A REMARK ON ROBUST STABILIZATION OF GENERAL ASYMPTOTICALLY CONTROLLABLE SYSTEMS

Yu. S.Ledyaev*
Steklov Institute of Mathematics
Moscow 117966
Russia
ledyaev@class.mi.ras.ru

Eduardo Sontag†
Dept. of Mathematics
Rutgers University
New Brunswick, NJ 08903
sontag@control.rutgers.edu

Abstract

It was shown recently by Clarke, Ledyaev, Sontag and Subbotin that any asymptotically controllable system can be stabilized by means of a certain type of discontinuous feedback. The feedback laws constructed in that work are robust with respect to actuator errors as well as to perturbations of the system dynamics. A drawback, however, is that they may be highly sensitive to errors in the measurement of the state vector. This paper addresses this shortcoming, and shows how to design a dynamic hybrid stabilizing controller which, while preserving robustness to external perturbations and actuator error, is also robust with respect to measurement error. This new design relies upon a controller which incorporates an internal model of the system driven by the previously constructed feedback.

1. Introduction

Let the nonlinear control system

$$\dot{x} \;=\; f(x,u) \tag{1}$$

be (null)-asymptotically controllable. This means that for each point $x_0 \in \mathbb{R}^n$ there exists an "open loop" control $u : [0,+\infty) \to \mathbb{U}$ which drives the state vector $x(t)$ to the origin in $\mathbb{R}^n$. It is clear that the control system (1) is asymptotically controllable if there exists a "closed-loop", or feedback, control $k : \mathbb{R}^n \to \mathbb{U}$ which stabilizes all trajectories of the system

$$\dot{x} \;=\; f(x,k(x)) \tag{2}$$

with respect to the origin. The answer to the converse question: "Does asymptotic controllability imply feedback stabilization?" is not so obvious. In fact, the answer is negative if we consider only *continuous* feedback laws, as remarked in [15] and [2]. One must allow for discontinuous feedback. Unfortunately, allowing discontinuous feedback leads to an immediate difficulty: how should one define the meaning of *solution* $x(\cdot)$ of the differential equation (2) with discontinuous right-hand side? One possibility would be to interpret solutions in terms of differential inclusions, and in particular solutions in the Filippov sense [7]. However, as shown in [12, 6], there are examples of asymptotically controllable systems for which there is no possible feedback that stabilizes in the Filippov sense; in fact, [6] shows that the existence of a stabilizing feedback in the Filippov sense is equivalent to the existence of a continuous stabilizing one, in the case of systems affine in controls. Thus a totally different notion is needed.

A natural and physically meaningful concept of solution of (2), which can be interpreted as stabilization by sampling at arbitrarily high sampling rates, was introduced in [4]. This concept allowed the proof of the general result: asymptotic controllability implies the existence of stabilizing feedback. The feedback laws constructed in that work are robust with respect to actuator errors as well as to perturbations of the system dynamics (1). A drawback, however, is that they may be highly sensitive to errors in the measurement of the state vector $x(t)$. This paper addresses this shortcoming, and shows how to design a dynamic hybrid stabilizing controller which, while preserving robustness to external perturbations and actuator error, is also robust with respect to measurement error. This new design relies upon a controller which incorporates an internal model of the system driven by the previously constructed feedback. We start by reviewing the relevant definitions and results from [4].

1.1. Definition of Feedback Solution

From now on, we assume that $\mathbb{U}$ is a compact metric space and that the mapping $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n : (x,u) \mapsto f(x,u)$ is continuous, and locally Lipschitz on x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{U}$. (The compactness assumption on $\mathbb{U}$ is made here only to clarify the exposition and it can be replaced by more general assumption that $\mathbb{U}$ is a locally compact metric space.) We use $|x|$ to denote the usual Euclidean norm of $x \in \mathbb{R}^n$, and $\langle x,y \rangle$ for the inner product of two such vectors.

By a *sampling partition* (of $[0,+\infty)$) we mean any infinite sequence $\pi = \{t_i\}_{i\geq 0}$ consisting of numbers

$$0 = t_0 < t_1 < t_2 < \ldots$$

with $\lim_{i\to\infty} t_i = \infty$; its *diameter* is the number

$$d(\pi) \;:=\; \sup_{i\geq 0}(t_{i+1} - t_i).$$

An arbitrary function $k : \mathbb{R}^n \to \mathbb{U}$ will be called *feedback*.

