Logarithmic Lipschitz norms and diffusion-induced instability

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A B S T R A C T

This paper proves that ordinary differential equation systems that are contractive with respect to L^p norms remain so when diffusion is added. Thus, diffusive instabilities, in the sense of the Turing phenomenon, cannot arise for such systems, and in fact any two solutions converge exponentially to each other. The key tools are semi inner products and logarithmic Lipschitz constants in Banach spaces. An example from biochemistry is discussed, which shows the necessity of considering non-Hilbert spaces. An analogous result for graph-defined interconnections of systems defined by ordinary differential equations is given as well.

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1. Introduction

In this work, we study reaction–diffusion PDE systems

$$\frac{\partial u}{\partial t} = F(u, t) + D\Delta u$$

as well as their discrete analogues ("compartmental-systems"). Here,

$$u(\omega, t) = (u_1(\omega, t), \ldots, u_n(\omega, t)), \qquad \frac{\partial u}{\partial t} = \left(\frac{\partial u_1}{\partial t}, \ldots, \frac{\partial u_n}{\partial t} \right),$$

Δ is the Laplacian operator on a suitable spatial domain Ω, and no flux (Neumann) boundary conditions are assumed.

In biology, a PDE system of this form describes individuals (particles, chemical species, etc.) of n different types, with respective abundances $u_i(\omega, t)$ at time t and location $\omega \in \Omega$, that can react instantaneously, guided by the interaction rules encoded into the vector field F, and can diffuse due to random motion. Reaction–diffusion PDEs play a key role in modeling intracellular dynamics and protein localization in cell processes such as cell division and eukaryotic chemotaxis (e.g., [1–4]) as well as in the modeling of differentiation in multi-cellular organisms, through the diffusion of morphogens which control heterogeneity in gene expression in different cells (e.g. [5,6]). From a bioengineering perspective, reaction–diffusion models can be used to model artificial mechanisms for achieving cellular heterogeneity in tissue homeostasis (e.g., [7,8]).

The "symmetry breaking" phenomenon of diffusion-induced, or Turing, instability refers to the case where a dynamic equilibrium $\bar{u}$ of the non-diffusing ODE system $\frac{\partial u}{\partial t} = F(u, t)$ is stable, but, at least for some diagonal positive matrices D, the corresponding uniform state $u(\omega) = \bar{u}$ is unstable for the PDE system $\frac{\partial u}{\partial t} = F(u, t) + D\Delta u$. This phenomenon has been studied at least since Turing's seminal work on pattern formation in morphogenesis [9], where he argued that chemicals might react and diffuse so as result in heterogeneous spatial patterns. Subsequent work by Gierer and Meinhardt [10,11] produced a molecularly plausible minimal model, using two substances that combine local autocatalysis and long-ranging inhibition. Since that early work, a variety of processes in physics, chemistry, biology, and many other areas have been studied from the point of view of diffusive instabilities, and the mathematics of the process has been extensively studied [12–16,5,17,18, 6,19]. Most past work has focused on local stability analysis, through the analysis of the instability of nonuniform spatial

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modes of the linearized PDE. Nonlinear, global, results are usually proved under strong constraints on diffusion constants as they compare to the growth of the reaction part.

In this work, we are interested in conditions on the reaction part F that guarantee that no diffusion instability will occur, no matter what is the size of the diffusion matrix D. We show that if the reaction system is "contractive" in the sense that trajectories globally and exponentially converge to each other with respect to a diagonally weighted L^p norm, then the same property is inherited by the PDE. In particular, if there is an equilibrium, $\bar{u}$, $F(\bar{u}, t) = 0$, it will follow that this equilibrium is globally exponentially stable for the PDE system. A similar result is also established for a discrete analog, in which a set of ODE systems are diffusively interconnected. We were motivated by the desire to understand the important biological systems described in [20,21] for which, as we will show, contractivity holds for diagonally weighted L^1 norms, but not with respect to diagonally weighted L^p norms, for any $1 < p \leq \infty$.

There have been other works that imposed conditions on F that insure no diffusion instability. Particularly relevant are [22,23], both of which provide conditions for asymptotic stability of solutions of the PDE based on properties of F. The first of these references uses a condition based on "contracting rectangles" (a condition should not be confused with our contractivity notion, which refers to infinitesimal contractivity of the vector field), and the second one an L^∞-like Lyapunov function. Our results, in contrast, provide *global* asymptotic stability. In fact, for systems satisfying our assumptions we show that solutions exponentially converge to each other, a property that is considerably stronger than global asymptotic stability, see Remark 8. In addition, we can also allow our systems to be time-dependent, which permits one to obtain conclusions about limit cycles, see Remark 8.

Closely related work in the literature has dealt with the synchronization problem, in which one is interested in the convergence of trajectories to their space averages in weighted L^2 norms, for appropriate diffusion coefficients and Laplacian eigenvalues, specifically [24], which used passivity ideas from control theory for systems with special structures such as cyclic systems, [25] which extended this approach to more general passive structures, and [26] which obtained a generalization involving a contraction-like diagonal stability condition. For contractions with respect to L^2 norm, a similar result had also been obtained in [27]. Our work uses very different techniques, from nonlinear functional analysis for normed spaces, than the quadratic Lyapunov function approaches, appropriate for Hilbert spaces, followed in these references.

2. Logarithmic Lipschitz constants and norms

We start by reviewing several useful concepts from nonlinear functional analysis, and proving certain technical properties for them.

2.1. General normed spaces

Definition 1 ([28,29])**.** Let $(X, \|\cdot\|_X)$ be a normed space. For $x_1, x_2 \in X$, the right and left semi inner products are defined by

$$(x_1, x_2)_\pm = \|x_1\|_X \lim_{h \to 0^\pm} \frac{1}{h} \left(\|x_1 + hx_2\|_X - \|x_1\|_X \right). \tag{1}$$

Remark 1. As every norm possesses left and right Gâteaux-differentials, the limits in (1) exist and are finite. For more details, see [30].

Remark 2. The right and left semi inner products $(\cdot, \cdot)_\pm$, induce the norm $\|\cdot\|_X$ in the usual way: $(x, x)_\pm = \|x\|_X^2$. Conversely, if the norm arises from an inner product $(\cdot, \cdot)$, as when X is a Hilbert space, $(x_1, x_2)_+ = (x_1, x_2)_- = (x_1, x_2)$. Moreover the right and left semi inner products satisfy the Cauchy–Schwarz inequalities:

$$-\|x\| \cdot \|y\| \leq (x, y)_\pm \leq \|x\| \cdot \|y\|.$$

The following elementary properties of semi inner products are consequences of the properties of norms. See [28,29] for the proof.

Proposition 1. *For $x, y, z \in X$ and $\alpha \geq 0$,*
1. $(x, -y)_\pm = -(x, y)_\mp$;
2. $(x, \alpha y)_\pm = \alpha(x, y)_\pm$;
3. $(x, y)_- + (x, z)_\pm \leq (x, y + z)_\pm \leq (x, y)_+ + (x, z)_\pm$.

Remark 3. In general, the semi inner product is not symmetric:

$$(x, y)_\pm \neq (y, x)_\pm.$$

Definition 2 ([29])**.** Let $(X, \|\cdot\|_X)$ be a normed space and $f : Y \to X$ be a function, where $Y \subseteq X$. The strong least upper bound logarithmic Lipschitz constants of f induced by the norm $\|\cdot\|_X$, on Y, are defined by

$$M_{Y,X}^\pm[f] = \sup_{u \neq v \in Y} \frac{(u - v, f(u) - f(v))_\pm}{\|u - v\|_X^2},$$

or equivalently

$$M_{Y,X}^{\pm}[f] = \sup_{u \neq v \in Y} \lim_{h \to 0^{\pm}} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right). \tag{2}$$

If $X = Y$, we write $M_X^{\pm}$ instead of $M_{X,X}^{\pm}$.

Note that in [29], M^+ is called the least upper bound logarithmic Lipschitz constants with respect to the semi inner product $(\cdot, \cdot)_+$.

We cite the following facts from [29].

Proposition 2. *Let* $(X, \| \cdot \|_X)$ *be a normed space. For any* $f, g \colon Y \to X$ *and any* $Y \subseteq X$:
1. $M_{Y,X}^+[f + g] \leq M_{Y,X}^+[f] + M_{Y,X}^+[g]$;
2. $M_{Y,X}^{\pm}[\alpha f] = \alpha M_{Y,X}^{\pm}[f]$ *for* $\alpha \geq 0$.

Definition 3 (*[29]*)**.** Let $(X, \| \cdot \|_X)$ be a normed space and $f \colon Y \to X$ be a function, where $Y \subseteq X$. The least upper bound Lipschitz constant of f induced by the norm $\| \cdot \|_X$, on Y, is defined by

$$L_{Y,X}[f] = \sup_{u \neq v \in Y} \frac{\|f(u) - f(v)\|_X}{\|u - v\|_X}.$$

Note that $L_{Y,X}[f] < \infty$ if and only if f is Lipschitz on Y.

Definition 4 (*[29]*)**.** Let $(X, \| \cdot \|_X)$ be a normed space and $f \colon Y \to X$ be a Lipschitz function. The least upper bound logarithmic Lipschitz constant of f induced by the norm $\| \cdot \|_X$, on $Y \subseteq X$, is defined by

$$M_{Y,X}[f] = \lim_{h \to 0+} \frac{1}{h} \left(L_{Y,X}[I + hf] - 1 \right),$$

or equivalently

$$M_{Y,X}[f] = \lim_{h \to 0+} \sup_{u \neq v \in Y} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right). \tag{3}$$

If $X = Y$, we write M_X instead of $M_{X,X}$.

Lemma 1. *Since for any* $u \neq v \in Y$,

$$\lim_{h \to 0+} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right) \leq \lim_{h \to 0+} \sup_{u \neq v \in Y} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right),$$

by definition, $M_{Y,X}^+[f] \leq M_{Y,X}[f]$.

