\sigma)$ belongs to B_q—note that we have assumed that the I/S quasi-characteristic exists, and its definition includes the assumption that $x(t,\xi,v_k^\sigma)$ is well-defined for all $t \geqslant 0$, even if ξ does not belong to a pre-specified "good" set of states.

Each flow $F_{n,v_k^\sigma} = x(n,\cdot,v_k^\sigma)$ induces a diffeomorphism of X with its image. Therefore, the set $F^{-1}_{n,v_k^\sigma}(B_q) = x(-n,B_q,v_k^\sigma)$ has zero measure (diffeomorphisms preserve zero-measure sets); thus, the countable union $\mathscr{B}$ also has zero measure.

Let $\xi \in X \setminus \mathscr{B}$ be arbitrary and $u(\cdot)$ be any bounded input. By definition of lim sup, for each $\varepsilon > 0$ there exist $\bar{n} \in \mathbb{N}$ and $u_{\mathbb{Q}} \in \mathscr{U}_0$, so that $u_{\sup} + \varepsilon\mathbf{1} \succeq u_{\mathbb{Q}} \succeq u(t)$ for all $t \geqslant \bar{n}$.

Moreover, the choice of the $v_k^{\inf}$ and $v_k^{\sup}$ in Lemma A.3 implies existence of $k \in \mathbb{N}$ so that $v_k^{\inf} \preceq u(t) \preceq v_k^{\sup}$ for any $t \geqslant 0$. Thus, exploiting monotonicity we get for $t \geqslant \bar{n}$:

$$x(t,\xi,u(\cdot)) = x(t-\bar{n}, x(\bar{n},\xi,u(\cdot)), u(\cdot+\bar{n})) \preceq x(t-\bar{n}, x(\bar{n},\xi,v_k^{\sup}), u_{\mathbb{Q}}). \quad (7)$$

An analogous inequality holds for outputs:

$$y(t,\xi,u(\cdot)) \preceq y(t-\bar{n}, x(\bar{n},\xi,v_k^{\sup}), u_{\mathbb{Q}}). \quad (8)$$

Notice that $\xi \notin \mathscr{B} \Rightarrow x(\bar{n},\xi,v_k^{\sup}) \notin B_{u_{\mathbb{Q}}}$, therefore, taking lim sup in both sides of (8) yields

$$\limsup_{t\to+\infty} y(t,\xi,u(\cdot)) \preceq \limsup_{t\to+\infty} y(t-\bar{n}, x(\bar{n},\xi,v_k^{\sup}), u_{\mathbb{Q}}) = k_y(u_{\mathbb{Q}}). \quad (9)$$

By arbitrariness of ε and continuity of k_y then $\limsup_{t\to+\infty} y(t,\xi,u(\cdot)) \preceq k_y(u_{\sup})$. By a symmetric argument the lim inf inequality can also be proved; therefore this concludes the proof of (5). Similarly, taking any sequence $t_n \to +\infty$ so that $x(t_n,\xi,u(\cdot))$ admits a limit, we have from (7), and closure of the positivity cone:

$$\lim_{n\to+\infty} x(t_n,\xi,u(\cdot)) \preceq k_x(u_{\mathbb{Q}}). \quad (10)$$

As the sequence t_n was arbitrary, we conclude that $\Omega(x(t,\xi,u)) \preceq k_x(u_{\mathbb{Q}})$. By arbitrariness ε and continuity of k_x then $\Omega(x(t,\xi,u)) \preceq k_x(u_{\sup})$. A symmetric argument shows $\Omega(x(t,\xi,u)) \succeq k_x(u_{\inf})$. This concludes the proof of the proposition. □

3. The small gain theorem

Based on Proposition 2.2, one of our main results will be the formulation of a small-gain theorem for almost-global asymptotic stability of the "negative" feedback interconnection of systems with monotonically increasing I/O quasi-characteristics.

By a precompact solution $x(t)$ of a differential equation, we mean one for which $x(t) \in K$ for all t, for some compact subset K of the state space. (Such solutions are necessarily defined for all $t \geqslant 0$.)