---

*Visiting Rutgers University. Supported in part by Russian Fund for Fundamental Research Grant 96-01-00219 and by the Rutgers Center for Systems and Control (SYCON)

†Supported in part by US Air Force Grant AFOSR-94-0293

π, and an $x_0 \in \mathbb{R}^n$. Recursively solve

$$\dot{x}(t) = f(x(t), k(x(t_i))), \; t \in [t_i, t_{i+1}] \qquad (3)$$

for $i = 0, 1, 2, \ldots$, using as initial value $x(t_i)$ the endpoint of the solution on the preceding interval (and starting with $x(t_0) = x_0$). The function $x(\cdot)$ thus obtained is the *π-trajectory* of (2) *starting from* x_0.

The above solution may fail to be defined on all of $[0, +\infty)$, because of possible finite escape times in one of the intervals, in which case we only have a trajectory defined on some maximal interval. We say that the trajectory *is well-defined* if this interval is all of $[0, +\infty)$.

Definition 1.2 The feedback $k : \mathbb{R}^n \to \mathbb{U}$ *s-stabilizes* the system (1) if for each pair

$$0 < r < R$$

there exist numbers $M = M(R) > 0$, $\delta = \delta(r, R) > 0$, and $T = T(r, R) > 0$ such that, for each partition π with $d(\pi) < \delta$, and for each initial state x_0 such that $|x_0| \le R$, the π-trajectory $x(\cdot)$ of (2) starting from x_0 is well-defined and the following properties are satisfied:

1. (**uniform attractiveness**)

$$|x(t)| \le r \quad \text{for all } t \ge T\,. \qquad (4)$$

2. (**overshoot boundedness**)

$$|x(t)| \le M(R) \quad \text{for all} \quad t \ge 0; \qquad (5)$$

3. (**Lyapunov stability**)

$$\lim_{R \downarrow 0} M(R) = 0. \qquad (6)$$

Thus, a stabilizing feedback is one such that, for all fast enough sampling, drives states asymptotically to the origin and with small overshoot. Observe that, implicit in this definition are the facts that a faster sampling rate may be required near the origin (since smaller steps are needed, to preserve stability), as well as very far from the origin (to avoid for instance lack of existence of solutions). This definition is consistent with more classical notions of stabilization: if a continuous feedback k stabilizes the system (1) in the usual sense of making the origin of (2) globally asymptotically stable, then it also s-stabilizes (cf. [4]).

The following definition of (global, null-) asymptotic controllability generalizes the classical concept of uniform asymptotic stability for control systems; it was introduced in [13]. Given any *control*, that is, a measurable function $u : [0, +\infty) \to \mathbb{U}$, the solution of (1) at time $t \ge 0$, with initial condition x_0, defined on some maximal interval $[0, t_{\max}(x_0, u))$, is denoted by $x(t; x_0, u)$.

Definition 1.3 The system (1) is *asymptotically controllable* if:

1. control u such that the trajectory $x(t) = x(t; x_0, u)$ is defined for all $t \ge 0$, and $x(t) \to 0$ as $t \to +\infty$;

2. (**Lyapunov stability**) for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$, there is a control u as in 1. such that $|x(t)| < \varepsilon$ for all $t \ge 0$.

A relation between asymptotic controllability and feedback s-stabilization was established in [4].

Theorem 1 *The system (1) is asymptotically controllable if and only if it admits an s-stabilizing feedback.* ∎

The interesting part of this result lies in the construction of an s-stabilizing feedback for any asymptotically controllable system (1). This construction was based on: (a) the notion of control-Lyapunov function [13], (b) methods of nonsmooth analysis [3], and (c) techniques from positional differential games [10, 5].

1.2. Robustness Properties of Feedback

The advantages of feedback over open-loop control are usually expressed in terms of robustness properties. The most obvious such property is the compensation for sudden changes in states. Often, in addition, one is interested in good performance even with respect to *persistent* disturbances. Consider the following "perturbed" model associated to (2):

$$\dot{x} = f(x, k(x + e(t)) + a(t)) + d(t). \qquad (7)$$

We think of $d(t)$ as an *external disturbance* of the dynamics, $a(t)$ as an *actuator error*, and $e(t)$ as a *measurement error* for the state vector $x(t)$. For continuous stabilizing feedback, one has the following classical results on stability for persistent disturbances (cf. [9]): that there is a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that $\rho(x) > 0$ for $x \ne 0$, and so that, for any disturbances $d(\cdot)$, $a(\cdot)$, and $e(\cdot)$ satisfying

$$|d(t)| \le \rho(x(t)), \;\; |a(t)| \le \rho(x(t)), \;\; |e(t)| \le \rho(x(t)), \quad (8)$$

the trajectories of (7) are uniformly asymptotically stable with respect to the origin. Moreover, it can be shown that a continuous stabilizing feedback provides *robust* practical semi-global stabilization of the perturbed system (7), in the following sense: for any $0 < r < R$ there exists positive $\eta = \eta(r, R)$, $T = T(r, R)$, and $M(R)$ such that for any disturbances $d(\cdot)$, $a(\cdot)$, $e(\cdot)$ satisfying

$$|d(t)| \le \eta, \;\; |a(t)| \le \eta, \;\; |e(t)| \le \eta,$$

any trajectory of (7) with $|x(0)| \le R$ satisfies (4), (5) (and (6) holds for M).