2.2. Finite dimensional case

The least upper bound (lub) logarithmic Lipschitz constant generalizes the usual logarithmic norm; for every matrix A we have $M_X[A] = \mu_X[A]$. For ease of reference, we review next the basic properties of logarithmic norms for finite dimensional operators.

Definition 5. Let $(X, \| \cdot \|_X)$ be a finite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. The space $\mathscr{L}(X, X)$ of linear transformations $A \colon X \to X$ is also a normed vector space with the induced operator norm

$$\|A\|_{X \to X} = \sup_{\|x\|_X = 1} \|Ax\|_X.$$

The logarithmic norm $\mu_X(\cdot)$ induced by $\| \cdot \|_X$ is defined as the directional derivative of the matrix norm, that is,

$$\mu_X(A) = \lim_{h \to 0^+} \frac{1}{h} \left(\|I + hA\|_{X \to X} - 1 \right),$$

where I is the identity operator on X.

Remark 4. Since $\sup_{s \in S}(as + b) = a \sup_{s \in S}(s) + b$, whenever $a > 0$ and $S \subseteq \mathbb{R}$, it follows that

$$\mu_X(A) = \lim_{h \to 0^+} \sup_{\|x\|_X = 1} \frac{1}{h} \left(\|x + hAx\|_X - 1 \right).$$

Proposition 3. *For any matrix* A, $\mu(A) = \sup_{\|v\|=1} \lim_{h \to 0^+} \frac{1}{h} (\|v + hAv\| - 1)$.

Table 1
Standard matrix measures for a real $n \times n$ matrix, $A = [a_{ij}]$.

Vector norm, $\|\cdot\|$	Induced matrix measure, $\mu(A)$
$\|x\|_1 = \sum_{i=1}^n |x_i|$	$\mu_1(A) = \max_j \left(a_{jj} + \sum_{i \neq j} |a_{ij}| \right)$
$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}$	$\mu_2(A) = \max_{\lambda \in \text{spec } \frac{1}{2}(A+A^T)} \left(\lambda \left\{ \frac{A+A^T}{2} \right\} \right)$
$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$	$\mu_\infty(A) = \max_i \left(a_{ii} + \sum_{i \neq j} |a_{ij}| \right)$

Table 2
Basic concepts.

Notation	Definition	Equivalent definition	Equivalent definition
$\mu_X(A)$	$\lim_{h \to 0^+} \frac{1}{h} \left(\|I + hA\|_{X \to X} - 1 \right)$	$\lim_{h \to 0^+} \sup_{\|x\|_X = 1} \frac{1}{h} \left(\|x + hAx\|_X - 1 \right)$	$\sup_{\|x\|_X = 1} \lim_{h \to 0^+} \frac{1}{h} \left(\|x + hAx\|_X - 1 \right)$
$L_{Y,X}[f]$	$\sup_{u \neq v \in Y} \frac{\|f(u) - f(v)\|_X}{\|u - v\|_X}$		
$M_{Y,X}[f]$	$\lim_{h \to 0^+} \frac{1}{h} \left(L_{Y,X}[I + hf] - 1 \right)$	$\lim_{h \to 0^+} \sup_{u \neq v \in Y} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right)$	
$(x, y)_+$	$\|x\|_X \lim_{h \to 0^+} \frac{1}{h} \left(\|x + hy\|_X - \|x\|_X \right)$		
$M_{Y,X}^+[f]$	$\sup_{u \neq v \in Y} \frac{(u - v, f(u) - f(v))_+}{\|u - v\|_X^2}$		$\sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right)$

See the Appendix for the proof.

Corollary 1. *Let $(X, \|\cdot\|_X)$ be a finite dimensional normed space. For any linear operator $A \colon X \to X$,*

$$\mu_X(A) = M_X^+[A] = M_X[A].$$

Proof. The proof is immediate from the definition of M_X, M_X^+, and Proposition 3. $\quad \square$

When $X = \mathbb{R}^n$ or $\mathbb{C}^n$, we identify operators and their matrix representations on the standard basis, and we call the logarithmic norm the *matrix measure*. In Table 1, the algebraic expression of the logarithmic norms induced by the L^p norm for $p = 1, 2$, and ∞ are shown for matrices. For proofs, see for instance [31].

For ease of reference, we summarize the main notations and definitions in Table 2.

3. Weighted L^p norms

Suppose Ω, a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial \Omega$ and outward normal $\mathbf{n}$, and a subset $V \subseteq \mathbb{R}^n$ have been fixed. We denote

$$\mathbf{Y} = \left\{ v \colon \bar{\Omega} \to V \mid v = (v_1, \ldots, v_n), v_i \in C_{\mathbb{R}}^2 \left(\bar{\Omega} \right), \frac{\partial v_i}{\partial \mathbf{n}}(\xi) = 0, \ \forall \xi \in \partial \Omega \ \forall i \right\};$$

where $C_{\mathbb{R}}^2 \left(\bar{\Omega} \right)$ is the set of twice continuously differentiable functions $\bar{\Omega} \to \mathbb{R}$. In addition, we denote $\mathbf{X} = C_{\mathbb{R}^n} \left(\bar{\Omega} \right)$, where $C_{\mathbb{R}^n} \left(\bar{\Omega} \right)$ is the set of all continuous functions $\bar{\Omega} \to \mathbb{R}^n$.

For any $1 \leq p \leq \infty$, and any nonsingular, diagonal matrix $Q = \text{diag}(q_1, \ldots, q_n)$, we introduce a Q-weighted norm on $\mathbf{X}$ as follows:

$$\|v\|_{p,Q} := \left\| Q \left(\|v_1\|_p, \ldots, \|v_n\|_p \right)^T \right\|_p. \tag{4}$$

Since

$$\|v\|_{p,Q} = \begin{cases} \left(\sum_i |q_i|^p \|v_i\|_p^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_i |q_i| \|v_i\|_p & p = \infty, \end{cases}$$

without loss of generality we will assume $q_i > 0$ for each i. Note that $\|v\|_{p,Q}$ is finite, for any p, Q, because each v_i is a continuous function on $\bar{\Omega}$ and $\bar{\Omega}$ is a compact subset of $\mathbb{R}^m$.

With a slight abuse of notation, we use the same symbol for a norm in $\mathbb{R}^n$:

$$\|x\|_{p,Q} := \|Qx\|_p.$$

Lemma 2. *For any $v \in \mathbf{X}$, $\|v\|_{p,Q} = \|v\|_{p,Q}^*$, where*

$$\|v\|_{p,Q}^* = \begin{cases} \left(\int_\Omega \|Q\,v(\omega)\|_p^p \, d\omega \right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sup_\omega \|Q\,v(\omega)\|_\infty & p = \infty. \end{cases} \tag{5}$$

Note that $\|Q\,v(\omega)\|_p^p = \sum_{i=1}^n |q_i v_i(\omega)|^p$ and $\|Q\,v(\omega)\|_\infty = \sup_i |q_i v_i(\omega)|$.

Proof. Let $Q = \text{diag}\,(q_1, \ldots, q_n)$, $q_i > 0$. For $1 \le p < \infty$ (the proof is analogous when $p = \infty$), by the definitions of $\|\cdot\|_{p,Q}$ and $\|\cdot\|_{p,Q}^*$

$$\begin{aligned} \|v\|_{p,Q}^* &= \left(\int_\Omega \|Q\,v(\omega)\|_p^p \, d\omega \right)^{\frac{1}{p}} \\ &= \left(\int_\Omega \|(q_1 v_1(\omega), \ldots, q_n v_n(\omega))^T\|_p^p \, d\omega \right)^{\frac{1}{p}} \\ &= \left(\int_\Omega |q_1 v_1(\omega)|^p + \cdots + |q_n v_n(\omega)|^p \, d\omega \right)^{\frac{1}{p}} \\ &= \left(\|q_1 v_1\|_p^p + \cdots + \|q_n v_n\|_p^p \right)^{\frac{1}{p}} \\ &= \left\| \left(q_1 \|v_1\|_p, \ldots, q_n \|v_n\|_p \right)^T \right\|_p \\ &= \|Q(\|v_1\|_p, \ldots, \|v_n\|_p)^T\|_p \\ &= \|v\|_{p,Q} . \quad \square \end{aligned}$$

Note that this equality between weighted p norms of functions and of vectors depends on our having taken the matrix Q to be diagonal. This is the key place where the assumption that Q is diagonal is being used.

4. Main result

In this section, we study the reaction–diffusion PDE:

$$\frac{\partial u}{\partial t}(\omega, t) = F_t(u(\omega, t)) + D\Delta u(\omega, t) \tag{6}$$

subject to the Neumann boundary condition:

$$\frac{\partial u}{\partial \mathbf{n}}(\xi, t) = 0 \quad \forall \xi \in \partial\Omega, \ \forall t \in [0, \infty). \tag{7}$$

Assumption 1. In (6)–(7) we assume:

- $F_t(x) = F(x, t)$ and $F \colon V \times [0, \infty) \to \mathbb{R}^n$ is a (globally) Lipschitz vector field with components F_i:

$$F(x, t) = (F_1(x, t), \ldots, F_n(x, t))^T,$$

  for some functions $F_i \colon V \times [0, \infty) \to \mathbb{R}$, where V is a convex subset of $\mathbb{R}^n$.
- $D = \text{diag}\,(d_1, \ldots, d_n)$, with $d_i > 0$, is called the diffusion matrix.
- Ω is a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and outward normal $\mathbf{n}$.

Definition 6. By a solution of the PDE

$$\frac{\partial u}{\partial t}(\omega, t) = F_t(u(\omega, t)) + D\Delta u(\omega, t)$$

$$\frac{\partial u}{\partial \mathbf{n}}(\xi, t) = 0 \quad \forall \xi \in \partial\Omega, \ \forall t \in [0, \infty),$$

on an interval $[0, T)$, where $0 < T \le \infty$, we mean a function $u = (u_1, \ldots, u_n)^T$, with $u \colon \bar\Omega \times [0, T) \to V$, such that:

1. for each $\omega \in \bar\Omega$, $u(\omega, \cdot)$ is continuously differentiable;
2. for each $t \in [0, T)$, $u(\cdot, t)$ is in $\mathbf{Y}$; and
3. for each $\omega \in \bar\Omega$, and each $t \in [0, T)$, u satisfies the above PDE.