Theorem 1. *Consider the following interconnection of two monotone dynamical systems*:

$$\dot{x}_1 = f_1(x_1, u_1), \quad y_1 = h_1(x_1),$$

$$\dot{x}_2 = f_2(x_2, u_2), \quad y_2 = h_2(x_2),$$

$$u_2 = y_1,$$

$$u_1 = -y_2 \tag{11}$$

with $U_1 \supset -Y_2$ and $U_2 \supset Y_1$ (well-posedness of the feedback loop). Suppose that

- *U_1, U_2, Y_1, Y_2 are ordered with respect to the positive orthants (of compatible dimensions), closed (as sets) and are closed under componentwise maximization and minimization,*
- *the respective I/S quasi-characteristics $k_{x_1}(\cdot)$ and $k_{x_2}(\cdot)$ exist and are continuous (thus, the I/O quasi-characteristics $k_1(\cdot)$* and *$k_2(\cdot)$ exist too),*
- *every solution of the closed-loop system is precompact.*

Then, system (11) has an almost globally attractive equilibrium $[k_{x_1}(\bar{u}), k_{x_2}(k_1(\bar{u}))]$, provided that the following discrete time dynamical system, evolving in U_1:

$$u_{k+1} = -(k_2 \circ k_1)(u_k) \tag{12}$$

has a unique globally attractive equilibrium $\bar{u}$.

Proof. Equilibria of (11) are in one-to-one correspondence with solutions of $k_2 \circ k_1(u) = -u$, i.e., equilibria of (12). Thus, existence and uniqueness of the equilibrium follows from the GAS assumption on (12).

We need to show that such an equilibrium is almost globally attractive. Denote by X_1 and X_2 the state spaces of the x_1 and x_2 subsystems, respectively. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the corresponding sets of initial conditions which give rise to non-converging solutions, as defined in Proposition 2.2. By virtue of the proposition both sets have zero measure.

Let $\xi \in (X_1 \setminus \mathscr{B}_1) \times (X_2 \setminus \mathscr{B}_2)$ be arbitrary. By assumption trajectories of the closed-loop system are bounded. Therefore, both $u_1^+ := \limsup_{t \to +\infty} u_1(t, \xi)$ and $u_1^- := \liminf_{t \to +\infty} u_1(t, \xi)$ exist and are finite. Then, $u_2^+ := \limsup_{t \to +\infty} u_2(t, \xi)$ and $u_2^- := \liminf_{t \to +\infty} u_2(t, \xi)$ satisfy by virtue of Proposition 2.2, applied to the x_1-subsystem:

$$k_1(u_1^-) \preceq u_2^- \preceq u_2^+ \preceq k_1(u_1^+). \tag{13}$$

An analogous argument, applied to the x_2-subsystem, yields $-k_2(u_2^+) \preceq u_1^- \preceq u_1^+ \preceq -k_2(u_2^-)$ and by combining this with the inequalities in (13) we end up with

$$-k_2(k_1(u_1^+)) \preceq u_1^- \preceq u_1^+ \preceq -k_2(k_1(u_1^-)).$$

By induction we have, after an even number $2n$ of iterations of the above argument

$$(-k_2 \circ k_1)^{2n}(u_1^-) \preceq u_1^- \preceq u_1^+ \preceq (-k_2 \circ k_1)^{2n}(u_1^+).$$

By letting $n \to +\infty$ and exploiting global attractivity of (12) we have $u_1^- = u_1^+$. Eq. (13) yields $u_2^- = u_2^+$. Thus there exists $\bar{u}$, such that

$$\bar{u} = \lim_{t \to +\infty} u_1(t, \xi),$$

$$k_1(\bar{u}) = \lim_{t \to +\infty} u_2(t, \xi). \tag{14}$$

Let e_1 be the (almost globally asymptotically stable) equilibrium (for the x_1-subsystem) corresponding to the constant input $u_1(t) \equiv \bar{u}$ and e_2 the equilibrium for the x_2-subsystem relative to the input $u_2(t) \equiv k_1(\bar{u})$. Clearly $\eta := [e_1, e_2]$ is an equilibrium of (11). The fact that $[x_1(t), x_2(t)] \to \eta$ now follows from Proposition 2.2.

Notice that $(X_1 \setminus \mathscr{B}_1) \times (X_2 \setminus \mathscr{B}_2) = (X_1 \times X_2) \setminus \mathscr{B}$ provided that we denote $\mathscr{B} := (\mathscr{B}_1 \times X_2) \cup (X_1 \times \mathscr{B}_2)$. Clearly $\mathscr{B}$ has zero measure. This concludes the proof of the theorem. □

The following result is a simple but interesting consequence of Proposition 2.2.