The situation concerning robustness properties of s-stabilizing discontinuous feedback is more complicated.

We will say that the feedback k is *robustly s-stabilizing with respect to external disturbances, and actuator errors,* tinuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\rho(x) > 0$ for $x \neq 0$, such that, for any disturbances $a(\cdot)$, $d(\cdot)$ satisfying (8), and $e(\cdot) \equiv 0$, k is s-stabilizing for the perturbed system (7).

We will simply say that k is *robustly s-stabilizing* if for any disturbances $a(\cdot)$, $d(\cdot)$ and errors $e(\cdot)$ satisfying (8) k is s-stabilizing for the perturbed system (7).

It was shown in [4] that the s-stabilizing feedback constructed there is robust with respect to external disturbances and actuator errors in the absence of measurement errors. But the assumption that there is no measurement error, i.e. $e(\cdot) \equiv 0$, is essential for establishing that fact. In general, s-stabilizing feedback can be too sensitive with respect to errors in measurement of state vector $x(t)$. The following result from [11] (which was motivated by an analogous result for classical solutions given in [8]) imposes strong conditions on robust stabilization:

Theorem 2 *Let $k : \mathbb{R}^n \to \mathbb{U}$ be a robust s-stabilizing feedback. Then the control system (1) admits a smooth control-Lyapunov function.*

The existence of a smooth control-Lyapunov function for control system (1) is a strong requirement, and fails in general for asymptotically controllable systems. For instance, if (1) is affine in controls, the existence of such a control-Lyapunov function implies that there is also a continuous stabilizing feedback, which as discussed earlier is in general not the case. More generally, the existence of smooth control-Lyapunov function is equivalent to the existence of stabilizing discontinuous feedback with solutions of (2) defined in terms of a suitable differential inclusion.

The main contribution of this paper lies in the construction, for any asymptotically controllable system, of a *dynamic hybrid* controller, which provides robust stabilization in spite of external disturbances, actuator errors, and measurement errors. An important component of this controller is an *internal model* of the system (1) under the action of the s-stabilizing feedback constructed in [4].

Our notion of s-stabilizing discontinuous feedback was based upon the notion of discontinuous feedback studied in the context of positional differential games by N.N. Krasovskii and A.I. Subbotin in [10]. Similarly, our design of a stabilizing robust controller in this paper is closely patterned after the "control with guide" algorithm, which was introduced in their work also in order to provide robustness of discontinuous feedback with respect to measurement errors.

Remark 1.4 Mathematically, because of the uniform continuity of $f(x, u)$ on bounded sets, any perturbation of the system (1) by actuator errors can be considered and accounted for by the external disturbances. So from now on we omit any explicit reference on actuator errors. □

This paper is organized as follows. In the next section we provide a construction of the robust controller auxiliary results. Later these results are used in the proof of the robustness properties of this controller.

2. Robust Dynamic Hybrid Controller

Let the control system (1) be asymptotically controllable and let $k : \mathbb{R}^n \to \mathbb{U}$ be a s-stabilizing feedback for (1).

In the presence of measurement errors $e(t)$, only values $x'(t)$ of the measured estimate of state vector $x(t)$

$$x'(t) := x(t) + e(t) \tag{9}$$

can be used for control.

Define a *tracking controller* $k_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{U}$ as follows:

$$\langle z - x', f(x', k_0(x', z))\rangle = \max_{u \in \mathbb{U}} \langle z - x', f(x', u)\rangle. \tag{10}$$

The most important feature of the tracking controller (10) is the fact that trajectories of the system

$$\dot{x} = f(x, k_0(x', z)) + d_1(t) \tag{11}$$

can track any trajectory of the control system

$$\dot{z} = f(z, u(t)) + d_2(t) \tag{12}$$

for any control $u(\cdot)$ if the measurement errors $e(\cdot)$, disturbances $d_1(\cdot)$, $d_2(\cdot)$, and diameter of the sampling partition π_x for (11) are small enough. We postpone a detailed discussion to the end of this Section, in order to first define a robust dynamic hybrid controller.