Under the additional assumptions that $F(x, t)$ is twice continuously differentiable with respect to x and continuous with respect to t, theorems on existence and uniqueness for PDEs such as (6)–(7) can be found in standard references, e.g. [32,33]. One must impose appropriate conditions on the vector field, on the boundary of V, to insure invariance of V. Convexity of V insures that the Laplacian also preserves V. Since we are interested here in estimates relating pairs of solutions, we will not deal with existence and well-posedness. Our results will refer to solutions already assumed to exist.

Pick any $0 < T \leq \infty$ and suppose that u is a solution of (6)–(7) defined on $\bar{\Omega} \times [0, T)$. Define $\hat{u} \colon [0, T) \to \mathbf{Y}$ by

$$\hat{u}(t)(\omega) := u(\omega, t).$$

Also define the function $\tilde{F}_t \colon \mathbf{Y} \to \mathbf{X}$ as follows: for any $u \in \mathbf{Y}$,

$$\tilde{F}_t(u)(\omega) := F_t(u(\omega)) \quad \text{for each } \omega \in \bar{\Omega}.$$

Let $A_{p,Q} \colon \mathbf{Y} \to \mathbf{X}$ denote an $n \times n$ diagonal matrix of operators on $\mathbf{Y}$ with the operators $d_i \Delta$ on the diagonal.

Lemma 3. *Suppose that u solves the PDE (6)–(7), on an interval $[0, T)$, for some $T \in (0, \infty]$, and let*

$$v(\omega, t) := \frac{\partial u}{\partial t}(\omega, t)$$

for each $t \geq 0$ and $\omega \in \bar{\Omega}$. We introduce $\hat{v} \colon [0, T) \to \mathbf{X}$ by $\hat{v}(t)(\omega) = v(\omega, t)$. Then, $\hat{v}(t)$ is the derivative of $\hat{u}(t)$ in the space $(\mathbf{X}, \|\cdot\|_{p,Q})$, that is:

$$\lim_{h \to 0} \left\| \frac{1}{h} \left[\hat{u}(t + h) - \hat{u}(t) \right] - \hat{v}(t) \right\|_{p,Q} = 0,$$

for all $t \in [0, T)$. Moreover,

$$\hat{v}(t) = \tilde{F}_t(\hat{u}(t)) + A_{p,Q}(\hat{u}(t)). \tag{8}$$

Proof. Fix $t \in [0, T)$ and $i \in \{1, \ldots, n\}$. Using the definition of v, we have:

$$\lim_{h \to 0} \left| \frac{1}{h} \left[u_i(\omega, t + h) - u_i(\omega, t) \right] - v_i(\omega, t) \right| = 0,$$

for any $\omega \in \bar{\Omega}$. Hence for any $\epsilon > 0$, there exists $h_\omega > 0$ such that for any $0 < h < h_\omega$,

$$\left| \frac{1}{h} \left[u_i(\omega, t + h) - u_i(\omega, t) \right] - v_i(\omega, t) \right| < \frac{\epsilon}{2}.$$

Now since u_i is a continuous function of ω, there exists a ball B_ω centered at ω such that for any $0 < h < h_\omega$,

$$\left| \frac{1}{h} \left[u_i(\tilde{\omega}, t + h) - u_i(\tilde{\omega}, t) \right] - v_i(\tilde{\omega}, t) \right| < \epsilon$$

for all $\tilde{\omega} \in B_\omega$. Since $\{B_\omega : \omega \in \bar{\Omega}\}$ is an open cover of $\bar{\Omega}$ and $\bar{\Omega}$ is a compact subset of $\mathbb{R}^m$, finitely many of these balls, namely $B_{\omega_1}, \ldots, B_{\omega_k}$, cover $\bar{\Omega}$. Now let $h_0 = \min(h_{\omega_1}, \ldots, h_{\omega_k})$. Then, for any $0 < h < h_0$ and any $\omega \in \bar{\Omega}$, we have

$$\left| \frac{1}{h} \left[u_i(\omega, t + h) - u_i(\omega, t) \right] - v_i(\omega, t) \right| < \epsilon.$$

Raising to the p-th power and taking the integral over Ω of the above inequality, we get

$$\int_\Omega \left| \frac{1}{h} \left[u_i(\omega, t + h) - u_i(\omega, t) \right] - v_i(\omega, t) \right|^p d\omega < |\Omega| \epsilon^p,$$

which by the definition of $\|\cdot\|_{p,Q}$, it implies that for any $0 < h < h_0$,

$$\left\| \frac{1}{h} \left[u(\cdot, t + h) - u(\cdot, t) \right] - v(\cdot, t) \right\|_{p,Q} < c\epsilon,$$

where $c = \left(|\Omega| \sum_{i=1}^n q_i^p \right)^{\frac{1}{p}}$. Since $\epsilon > 0$ was arbitrary, we have proved that

$$\lim_{h \to 0} \left\| \frac{1}{h} \left[\hat{u}(t + h) - \hat{u}(t) \right] - \hat{v}(t) \right\|_{p,Q} = 0.$$

For a fixed $t \in [0, T)$ and any $\omega \in \bar{\Omega}$:

$$\hat{v}(t)(\omega) = v(t, \omega) = \frac{\partial u}{\partial t}(\omega, t) = F_t(u(\omega, t)) + D\Delta u(\omega, t)$$
$$= \tilde{F}_t(\hat{u}(t))(\omega) + A_{p,Q}(\hat{u}(t))(\omega),$$

and therefore Eq. (8) holds. □

In this section we show that (6)–(7) is contracting (meaning that solutions converge exponentially to each other, as $t \to +\infty$) if $\sup_{t \in [0,\infty)} M_{p,Q}[F_t] < 0$, where, as defined before,

$$M_{p,Q}[F_t] = \lim_{h \to 0^+} \sup_{x \neq y \in V} \frac{1}{h} \left(\frac{\|x - y + h(F_t(x) - F_t(y))\|_{p,Q}}{\|x - y\|_{p,Q}} - 1 \right).$$

Note that for any t, $M_{p,Q}[F_t] < \infty$ because F_t is (globally) Lipschitz.

Now we state the main result of this section.

Theorem 1. *Consider the PDE (6)–(7) and suppose Assumption 1 holds. Let $c = \sup_{t \in [0,\infty)} M_{p,Q}[F]$ for some $1 \leq p \leq \infty$, and some positive, diagonal matrix Q. Then for every two solutions u, v of the PDE (6)–(7) and all $t \in [0, T)$:*

$$\left\| \hat{u}(t) - \hat{v}(t) \right\|_{p,Q} \leq e^{ct} \left\| \hat{u}(0) - \hat{v}(0) \right\|_{p,Q}.$$

Remark 5. In terms of the PDE (6)–(7), this last estimate can be equivalently written as:

$$\|u(\cdot, t) - v(\cdot, t)\|_{p,Q} \leq e^{ct} \|u(\cdot, 0) - v(\cdot, 0)\|_{p,Q}.$$

Before proving the theorem, we prove a few technical lemmas.

Definition 7. The upper left and right Dini derivatives for any continuous function, $\Psi : [0, \infty) \to \mathbb{R}$, are defined by

$$\left(D^{\pm}\Psi\right)(t) = \lim_{h \to 0^{\pm}} \sup \frac{1}{h} \left(\Psi(t + h) - \Psi(t)\right).$$

Note that $D^+\Psi$ and/or $D^-\Psi$ might be infinite.

Lemma 4. *Let $(X, \|\cdot\|_X) = \left(C_{\mathbb{R}^n}\left(\bar{\Omega}\right), \|\cdot\|_{p,Q}\right)$. Let $G: Y \times [0, \infty) \to X$ be a (globally) Lipschitz function, where $Y \subseteq X$. Let $u, v : [0, \infty) \to Y$ be two solutions of $\frac{du(t)}{dt} = G_t(u(t))$, where $G_t(u) = G(u, t)$. Then for all $t \in [0, \infty)$,*

$$D^+\|(u - v)(t)\|_X = \frac{((u - v)(t), G_t(u(t)) - G_t(v(t)))_+}{\|(u - v)(t)\|_X^2} \|(u - v)(t)\|_X. \tag{9}$$

When $u(t) = v(t)$, we understand the right hand side through the limit in (10).

Proof. By the definition of right semi inner product, the right hand side of (9) is:

$$\lim_{h \to 0^+} \frac{1}{h} \left(\|(u - v)(t) + h(G_t(u(t)) - G_t(v(t)))\|_X - \|(u - v)(t)\|_X\right), \tag{10}$$

hence we just need to show that

$$D^+\|(u - v)(t)\|_X = \lim_{h \to 0^+} \frac{1}{h} \left(\|(u - v)(t) + h(G_t(u(t)) - G_t(v(t)))\|_X - \|(u - v)(t)\|_X\right).$$

Now using the definition of Dini derivative, we have:

$$D^+\|(u - v)(t)\|_X = \lim_{h \to 0^+} \sup \frac{1}{h} \left(\|(u - v)(t + h)\|_X - \|(u - v)(t)\|_X\right)$$
$$= \lim_{h \to 0^+} \sup \frac{1}{h} \left(\|(u - v)(t) + h(\dot{u} - \dot{v})(t) + o(h)\|_X - \|(u - v)(t)\|_X\right)$$
$$= \lim_{h \to 0^+} \sup \frac{1}{h} \left(\|(u - v)(t) + h(\dot{u} - \dot{v})(t)\|_X - \|(u - v)(t)\|_X\right)$$
$$= \lim_{h \to 0^+} \frac{1}{h} \left(\|(u - v)(t) + h(\dot{u} - \dot{v})(t)\|_X - \|(u - v)(t)\|_X\right)$$
$$= \lim_{h \to 0^+} \frac{1}{h} \left(\|(u - v)(t) + h(G_t(u(t)) - G_t(v(t)))\|_X - \|(u - v)(t)\|_X\right),$$

where $\dot{u} = \frac{du}{dt}$. Note that the fourth equality holds because of Remark 1. □

Corollary 2. *Under the assumptions of Lemma 4, for any $t \in [0, \infty)$ we have:*

$$D^+ \|u(t) - v(t)\|_X \leq M^+_{Y,X}[G_t] \|u(t) - v(t)\|_X. \tag{11}$$

Proof. By the definition of the strong least upper bound logarithmic Lipschitz constant,

$$\frac{((u-v)(t), G_t(u(t)) - G_t(v(t)))_+}{\|(u-v)(t)\|_X^2} \leq M^+_{Y,X}[G_t].$$

Now apply Lemma 4 to the above inequality. □

Corollary 3. *Under the assumptions of Lemma 4, for any $t \in [0, \infty)$ we have:*

$$\|u(t) - v(t)\|_X \leq e^{ct} \|u(0) - v(0)\|_X,$$

where $c = \sup_{t \in [0, \infty)} M^+_{Y,X}[G_t]$.