Proposition 3.1. *Consider the following cascade of monotone systems*:

$$\dot{x}_1 = f_1(x_1, u_1), \quad y_1 = h_1(x_1),$$
$$\dot{x}_2 = f_2(x_2, u_2), \quad y_2 = h_2(x_2),$$
$$u_1 = y_2, \tag{15}$$

both endowed with I/S and I/O quasi-characteristics, $k_{x_1} : U_1 \to X_1$, $k_{x_2} : U_2 \to X_2$, *and* $k_1 : U_1 \to Y_1$, $k_2 : U_2 \to Y_2$, *respectively. Then, system* (15) *is monotone and admits I/S and I/O quasi-characteristic*, $k_x(u) := [k_{x_1}(k_2(u)), k_{x_2}(u)]$, $k := k_1 \circ k_2$ *provided that the partial orders of* Y_2 *and* U_1 *are compatible.*

Proof. By compatibility of the orders the cascaded interconnection is again a monotone dynamical system (see [3]). We next show that the I/S and I/O quasi-characteristics are well defined. Let X_1 and X_2 be the state-spaces relative to subsystem x_1 and x_2, respectively. By Proposition 2.2 there exists a zero-measure set $\mathscr{B}_2 \subset X_2$ such that for all constant inputs $u_2(\cdot) \equiv u_2$ and all initial conditions $\xi_2 \in X_2 \setminus \mathscr{B}_2$, the following holds:

$$\lim_{t \to +\infty} x_2(t, \xi_2, u_2) = k_{x_2}(u_2) \tag{16}$$

and consequently $y_2(t, \xi_2, u_2) \to k_2(u_2)$. As $u_1 = y_2$ we can complete our argument by applying Proposition 2.2 to the x_1 subsystem. In fact, there exists $\mathscr{B}_1 \subset X_1$ of zero-measure, such that for any $\xi_1 \in X_1 \setminus \mathscr{B}_1$ we have

$$\Omega(x_1(\cdot, \xi_1, u_1)) \subseteq [k_{x_1}(\liminf u_1), k_{x_1}(\limsup u_1)]$$
$$= \{k_{x_1}(k_2(u_2))\}. \tag{17}$$

Notice, defining $\mathscr{B} = (X_1 \times \mathscr{B}_2) \cup (\mathscr{B}_1 \times X_2)$ that $X_1 \times X_2 \setminus \mathscr{B} = (X_1 \setminus \mathscr{B}_1) \times (X_2 \setminus \mathscr{B}_2)$; moreover $\mathscr{B}$ has zero-measure in the product space $X_1 \times X_2$. Now, combining (17) and (16) yields, for all $\xi := [\xi_1, \xi_2] \in (X_1 \setminus \mathscr{B}_1) \times (X_2 \setminus \mathscr{B}_2)$ and all constant inputs $u_2(\cdot)$,

$$\lim_{t \to +\infty} x(t, \xi, u_2) = [k_{x_1}(k_2(u_2)), \ k_{x_2}(u_2)], \tag{18}$$

and, taking output maps:

$$\lim_{t \to +\infty} h_2(x(t, \xi, u_2)) = k_1(k_2(u_2)). \tag{19}$$

This completes the proof of the proposition. □

4. An application to Lotka–Volterra systems

An interesting class of models for which the small gain theorem applies is the class of Lotka–Volterra systems given below:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \mathrm{diag}\,(x, z) \left[\begin{pmatrix} A & B \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right]. \tag{20}$$

Lotka–Volterra systems evolve in the positive orthant and, it is easy to show, the boundary of the state space, is an invariant set for (20). Consequently, any attractor can be at most almost globally attractive as the boundary of X is usually left apart from the basin of attraction, either completely or partially. Moreover, it is shown in [7], that if A and D are Metzler and Hurwitz, and B, C are nonnegative matrices, then system (20) can be seen as the "negative" feedback interconnection of two cooperative systems, each of which admits a piecewise linear I/S quasi-characteristic. Explicit computation of the characteristic is usually possible and therefore, the small gain condition can be checked rigorously. See [7] for a detailed analysis. It is shown in [7], through simulations and bifurcation analysis that high feedback gains may lead to instability and a Hopf bifurcation may occur, causing self-sustained oscillations for the class of systems (20). The small-gain theorem proved here allows one to determine upper bounds on feedback under which such oscillations will not occur.

5. Conclusions

This note extends the small gain theorem for negative feedback interconnections of monotone systems with input-state static characteristics (first appeared in [3]) to an almost global set-up. Namely, the static asymptotic value corresponding to any constant input is for almost all initial conditions unique and well-defined. A zero measure set of initial conditions which give rise to non convergent trajectories is therefore allowed as well as the presence of multiple equilibria provided that only one is asymptotically stable and the basins of attractions of the unstable equilibria have zero measure. Results of convergence under cascading are not new in the literature, see for instance [16,19,21], or even in the almost global set-up [1]. Small-gain theorems are instead somewhat new in the almost global context and still an area of active research. The theorem presented here heavily relies on monotonicity in order to show that the set of initial conditions which do not produce a convergent trajectory has zero measure.