Proof. Apply Gronwall's inequality, [34], to (11). □

Remark 6. Note that Lemma 4 says in particular that for any bounded linear operator $A \colon X \to X$, and any solution $u \colon [0, T) \to X$ of $\frac{du}{dt} = Au$,

$$D^+ \|u(t)\|_X = \frac{(u(t), Au(t))_+}{\|u(t)\|_X^2} \|u(t)\|_X \leq M^+_X[A] \|u(t)\|_X,$$

for all $t \in [0, T)$.

Lemma 5. *Let $A_{p,Q}$, as defined above, denote an $n \times n$ diagonal matrix of operators on $\mathbf{Y}$ with the operators $d_i \Delta$ on the diagonal. Then $M^+_{\mathbf{Y},\mathbf{X}}[A_{p,Q}] \leq 0$.*

Proof. To prove the lemma, we consider the following three cases:

Case 1. $1 < p < \infty$. By the definition of $M^+_{\mathbf{Y},\mathbf{X}}[A_{p,Q}]$, it is enough to show that for any $u \in \mathbf{Y}$ with $\|u\|_{p,Q} \neq 0$, and any $\epsilon > 0$, there exists $h_\epsilon > 0$, depending on ϵ, such that for $0 < h < h_\epsilon$,

$$\frac{1}{h} \left(\frac{\|u + hD\Delta u\|_{p,Q}}{\|u\|_{p,Q}} - 1 \right) = \frac{1}{h} \left(\frac{\left(\sum_i q_i^p \|u_i + hd_i \Delta u_i\|_p^p \right)^{\frac{1}{p}}}{\left(\sum_i q_i^p \|u_i\|_p^p \right)^{\frac{1}{p}}} - 1 \right) < \epsilon.$$

(As $A_{p,Q} u = D\Delta u$, we write $D\Delta u$ instead of $A_{p,Q} u$.)

Therefore we will show that for h small enough

$$\sum_i q_i^p \|u_i + hd_i \Delta u_i\|_p^p < (1 + \epsilon h)^p \sum_i q_i^p \|u_i\|_p^p. \tag{12}$$

Let us define $k \colon [0, 1] \to \mathbb{R}$ as follows:

$$k(h) = \sum_i q_i^p \|u_i + hd_i \Delta u_i\|_p^p - (1 + \epsilon h)^p \sum_i q_i^p \|u_i\|_p^p.$$

Observe that k is continuously differentiable:

$$k'(h) = \frac{d}{dh} \sum_i q_i^p \int_\Omega |u_i(\omega) + hd_i \Delta u_i(\omega)|^p \, d\omega - p\epsilon (1 + \epsilon h)^{p-1} \sum_i q_i^p \|u_i\|_p^p$$

$$= \sum_i q_i^p \int_\Omega p |u_i(\omega) + hd_i \Delta u_i(\omega)|^{p-2} (u_i(\omega) + hd_i \Delta u_i(\omega)) d_i \Delta u_i(\omega) \, d\omega - p\epsilon (1 + \epsilon h)^{p-1} \sum_i q_i^p \|u_i\|_p^p.$$

Note that in general $|g|^p$ is differentiable for $p > 1$ and its derivative is $p|g|^{p-2} gg'$. Now by Green's identity, the Neumann boundary condition, and by the assumption that $\sum_i q_i^p \|u_i\|_p^p \neq 0$, it follows integrating by parts that:

$$k'(0) = p \sum_i q_i^p \int_\Omega |u_i(\omega)|^{p-2} u_i(\omega) d_i \Delta u_i(\omega) \, d\omega - p\epsilon \sum_i q_i^p \|u_i\|_p^p$$

$$= -p(p-1) \sum_i q_i^p d_i \int_\Omega |u_i(\omega)|^{p-2} \nabla u_i(\omega)^2 \, d\omega - p\epsilon \sum_i q_i^p \|u_i\|_p^p$$

$$< 0.$$

Since $k'(0) < 0$ and k' is continuous and $k(0) = 0$, $k(h) < 0$ for h small enough and therefore Inequality (12) holds.

Note that by the definition of **Y**, any $u \in$ **Y** satisfies the Neumann boundary condition.

Case 2. $p = 1$. Let

$$g(p) := \lim_{h \to 0^+} \frac{1}{h} \left(\frac{\left(\sum_i q_i^p \|u_i + hd_i \Delta u_i\|_p^p \right)^{\frac{1}{p}}}{\left(\sum_i q_i^p \|u_i\|_p^p \right)^{\frac{1}{p}}} - 1 \right).$$

Since $g(p)$ is a continuous function at $p = 1$, and since in Case 1, we showed that $g(p) \leq 0$ for any $p > 1$, we conclude that $g(1) \leq 0$.

Case 3. $p = \infty$. Before proving this case we need the following lemma, which is an easy exercise in real analysis.

Lemma 6. *Let $\Omega \subset \mathbb{R}^m$ be a Lebesgue measurable set with finite measure $|\Omega|$ and let f be a bounded, continuous function on $\mathbb{R}$. Then $F(p) := \left(\frac{1}{|\Omega|} \int_\Omega |f|^p \right)^{\frac{1}{p}}$ is an increasing function of p and its limit as $p \to \infty$ is $\|f\|_\infty$.*

For a fixed $p_0 > 1$, pick $u \in$ **Y** with $\|u\|_{p_0, Q} \neq 0$. By the definition of the norm, $\|u\|_{p_0, Q} \neq 0$ implies that for some $i_0 \in \{1, \ldots, n\}$, $\|u_{i_0}\|_{p_0} \neq 0$. Let $\varphi(p) := \frac{1}{|\Omega|^{\frac{1}{p}}} \|u_{i_0}\|_p$. By Lemma 6, φ is an increasing function of p. Hence for any $p > p_0$, $\|u_{i_0}\|_p \geq \|u_{i_0}\|_{p_0} > 0$. Now fix $i \in \{1, \ldots, n\}$, $p > p_0$, and $\epsilon > 0$. Define k as follows:

$$k(h) = \begin{cases} \|u_i + hd_i \Delta u_i\|_p^p - (1 + \epsilon h)^p \|u_{i_0}\|_p^p & \text{if } \|u_{i_0}\|_p \geq \|u_i\|_p \\ \|u_i + hd_i \Delta u_i\|_p^p - (1 + \epsilon h)^p \|u_i\|_p^p & \text{if } \|u_{i_0}\|_p \leq \|u_i\|_p. \end{cases}$$

In both cases $k(0) \leq 0$ and $k'(0) < 0$ (the proof is similar to the proof of $k'(0) < 0$ in Case 1, since both $\|u_{i_0}\|_p > 0$ and $\|u_{i_0}\|_p > 0$). Therefore, for some small h, $k(h) \leq 0$, which implies that:

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u_i + hd_i \Delta u_i\|_p}{\|u_i\|_p} - 1 \right) \leq 0.$$

Now by Lemma 6, since

$$\frac{1}{|\Omega|^{\frac{1}{p}}} \|u_i + hd_i \Delta u_i\|_p \to \|u_i + hd_i \Delta u_i\|_\infty \quad \text{and} \quad \frac{1}{|\Omega|^{\frac{1}{p}}} \|u_i\|_p \to \|u_i\|_\infty, \quad \text{as } p \to \infty,$$

we can conclude that

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u_i + hd_i \Delta u_i\|_\infty}{\|u_i\|_\infty} - 1 \right) \leq 0.$$

In other words, for a fixed $\epsilon > 0$, there exists $h_i > 0$ such that for any $0 < h < h_i$,

$$\|u_i + hd_i \Delta u_i\|_\infty \leq (1 + \epsilon h) \|u_i\|_\infty \quad \text{for any } i \in \{1, \ldots, n\}.$$

Let $h_0 = \min_i h_i$. Then for any $0 < h < h_0$,

$$\max_i q_i \|u_i + hd_i \Delta u_i\|_\infty =: q_j \|u_j + hd_j \Delta u_j\|_\infty \leq q_j (1 + \epsilon h) \|u_j\|_\infty \leq (1 + \epsilon h) \max_i q_i \|u_i\|_\infty,$$

which implies

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\max_i q_i \|u_i + hd_i \Delta u_i\|_\infty}{\max_i q_i \|u_i\|_\infty} - 1 \right) \leq 0. \quad \square$$

Lemma 7. *For any function F, any $1 \leq p \leq \infty$, and any positive diagonal matrix Q,*

$$M_{\mathbf{Y}, \mathbf{X}}^+ [\tilde{F}] \leq M_{p, Q}[F],$$

where $M_{p, Q}$ is the lub logarithmic Lipschitz constant induced by the norm $\| \cdot \|_{p, Q}$ defined on $\mathbb{R}^n$: $\|x\|_{p, Q} = \|Qx\|_p$.

Proof. By the definition of $c := M_{p, Q}[F]$, we have

$$\lim_{h \to 0^+} \frac{1}{h} \sup_{x \neq y \in V} \left(\frac{\|x - y + h(F(x) - F(y))\|_{p, Q}}{\|x - y\|_{p, Q}} - 1 \right) = c.$$

Fix an arbitrary $\epsilon > 0$. Then there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$\frac{1}{h} \sup_{x \neq y \in V} \left(\frac{\|x - y + h(F(x) - F(y))\|_{p,Q}}{\|x - y\|_{p,Q}} - 1 \right) < c + \epsilon.$$

Therefore, for any $x \neq y$, and $0 < h < h_0$

$$\frac{\|x - y + h(F(x) - F(y))\|_{p,Q}}{\|x - y\|_{p,Q}} < (c + \epsilon)h + 1. \tag{13}$$

For fixed $u \neq v \in \mathbf{Y}$, let $\Omega_1 = \{\omega \in \bar{\Omega} : u(\omega) \neq v(\omega)\}$. Fix $\omega \in \Omega_1$, and let $x = u(\omega)$ and $y = v(\omega)$. We give a proof for the case $p < \infty$; the case $p = \infty$ is analogous. Using Eq. (13), we have:

$$\frac{\left(\sum_i q_i^p |u_i(\omega) - v_i(\omega) + h(F_i(u(\omega)) - F_i(v(\omega)))|^p \right)^{\frac{1}{p}}}{\left(\sum_i q_i^p |u_i(\omega) - v_i(\omega)|^p \right)^{\frac{1}{p}}} < (c + \epsilon)h + 1. \tag{14}$$

Multiplying both sides by the denominator and raising to the power p, we have:

$$\sum_i q_i^p |u_i(\omega) - v_i(\omega) + h(F_i(u(\omega)) - F_i(v(\omega)))|^p < ((c + \epsilon)h + 1)^p \sum_i q_i^p |u_i(\omega) - v_i(\omega)|^p. \tag{15}$$

Since $\tilde{F}(u)(\omega) = F(u(\omega))$, Eq. (15) can be written as:

$$\sum_i q_i^p \left| u_i(\omega) - v_i(\omega) + h \left(\tilde{F}_i(u)(\omega) - \tilde{F}_i(v)(\omega) \right) \right|^p < ((c + \epsilon)h + 1)^p \sum_i q_i^p |u_i(\omega) - v_i(\omega)|^p. \tag{16}$$

Now by taking the integral over $\bar{\Omega}$, using Lemma 2, we get:

$$\left\| u - v + h \left(\tilde{F}(u) - \tilde{F}(v) \right) \right\|_{p,Q} < ((c + \epsilon)h + 1) \|u - v\|_{p,Q}.$$

(Note that for $\omega \notin \Omega_1$,

$$((c + \epsilon)h + 1)^p \sum_i q_i^p |u_i(\omega, t) - v_i(\omega, t)|^p = 0$$

which we can add to the right hand side of (16), and also

$$\sum_i q_i^p |u_i(\omega) - v_i(\omega) + h(F_i(u(\omega)) - F_i(v(\omega)))|^p = 0$$

which we can add to the left hand side of (16), and hence we can indeed take the integral over all $\bar{\Omega}$.)

Hence,

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\left\| u - v + h \left(\tilde{F}(u) - \tilde{F}(v) \right) \right\|_{p,Q}}{\|u - v\|_{p,Q}} - 1 \right) \leq c + \epsilon.$$

Now by letting $\epsilon \to 0$ and taking sup over $u \neq v \in \mathbf{Y}$, we get $M^+_{\mathbf{Y},\mathbf{X}}[\tilde{F}] \leq c$. □

Proof of Theorem 1. If $c = \sup_{t \in [0,\infty)} M_{p,Q}[F_t] = \infty$, there is nothing to prove. Suppose $c < \infty$. For any $1 \leq p \leq \infty$, by subadditivity of semi inner product, Lemmas 5 and 7,

$$M^+_{\mathbf{Y},\mathbf{X}}[\tilde{F}_t + A_{p,Q}] \leq M^+_{\mathbf{Y},\mathbf{X}}[\tilde{F}_t] \leq c.$$

Now using Corollary 3,

$$\left\| \hat{u}(t) - \hat{v}(t) \right\|_{p,Q} \leq e^{ct} \left\| \hat{u}(0) - \hat{v}(0) \right\|_{p,Q},$$

for all $t \in [0, \infty)$. □

Theorem 2. *Consider the reaction–diffusion system* (6)–(7) *and suppose Assumption 1 holds and for each t, $F_t(x)$ is continuously differentiable with respect to x. In addition suppose for some $1 \leq p \leq \infty$, $c \in \mathbb{R}$, and a positive diagonal matrix Q, $\mu_{p,Q}(J_{F_t}(x)) \leq$*

c for all $x \in V$ and $t \in [0, \infty)$, where $\mu_{p,Q}$ is the logarithmic norm induced by $\|\cdot\|_{p,Q}$ and $J_{F_t}(x) = \frac{\partial}{\partial x} F_t(x)$. Then, for any two solutions u, v of (6)–(7), we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{p,Q} \le e^{ct} \|u(\cdot, 0) - v(\cdot, 0)\|_{p,Q} .$$

To prove Theorem 2, we use the following proposition, from [35].

Proposition 4. *Let $(X, \| \cdot \|_X)$ be a normed space and Y is a connected subset of X. Then for any (globally) Lipschitz and continuously differentiable function $f : Y \to \mathbb{R}^n$,*

$$\sup_{x \in Y} \mu_X(J_f(x)) \le M_{Y,X}[f].$$

Moreover if Y is convex, then

$$\sup_{x \in Y} \mu_X(J_f(x)) = M_{Y,X}[f].$$

Proof of Theorem 2. The proof is immediate from Theorem 1 and Proposition 4. $\quad\square$

Corollary 4. *Consider the reaction–diffusion system (6)–(7) and suppose Assumption 1 holds and for each t, $F_t(x)$ is continuously differentiable with respect to x. In addition suppose for some $1 \le p \le \infty$, and a positive diagonal matrix Q, $\sup_{(x,t) \in V \times [0,\infty)} \mu_{p,Q}(J_{F_t}(x)) < 0$. Then (6)–(7) is contracting in **Y**, meaning that solutions converge (exponentially) to each other, as $t \to +\infty$.*

Remark 7. When the time-dependence of F on t is periodic (as in Example 1 below when $z(t)$ is periodic), there will be convergence to a (unique) globally asymptotically stable solution, uniform in space. This is because the corresponding ODE system admits a periodic limit cycle [21], which is also a solution of the associated PDE.

Remark 8. Note that our results provide a far stronger property than asymptotic stability of solutions. These properties are very different, even for ordinary differential equations. Consider for example the system

$$x_t = -x$$
$$y_t = (x - 1)y$$

which has the origin as a globally asymptotically stable state. This system cannot be contractive under any possible norm, since solutions starting with large x initially diverge from each other.

5. Examples

We first provide an example of a biochemical model which can be shown to be contractive by applying Corollary 4 when using a weighted L^1 norm, but which is not contractive in any weighted L^p norm, $p > 1$. This shows that the choice of norms is a key step in the application of contraction techniques. The example is of great interest in molecular systems biology [20], and contractivity in a weighted L^1 norm was shown for ODE systems in [21], but the PDE case was open. The variant with more enzymes discussed in [21] can also be extended to the PDE case in an analogous fashion.

Example 1. A typical biochemical reaction is one in which an enzyme X (whose concentration is quantified by the non-negative variable $x = x(t)$) binds to a substrate S (whose concentration is quantified by $s = s(t) \ge 0$), to produce a complex Y (whose concentration is quantified by $y = y(t) \ge 0$), and the enzyme is subject to degradation and dilution (at rate δx, where $\delta > 0$) and production according to an external signal $z = z(t) \ge 0$. An entirely analogous system can be used to model a transcription factor binding to a promoter, as well as many other biological processes of interest. The complete system of chemical reactions is given by:

$$0 \xrightarrow{z} X \xrightarrow{\delta} 0, \qquad X + S \underset{k_1}{\overset{k_2}{\rightleftharpoons}} Y.$$
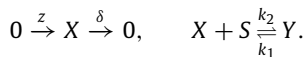
We let the domain Ω represent the part of the cytoplasm where these chemicals are free to diffuse. Taking equal diffusion constants for S and Y (which is reasonable since typically S and Y have approximately the same size), a natural model is given by a reaction–diffusion system

$$x_t = z(t) - \delta x + k_1 y - k_2 s x + d_1 \Delta x$$
$$y_t = -k_1 y + k_2 s x + d_2 \Delta y$$
$$s_t = k_1 y - k_2 s x + d_2 \Delta s.$$

If we assume that initially S and Y are uniformly distributed, it follows that $\frac{\partial}{\partial t}(y(\omega, t) + s(\omega, t)) = 0$, so $y(\omega, t) + s(\omega, t) = y(\omega, 0) + s(\omega, 0) = S_Y$ is a constant. Thus we can study the following reduced system:

$$x_t = z(t) - \delta x + k_1 y - k_2(S_Y - y)x + d_1 \Delta x$$
$$y_t = -k_1 y + k_2(S_Y - y)x + d_2 \Delta y.$$

Note that $(x(t), y(t)) \in V = [0, \infty) \times [0, S_Y]$ for all $t \geq 0$ (V is convex and forward-invariant), and S_Y, k_1, k_2, δ, d_1, and d_2 are arbitrary positive constants.

Let J be the Jacobian of $F = (z - \delta x + k_1 y - k_2(S_Y - y)x, -k_1 y + k_2(S_Y - y)x)^T$:

$$J = \begin{pmatrix} -\delta - k_2(S_Y - y) & k_1 + k_2 x \\ k_2(S_Y - y) & -(k_1 + k_2 x) \end{pmatrix}.$$

In [21], it has been shown that $\sup_{(x,y)\in V} \mu_{1,Q}(J(x, y)) < 0$ for $Q = \text{diag}\left(1, 1 + \frac{\delta}{k_2 S_Y} - \zeta\right)$, where $0 < \zeta < \frac{\delta}{k_2 S_Y}$. For ease of reference, we review the proof next.