Appendix A. Remarks on componentwise maximization

We present in this appendix some basic facts about sets which are closed under componentwise maximization (see Fig. 1).

Lemma A.1. *A closed set U is closed with respect to componentwise maximization (minimization), if and only if, for any bounded sequence $u_n \in U$,*

$$\limsup_{n\to+\infty} u_n \in U \qquad \left(\liminf_{n\to+\infty} u_n \in U\right).$$

Proof. We show first sufficiency. Let u_1, u_2 be arbitrary in U. We let $u_n := u_{1+n_{\text{mod } 2}}$. Clearly $\max\{u_1, u_2\} = \limsup_{n\to+\infty} u_n \in U$. This concludes the proof of sufficiency. Conversely, let $u_n \in U$ be an arbitrary sequence. We let $v_1^k := u_{k+1}$ and $v_n^k := \max\{v_{n-1}^k, u_{k+n}\}$ for all $k, n \in \mathbb{N}$. By induction v_n^k belongs to U for all $k, n \in \mathbb{N}$ (we are using of course closure with respect to componentwise maximization). Moreover, for all fixed $k \in \mathbb{N}$ each component of v_n^k gives a monotone nondecreasing and bounded sequence. Then it admits a limit and, by

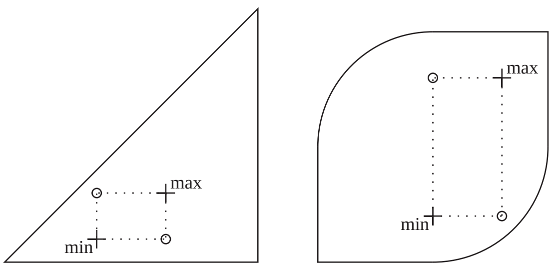

Fig. 1. Sets closed with respect to componentwise maximization and minimization.

closure of U we have

$$v_\infty^k := \lim_{n\to+\infty} v_n^k \in U.$$

It is straightforward from its definition that

$$\limsup_{n\to+\infty} u_n = \lim_{k\to+\infty} v_\infty^k.$$

Therefore, the claim follows by closure of U. An analogous proof applies for componentwise minimization (as a matter of fact $\min\{v_1, v_2\} = -\max\{-v_1, -v_2\}$). □

Lemma A.2. *Let U be compact, and closed with respect to componentwise maximization (minimization). Then, it contains a unique maximal (minimal) element $\bar{u}$,* i.e., *there is a unique $\bar{u} \in U$ so that $\bar{u} \succeq u (\preceq u)$ for all $u \in U$.*

Proof. Let $\mathcal{U}_0 = \{u_n, n = 1, 2, \ldots\}$ be a dense and countable subset of U. We define $v_1 := u_1$ and $v_n := \max\{v_{n-1}, u_n\}$ for all $n \geqslant 2$. By construction, $v_n \in U$ is a non-decreasing sequence. Therefore it admits a unique limit and, since U is closed, we have

$$\lim_{n\to+\infty} v_n = \bar{v} \in U. \tag{A.1}$$

We claim that $\bar{v}$ is the sought maximal element. In fact, by monotonicity of v_n we have that $\bar{v} \succeq v_n$ for all $n \in \mathbb{N}$ and in particular then $\bar{v} \succeq u$ for all $u \in \mathcal{U}_0$. By density of $\mathcal{U}_0$ then, the same holds true for all $u \in U$. Uniqueness of the maximal element follows because $\bar{v}_1 \succeq \bar{v}_2$ and $\bar{v}_2 \succeq \bar{v}_1$ implies $\bar{v}_1 = \bar{v}_2$. □

Lemma A.3. *Let U be closed and closed with respect to the componentwise maximization (minimization). Then, there exists a sequence $v_k^{\sup}$ ($v_k^{\inf}$) of elements of U such that given any compact subset $K \subset U$, $K \preceq v_k^{\sup}$ ($K \succeq v_k^{\inf}$) for all sufficiently large k's.*

Proof. Notice first of all that the intersection of sets closed with respect to componentwise maximization is again a set of the same kind. Thus, defining $U_k := U \cap k[-\mathbf{1}, \mathbf{1}]$, for all $k \in \mathbb{N}$, we have that U_k is compact and closed with respect to componentwise maximization (minimization). By Lemma A.2 it admits a unique maximal element $v_k^{\sup}$. Given now an arbitrary compact subset $K \subset U$ we have: $K \subset k[-\mathbf{1},\mathbf{1}]$ for all sufficiently large k's. Hence $K \subset U_k \preceq v_k^{\sup}$ for all such k's. This concludes the proof of the claim. □

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