Without loss of generality we assume $Q = \text{diag}(1, q)$. Then

$$QJQ^{-1} = \begin{pmatrix} -\delta - a & \dfrac{b}{q} \\ aq & -b \end{pmatrix},$$

where $a = k_2(S_Y - y) \in [0, k_2 S_Y]$ and $b = k_1 + k_2 x \in [k_1, \infty)$. Since $a \geq 0$, $b > 0$, and $q > 0$, by Table 1, we have:

$$\begin{aligned} \mu_{1,Q}(J) &= \mu_1(QJQ^{-1}) \\ &= \max\left\{-\delta - a + |aq|, -b + \left|\frac{b}{q}\right|\right\} \\ &= \max\left\{-\delta + a(q - 1), b\left(\frac{1}{q} - 1\right)\right\}. \end{aligned}$$

So to show that $\mu_{1,Q}(J) < 0$, we need to find a range for the values of q such that:

$$-\delta + a(q - 1) < 0, \tag{17}$$

and

$$b\left(\frac{1}{q} - 1\right) < 0. \tag{18}$$

Eq. (18) holds if and only if $q > 1$. So we need to find an appropriate $q > 1$ such that Eq. (17) holds:

$$-\delta + a(q - 1) < 0 \quad \text{iff} \quad q < 1 + \frac{\delta}{a} = 1 + \frac{\delta}{k_2(S_Y - y)} < 1 + \frac{\delta}{k_2 S_Y}.$$

Hence for $Q = \text{diag}(1, q)$, with $1 < q < 1 + \frac{\delta}{k_2 S_Y}$, $\mu_{1,Q}(J) < 0$. Therefore, by Corollary 4, the system is contracting. Note that a *weighted* norm L^1 is necessary, since with $Q = I$ we obtain $\mu_1 = 0$.

We will show that for any $p > 1$ and any diagonal Q, it is not true that $\mu_{p,Q}(J(x, y)) < 0$ for all $(x, y) \in V$.

We first consider the case $p \neq \infty$. We will show that there exists $(x_0, y_0) \in V$ such that for any small $h > 0$, $\|I + hQJ(x_0, y_0)Q^{-1}\|_p > 1$. This will imply $\mu_{p,Q}(J(x_0, y_0)) \geq 0$. Computing explicitly, we have:

$$\begin{aligned} \|I + hQJQ^{-1}\|_p &= \sup_{(\xi_1, \xi_2) \neq (0,0)} \frac{\left(\left|\xi_1 - h(\delta + a)\xi_1 + h\frac{b\xi_2}{q}\right|^p + |haq\xi_1 + \xi_2 - hb\xi_2|^p\right)^{\frac{1}{p}}}{(|\xi_1|^p + |\xi_2|^p)^{\frac{1}{p}}} \\ &\geq \frac{\left(\left|1 - h(\delta + a) + h\frac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p\right)^{\frac{1}{p}}}{(1 + |\lambda|^p)^{\frac{1}{p}}}, \end{aligned}$$

where we take a point of the form $(\xi_1, \xi_2) = (1, \lambda)$, for a $\lambda > 0$ which will be determined later. To show

$$\frac{\left(\left|1 - h(\delta + a) + h\frac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p\right)^{\frac{1}{p}}}{(1 + |\lambda|^p)^{\frac{1}{p}}} > 1,$$

we will equivalently show that for any small enough $h > 0$:

$$\frac{1}{h}\left(\left|1 - h(\delta + a) + h\frac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p - 1 - |\lambda|^p\right) > 0. \tag{19}$$

Note that the $\lim_{h\to 0^+}$ of the left hand side of the above inequality is $f'(0)$ where

$$f(h) = \left|1 + h\left(\frac{b\lambda}{q} - (\delta + a)\right)\right|^p + |\lambda + h(aq - b\lambda)|^p.$$

Therefore it suffices to show that $f'(0) > 0$ for some value $(x_0, y_0) \in C$ (because $f'(0) > 0$ implies that there exists $h_0 > 0$ such that for $0 < h < h_0$, (19) holds). Since $p > 1$, by assumption, f is differentiable and

$$f'(h) = p\left(\frac{b\lambda}{q} - (\delta + a)\right)\left|1 + h\left(\frac{b\lambda}{q} - (\delta + a)\right)\right|^{p-2}\left(1 + h\left(\frac{b\lambda}{q} - (\delta + a)\right)\right)$$

$$+ p(aq - b\lambda)|\lambda + h(aq - b\lambda)|^{p-2}(\lambda + h(aq - b\lambda)).$$

Hence, since $\lambda > 0$

$$f'(0) = p\left(\frac{b\lambda}{q} - (\delta + a)\right) + p(aq - b\lambda)\lambda^{p-1}$$

$$= p\left(\frac{b\lambda}{q} - a\right)(1 - \lambda^{p-1}q) - p\delta.$$

Choosing λ small enough such that $1 - \lambda^{p-1}q > 0$ and choosing x, or equivalently b, large enough, we can make $f'(0) > 0$.

For $p = \infty$, using Table 1, $\mu_p(QJQ^{-1}) = \max\left\{-\delta - a + \frac{b}{q}, -b + aq\right\}$. For large enough x, $-\delta - a + \frac{b}{q} > 0$ (and $-b + aq < 0$) and hence $\mu_\infty(QJQ^{-1}) > 0$.

The following example, from the literature on pattern formation, also illustrates the need to choose norms judiciously.

Example 2 ([5]). In this example, we study the Thomas mechanism, which is based on a specific reaction, involving the substrates oxygen v, and uric acid u. The dimensionless form of the reaction–diffusion equations for the oxygen and the uric acid concentrations are as follows:

$$u_t = a - u - \rho R(u, v) + d_1 \Delta u$$
$$v_t = \alpha(b - v) - \rho R(u, v) + d_2 \Delta v,$$

where $R(u, v) = \frac{uv}{1+u+Ku^2}$. We assume :

1. $a, b, \rho, \alpha, K, d_1$, and d_2 are all positive constants,
2. for all $t \geq 0$, $(u(t), v(t)) \in V = [0, 2a] \times [0, \infty)$,
3. $a < \frac{1}{2\sqrt{K}}$.

Note that V is convex and forward-invariant.

In this model, u and v are subject to production at constant rates a and αb, and are subject to degradation at rates $-u$ and $-\alpha v$ respectively; and both are used up in the reaction at a rate $\rho R(u, v)$.

Let J be the Jacobian of $F = (a - u - \rho R(u, v), \alpha(b - v) - \rho R(u, v))^T$:

$$J(u, v) = \begin{pmatrix} -1 - \rho R_u(u, v) & -\rho R_v(u, v) \\ -\rho R_u(u, v) & -\alpha - \rho R_v(u, v) \end{pmatrix},$$

where $R_u(u, v) = \frac{v(1 - Ku^2)}{(1+u+Ku^2)^2}$ and $R_v(u, v) = \frac{u}{1+u+Ku^2}$ are the partial derivatives of R with respect to u and v, respectively. Note that, by the second and third conditions above, both R_u and R_v are non-negative on V. Hence for any $(u, v) \in V$,

$$\mu_1(J(u, v)) = \max\{-1 - \rho R_u(u, v) + |-\rho R_u(u, v)|, -\alpha - \rho R_v(u, v) + |-\rho R_v(u, v)|\}$$

$$= \max\{-1 - \rho R_u(u, v) + \rho R_u(u, v), -\alpha - \rho R_v(u, v) + \rho R_v(u, v)\}$$

$$= \max\{-1, -\alpha\}$$

$$< 0.$$

Therefore, by Corollary 4, the system is contractive.

Remark 9. For a system

$$x_t = f(x, y) + d_1 \Delta x$$
$$y_t = g(x, y) + d_2 \Delta y,$$

with a steady state (x^*, y^*), a set of necessary and sufficient conditions for diffusive instability are as follows (for a proof see e.g. [5,6]):

1. $f_x + g_y < 0$,
2. $f_x g_y - f_y g_x > 0$,
3. $d_2 f_x + d_1 g_y > 0$, and
4. $(d_2 f_x + d_1 g_y)^2 - 4 d_2 d_1 (f_x g_y - f_y g_x) > 0$;

where f_x denote the partial derivative of f, with respect to x, at the steady state (x^*, y^*), etc.

The first two conditions say that (u^*, v^*) is (locally) stable before diffusion.

Note that the derivatives f_x and g_y must be of opposite sign.

In Example 2, the first two conditions hold for all $(u, v) \in V$, so if there exists a steady state in V, it must be asymptotically stable (without diffusion terms). But since R_u and R_v are both non-negative on V (because of the choice of V and the parameters), the third condition is violated. Hence if there exists a steady state in V, it remains locally asymptotically stable after diffusion; and we showed that it is in fact *globally* stable on V.

One may get diffusive instability with choosing parameters appropriately.

Remark 10. For any positive, diagonal matrix Q, and $p > 1$,

$$\sup_{(u,v) \in V} \mu_{p,Q} J(u, v) \geq 0.$$

Proof. Let $Q = \text{diag}(1, q)$ and $u = 0$. Then for any $v \in [0, \infty)$:

$$J_0(v) := I + h Q J(0, v) Q^{-1} = \begin{pmatrix} 1 - h(1 + \rho v) & 0 \\ -q \rho v h & 1 - h\alpha \end{pmatrix}.$$

We first consider $p \neq \infty$ and will show that $\mu_{p,Q} J(0, v) \geq 0$ for some $v \in [0, \infty)$. To this end, by the definition of the logarithmic norm, we show that there exists $v \in [0, \infty)$ such that for all small enough $h > 0$, $\|J_0(v)\|_p > 1$. Computing explicitly, we have:

$$\|J_0(v)\|_p = \sup_{(\xi_1, \xi_2) \neq (0,0)} \frac{\left(|\xi_1 - h(1 + \rho v)\xi_1|^p + |-qh\rho v \xi_1 + \xi_2 - \alpha h \xi_2|^p \right)^{\frac{1}{p}}}{(|\xi_1|^p + |\xi_2|^p)^{\frac{1}{p}}}$$

$$\geq \frac{\left(|1 - h(1 + \rho v)|^p + |-qh\rho v + \lambda - \alpha h \lambda|^p \right)^{\frac{1}{p}}}{(1 + |\lambda|^p)^{\frac{1}{p}}},$$

where we take a point of the form $(\xi_1, \xi_2) = (1, \lambda)$, for a $\lambda < 0$ which will be determined later. To show

$$\frac{\left(|1 - h(1 + \rho v)|^p + |-qh\rho v + \lambda - \alpha h \lambda|^p \right)^{\frac{1}{p}}}{(1 + |\lambda|^p)^{\frac{1}{p}}} > 1,$$

we will equivalently show that for any small enough $h > 0$:

$$\frac{1}{h} \left(|1 - h(1 + \rho v)|^p + |-qh\rho v + \lambda - \alpha h \lambda|^p - 1 - |\lambda|^p \right) > 0. \tag{20}$$

Note that the $\lim_{h \to 0^+}$ of the left hand side of the above inequality is $f'(0)$ where

$$f(h) = |1 - h(1 + \rho v)|^p + |-qh\rho v + \lambda - \alpha h \lambda|^p.$$

Therefore it suffices to show that $f'(0) > 0$ for some value $v \in [0, \infty)$ (because $f'(0) > 0$ implies that there exists $h_0 > 0$ such that for $0 < h < h_0$, (20) holds). Since $p > 1$, by assumption, f is differentiable and

$$f'(h) = -p(1 + \rho v) |1 - h(1 + \rho v)|^{p-2} (1 - h(1 + \rho v))$$
$$+ p(-q\rho v - \alpha\lambda) |-qh\rho v + \lambda - \alpha h \lambda|^{p-2} (-qh\rho v + \lambda - \alpha h \lambda).$$

Hence, since $\lambda < 0$

$$
\begin{aligned}
f'(0) &= -p(1 + \rho v) + p(-q\rho v - \alpha\lambda) |\lambda|^{p-2} \lambda \\
&= -p(1 + \rho v) + p(q\rho v + \alpha\lambda)(-\lambda)^{p-1} \\
&= p\rho \left(-1 + q(-\lambda)^{p-1}\right) v - p \left(1 + \alpha(-\lambda)^{p-1}\right).
\end{aligned}
$$

Choosing λ small enough such that $-1 + q(-\lambda)^{p-1} > 0$ and choosing v large enough, we can make $f'(0) > 0$.

Now we show that for large v, $\mu_\infty(J_0(v)) > 0$:

By Table 1,

$$
\mu_\infty(J_0(v)) = \max\{-\alpha + q\rho v, -1 - \rho v\},
$$

which is positive for $v > \frac{\alpha}{q\rho}$. $\square$

6. Diffusive interconnection of ODEs

In this section, we derive a result analogous to that for PDEs for a network of identical ODE models which are diffusively interconnected. We study systems of ODEs as follows:

$$
\dot{u}(t) = \tilde{F}(u(t), t) - (L \otimes D)u(t). \tag{21}
$$

Assumption 2. In (21), we assume:

- For a fixed convex subset of $\mathbb{R}^n$, say V, $\tilde{F} : V^N \times [0, \infty) \to \mathbb{R}^{nN}$ is a function of the form:

$$
\tilde{F}(u, t) = \left(F(u_1, t)^T, \ldots, F(u_N, t)^T\right)^T,
$$

where $u = \left(u_1^T, \ldots, u_N^T\right)^T$, with $u_i \in V$ for each i, and $F : V \times [0, \infty) \to \mathbb{R}^n$ is a (globally) Lipschitz function.
- For any $u \in V^N$ we define $\|u\|_{p,Q}$ as follows:

$$
\|u\|_{p,Q} = \left\| \left(\|Qu_1\|_p, \ldots, \|Qu_N\|_p\right)^T \right\|_p,
$$

where $Q = \text{diag}(q_1, \ldots, q_n)$ is a positive diagonal matrix and $1 \le p \le \infty$.

With a slight abuse of notation, we use the same symbol for a norm in $\mathbb{R}^n$:

$$
\|x\|_{p,Q} := \|Qx\|_p.
$$

- $u : [0, \infty) \to V^N$ is a continuously differentiable function.
- $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i > 0$, which we call the diffusion matrix.
- $L \in \mathbb{R}^{N \times N}$ is a symmetric matrix with non-positive off-diagonal entries, and $L\mathbf{1} = 0$, where $\mathbf{1} = (1, \ldots, 1)^T$. We think of L as the Laplacian of a graph that describes the interconnections among component subsystems.

Theorem 3. *Consider the system* (21) *and suppose Assumption 2 holds. Let* $c = \sup_{t \in [0,\infty)} M_{p,Q}[F_t]$, *where* $M_{p,Q}$ *is the lub logarithmic Lipschitz constant induced by the norm* $\| \cdot \|_{p,Q}$ *on* $\mathbb{R}^n$ *defined by* $\|x\|_{p,Q} := \|Qx\|_p$. *Then for any two solutions* u, v *of* (21), *we have*

$$
\|u(t) - v(t)\|_{p,Q} \le e^{ct} \|u(0) - v(0)\|_{p,Q}.
$$

This theorem is proved by following the same steps as in the PDE case and using Lipschitz norms and properties of discrete Laplacians on finite graphs. For ODEs, we can make some of the steps more explicit, and for purposes of exposition, we do so next. We start with several technical lemmas.

The following elementary property of logarithmic norms is well-known. For more properties of logarithmic norms, see e.g. [31].

Lemma 8. *Let* λ *be the largest real part of an eigenvalue of* A. *Then,* $\mu_{p,Q}(A) \ge \lambda$.

We recall that if $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then the Kronecker product, denoted by $A \otimes B$, is the $mp \times nq$ block matrix defined as follows:

$$
A \otimes B := \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{bmatrix},
$$

where $a_{ij}B$ denote the following $p \times q$ matrix:

$$
a_{ij}B := \begin{bmatrix} a_{ij}b_{11} & \ldots & a_{ij}b_{1q} \\ \vdots & \ddots & \vdots \\ a_{ij}b_{p1} & \ldots & a_{ij}b_{pq} \end{bmatrix}.
$$

The following are some properties of Kronecker product (for more properties see e.g. [36]):

1. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;
2. If A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Proposition 5. *For any* $1 \leq p \leq \infty$, $M_p^+(-L \otimes D) = 0$, *where* M_p^+ *is the strong least upper bound logarithmic Lipschitz constant induced by the* L^p *norm.*

Proof. Let $\mathcal{L} = -L \otimes D = (\mathcal{L}_{ij})$. Note that since $L\mathbf{1} = 0$, by the definition of Kronecker product, $\mathcal{L}\mathbf{1} = 0$. In addition because L is symmetric and D is diagonal, $\mathcal{L}$ is also symmetric and therefore $\mathcal{L}\mathbf{1} = \mathbf{1}\mathcal{L} = 0$. Also the off diagonal entries of $\mathcal{L}$, like those of $-L$, are positive. By Corollary 1, it suffices to show that $\mu_p(\mathcal{L}) = 0$ for any p. We first show that $\mu_p(\mathcal{L}) = 0$ for $p = 1, \infty$. For $p = 1$,

$$\mu_1(\mathcal{L}) = \max_j \sum_{i \neq j, i=1,\ldots,nN} (\mathcal{L}_{ii} + |\mathcal{L}_{ij}|) = \max_j 0 = 0.$$

Similarly for $p = \infty$,

$$\mu_\infty(\mathcal{L}) = \max_i \sum_{i \neq j, j=1,\ldots,nN} (\mathcal{L}_{ii} + |\mathcal{L}_{ij}|) = \max_j 0 = 0.$$

Now suppose $p \neq 1, \infty$. By Lemma 8, $\mu_p(\mathcal{L}) \geq \Re\lambda$, where λ is an eigenvalue of $\mathcal{L}$. Because $\mathcal{L}\mathbf{1} = 0$, $\lambda = 0$ is an eigenvalue of $\mathcal{L}$; therefore $\mu_p(\mathcal{L}) \geq 0$. To show that $\mu_p(\mathcal{L}) \leq 0$, by Remark 6, it suffices to show that $D^+\|u\|_p \leq 0$ where u is the solution of $\dot{u} = \mathcal{L}u$. By the definition of Dini derivative, it suffices to show that $\|u(t)\|_p$ is a non-increasing function of t. Let $\Phi(u(t)) := \|u(t)\|_p^p$, where $u = (u_1^T, \ldots, u_N^T)^T$ with $u_i = (u_i^1, \ldots, u_i^n)^T \in V^n$. Here we abuse the notation and assume that $u = (u_1, \ldots, u_{nN})^T$. We will show that $\frac{d\Phi}{dt}(u(t)) \leq 0$.

We will use the following simple fact:

Lemma 9. *For any real* α *and* β *and* $1 \leq p$:

$$(|\alpha|^{p-2} + |\beta|^{p-2})\alpha\beta \leq |\alpha|^p + |\beta|^p.$$

As we explained above, $\mathcal{L}$ is symmetric and $\mathcal{L}\mathbf{1} = 0$. Using this information and the above inequality:

$$
\begin{aligned}
\frac{d\Phi}{dt}(u(t)) &= \sum_{i=1}^{nN} \frac{d\Phi}{du_i}\frac{du_i}{dt} \\
&= \nabla\Phi \cdot \dot{u} \\
&= \nabla\Phi \cdot \mathcal{L}u \\
&= p(|u_1|^{p-2}u_1, \ldots, |u_{nN}|^{p-2}u_{nN})\mathcal{L}(u_1, \ldots, u_{nN})^T \\
&= p\sum_{i,j} |u_i|^{p-2}u_i\mathcal{L}_{ij}u_j \\
&= p\sum_i |u_i|^p\mathcal{L}_{ii} + p\sum_{i<j}\mathcal{L}_{ij}(|u_i|^{p-2} + |u_j|^{p-2})u_iu_j \\
&\leq p\sum_i |u_i|^p\mathcal{L}_{ii} + p\sum_{i<j}\mathcal{L}_{ij}(|u_i|^p + |u_j|^p) \\
&= p\sum_i |u_i|^p\mathcal{L}_{ii} + p\sum_{i \neq j}\left(\mathcal{L}_{ij}|u_i|^p + \mathcal{L}_{ji}|u_j|^p\right) \\
&= p\sum_i |u_i|^p\left(\mathcal{L}_{ii} + \sum_{i \neq j}\mathcal{L}_{ij}\right) \\
&= 0,
\end{aligned}
$$

since $\frac{\partial\Phi}{\partial u_i} = \frac{\partial}{\partial u_i}|u_i|^p = p|u_i|^{p-1}\frac{u_i}{|u_i|} = p|u_i|^{p-2}u_i$. Recall that $|x|^p$ is differentiable for $p > 1$. □

Lemma 10. *Let* μ_p *and* $\mu_{p,Q}$ *denote the logarithmic norms induced by* $\|\cdot\|_p$ *and* $\|\cdot\|_{p,Q}$ *respectively. Then*

$$\mu_{p,Q}(-L \otimes D) = \mu_p(-L \otimes D).$$

Proof. By the properties of Kronecker product mentioned above, we have:

$$
\begin{aligned}
\mu_{p,Q}(-L \otimes D) &= \mu_p[(I \otimes Q)(-L \otimes D)(I \otimes Q^{-1})] \\
&= \mu_p(-L \otimes QDQ^{-1}) \\
&= \mu_p(-L \otimes D).
\end{aligned}
$$

The last equality holds because both Q and D are diagonal, and thus they commute. Therefore $QDQ^{-1} = DQQ^{-1} = D$. □

Proposition 6. *Let $M_{p,Q}^+$ denote the strong least upper bound logarithmic Lipschitz constant induced by the norm $\| \cdot \|_{p,Q}$ on $\mathbb{R}^{nN}$. Then,*

$$M_{p,Q}^+[-L \otimes D] = 0.$$

Proof. By Proposition 5, Corollary 1 and Lemma 10,

$$M_{p,Q}^+[-L \otimes D] = \mu_{p,Q}(-L \otimes D) = \mu_p(-L \otimes D) = M_p^+[-L \otimes D] = 0. \quad \square$$

Lemma 11. *Let $M_{p,Q}^+$ denote the strong lub logarithmic Lipschitz constant induced by the norm $\| \cdot \|_{p,Q}$ on $\mathbb{R}^{nN}$ and let $M_{p,Q}$ denote the lub logarithmic Lipschitz constant induced by the norm $\| \cdot \|_{p,Q}$ on $\mathbb{R}^n$. Then,*

$$M_{p,Q}^+[\tilde{F}] \leq M_{p,Q}[F].$$

Proof. The proof is exactly the same as the proof of Lemma 7. $\square$

Proof of Theorem 3. By subadditivity of $M_{p,Q}^+$, Propostions 2, 6, and Lemma 11, for any $t > 0$:

$$M_{p,Q}^+[\tilde{F}_t - L \otimes D] \leq M_{p,Q}^+[\tilde{F}_t] + M_{p,Q}^+[-L \otimes D] \leq M_{p,Q}[F_t].$$

Therefore $\sup_{t \in [0,\infty)} M_{p,Q}^+[\tilde{F}_t - L \otimes D] \leq c$. Now using Corollary 3,

$$\|u(t) - v(t)\|_{p,Q} \leq e^{ct} \|u(0) - v(0)\|_{p,Q}. \quad \square$$

Lemma 12. *Assume F is a linear operator. Then*

$$\mu_{p,Q}(\tilde{F} - L \otimes D) \leq \mu_{q,Q}(F) \quad \text{if } p = q. \tag{22}$$

Proof. The proof is immediate by subadditivity of logarithmic norm, Lemma 11, and Corollary 1.

Remark 11. Note that (22) does not need to hold if $p \neq q$. Indeed, consider the following system:

$$\dot{x}_1 = Ax_1 + D(x_2 - x_1)$$
$$\dot{x}_2 = Ax_2 + D(x_1 - x_2),$$

where $x_i \in \mathbb{R}^2$, $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $D = \text{diag}(d_1, d_2)$. In this example $L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $F(u) = Au$, and $\tilde{F}(u) = \text{diag}(Au, Au)$. We will show that for $Q = \text{diag}(3, 1)$, $\mu_{2,Q}(A) < 0$ while $\mu_{1,Q}(\tilde{F} - L \otimes D) > 0$.

By Table 1,

$$\mu_{2,Q}(A) = \mu_2(QAQ^{-1}) = \mu_2 \begin{bmatrix} -2 & 3 \\ \frac{1}{3} & -2 \end{bmatrix} < 0,$$

and

$$\mu_{1,Q}(\tilde{F} - L \otimes D) = \mu_{1,Q} \begin{bmatrix} -2 - d_1 & 1 & d_1 & 0 \\ 1 & -2 - d_2 & 0 & d_2 \\ d_1 & 0 & -2 - d_1 & 1 \\ 0 & d_2 & 1 & -2 - d_2 \end{bmatrix}$$

$$= \mu_1 \begin{bmatrix} -2 - d_1 & 3 & d_1 & 0 \\ \frac{1}{3} & -2 - d_2 & 0 & d_2 \\ d_1 & 0 & -2 - d_1 & 3 \\ 0 & d_2 & \frac{1}{3} & -2 - d_2 \end{bmatrix}$$

$$= 1 > 0.$$

Theorem 4. *Consider the reaction–diffusion ODE (21) and suppose Assumption 2 holds. In addition assume that $F(x, t)$ is continuously differentiable with respect to x and $c = \sup_{(x,t) \in V \times [0,\infty)} \mu_{p,Q}(J_F(x, t))$. Then for any two solutions u, v of (21) we have*

$$\|u(t) - v(t)\|_{p,Q} \leq e^{ct} \|u(0) - v(0)\|_{p,Q}.$$

Proof. The proof is immediate by Theorem 3 and Proposition 4. $\square$

Acknowledgments

Appendix

A.1. Proof of Proposition 3

To prove this result, we first review some basic minimax optimization facts.

Proposition 7 ([37]). *Let X, Y be arbitrary sets and let $\varphi : X \times Y \to \mathbb{R}$ be an arbitrary function. For any $y \in Y$ and $c \in \mathbb{R}$, denote $H_{y,c} = \{x \in X : \varphi(x, y) \geq c\}$ and $\mathcal{C}$ the set of all real numbers c such that for all $y \in Y$, $H_{y,c} \neq \emptyset$, and let $c^* = \sup \mathcal{C}$. Then*

$$(B =) \sup_{x \in X} \inf_{y \in Y} \varphi(x, y) = \inf_{y \in Y} \sup_{x \in X} \varphi(x, y)(= J)$$

if and only if for every $c < c^*$, $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$. In this case $B = J = c^*$.

A proof is outlined in [37].

For a fixed arbitrary norm $\| \cdot \|$ on $\mathbb{R}^{nN}$ and a fixed arbitrary matrix $A \in \mathbb{R}^{nN \times nN}$, define $\varphi : S^{nN-1} \times (0, 1) \to \mathbb{R}$ by

$$\varphi(v, h) = \frac{1}{h} \left(\|v + hAv\| - 1 \right),$$

where $S^{nN-1} = \{v \in \mathbb{R}^{nN} : \|v\| = 1\}$. For any $h \in (0, 1)$ and $c \in \mathbb{R}$, let $H_{h,c} = \{v \in S^{nN-1} : \varphi(v, h) \geq c\}$ and let $\mathcal{C}$ be the set of all real numbers c such that $H_{h,c} \neq \emptyset$ whenever $h \in (0, 1)$. Let $c^* = \sup \mathcal{C}$.

The following two facts are easy to prove, see [31].

Lemma 13. $\varphi(v, h) = \frac{1}{h}(\|v + hAv\| - 1)$ *is non-increasing as $h \to 0^+$.*

Corollary 5. *For any matrix A, $\frac{1}{h}(\|I + hA\| - 1)$ is non-increasing as $h \to 0^+$.*

Proof of Proposition 3. Claim 1.

$$\sup_{v \in S^{nN-1}} \inf_{h \in (0,1)} \varphi(v, h) = \inf_{h \in (0,1)} \sup_{v \in S^{nN-1}} \varphi(v, h). \tag{23}$$

Proof of Claim 1. To apply Proposition 7, we will show that for $c < c^*$, $\bigcap_{h \in (0,1)} H_{h,c} \neq \emptyset$, where $H_{h,c} = \{v \in S^{nN-1} : \varphi(v, h) \geq c\}$ and c^* is defined as above. By Lemma 13, $\varphi(v, h)$ is decreasing in h which implies $H_{h_1,c} \subset H_{h_2,c}$ when $h_1 < h_2$. Also by the definition of c^*, $c < c^*$ implies that $H_{h,c} \neq \emptyset$ for any $h \in (0, 1)$. On the other hand, each $H_{h,c}$ is a closed subset of S^{nN-1}, so they are all compact. Hence their intersection is non-empty.

Claim 2.

$$\sup_{v \in S^{nN-1}} \lim_{h \to 0^+} \varphi(v, h) = \sup_{v \in S^{nN-1}} \inf_{h \in (0,1)} \varphi(v, h) \tag{24}$$

and

$$\lim_{h \to 0^+} \sup_{v \in S^{nN-1}} \varphi(v, h) = \inf_{h \in (0,1)} \sup_{v \in S^{nN-1}} \varphi(v, h). \tag{25}$$

Proof of Claim 2. By Lemma 13, since $f(v, h)$ is non-increasing as $h \to 0^+$, (24) holds. By Corollary 5, since $\frac{1}{h}(\|I + hA\| - 1)$ is non-increasing as $h \to 0^+$, (25) holds.

By Claim 1, the right hand side of the equalities in Claim 2 are equal, and therefore so are their left hand sides:

$$\sup_{v \in S^{nN-1}} \lim_{h \to 0^+} \varphi(v, h) = \lim_{h \to 0^+} \sup_{v \in S^{nN-1}} \varphi(v, h),$$

which implies $\mu(A) = \sup_{\|v\|=1} \lim_{h \to 0^+} \frac{1}{h}(\|v + hAv\| - 1)$. $\quad \square$

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