A *dynamic hybrid controller* consists of:

- a *tracking controller* k_0 (10) which drives system (11)
- an *internal model* of the system (1) driven by an s-stabilizing feedback

$$\dot{z} = f(z, k(z)) + d_2(t) \tag{13}$$

- a set of *re-initialization rules* which define moments t'_k from the sampling partition π_z for (13) for re-initialization of internal model (13)

$$z(t'_k) = x'(t'_k) \tag{14}$$

- a *sampling rule* to choose sampling moments $t_i \in \pi_x$ and $\tau_i \in \pi_z$

$$t_{i+1} - t_i \leq \delta_x(x'(t_i)) \tag{15}$$

$$\tau_{i+1} - \tau_i \leq \delta_z(z(\tau_i)). \tag{16}$$

Remark 2.5 Note that, by allowing a disturbance in the internal model (13), we can assume that the output $z(t)$ of the internal model is computed by an Euler method for the differential equation (13). In this sense, the internal model is the computational model of closed-loop system (2).

initialization of the internal model is determined by the stabilizing properties of s-stabilizing feedback k. Their precise definition will be given in (26)-(27), after a description of these properties.

A sampling rule means that sampling partitions π_x for (11) and π_z for (13) satisfying respectively (15) and (16) are considered. The continuous functions $\delta_x : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\delta_z : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are assumed to be positive outside 0. □

The main result of this paper is the following statement characterizing robustness properties of the dynamic hybrid controller defined in this section.

Theorem 3 *Let the system (1) be asymptotically controllable and k be an s-stabilizing feedback. Then, there exists continuous functions $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\delta_x : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $\delta_z : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, as well as re-initialization rules (26)-(27), such that for any measured estimate x' of state vector x, and any disturbances $d_1(\cdot)$, $d_2(\cdot)$ satisfying*

$$|x'(t) - x(t)| \leq \rho(x(t)), \quad \text{for any} \quad t \geq 0 \tag{17}$$

$$|d_1(t)| \leq \rho(x(t)), \quad |d_2(t)| \leq \rho(x(t)) \quad \text{for any} \quad t \geq 0, \tag{18}$$

The π_x-trajectories $x(\cdot)$ of (11) are stable in the following sense: for any $0 < r < R$ there exist positive $T = T(r, R)$ and $M(R)$ such that (6) holds, and any π_x-trajectory of (11) with $|x(0)| \leq R$ satisfies (4) and (5).

It is possible to derive from Theorem 3 that this dynamic hybrid controller provides robust practical semi-global stabilization of system (1) even for π_x-trajectories of (11) with sampling partitions π_x satisfying assumption $d(\pi_x) \leq \delta$, instead of (15), for some positive constant δ.

Corollary 2.6 For any $0 < r < R$ there exist positive $T = T(r, R)$, $\eta = \eta(r, R)$, $\delta = \delta(r, R)$, and $M(R)$ such that (6) holds and for any measured estimate x' and any disturbances $d_1(\cdot)$, $d_2(\cdot)$ satisfying

$$|x'(t) - x(t)| \leq \eta \quad \text{for any} \quad t \geq 0, \tag{19}$$

$$|d_1(t)| \leq \eta, \qquad |d_2(t)| \leq \eta \quad \text{for any} \quad t \geq 0, \tag{20}$$

any π_x-trajectory of (11) with $d(\pi_x) \leq \delta$ and $|x(0)| \leq R$ satisfies (4) end (5). □

This result means that the dynamic hybrid controller drives $x(t)$ from a bounded set in $\mathbb{R}^n$ uniformly to an arbitrary small neighborhood of the origin, if the measurements errors, external disturbances, and diameter of a sampling partition are small enough.

2.1. Re-Initialization Rules

In this paper we prove Theorem 3 for the case of s-stabilizing feedback k defined in [4]. Let us describe some properties of this feedback which are used in the statement of the re-initialization rules for the internal model. $\{R_j\}_{-\infty}^{+\infty}$ and closed sets $\{G_j\}_{-\infty}^{+\infty}$ such that

$$\lim_{j \to -\infty} R_j = 0, \qquad \lim_{j \to +\infty} R_j = +\infty,$$

$$G_j \subset B_{R_j} \subset B_{2R_j} \subset G_{j+1} \tag{21}$$

and the set G_{j+1} is invariant with respect to π_z-trajectories of the internal model:

$$z(t) \in G_{j+1} \quad \text{for all} \quad t \geq 0 \tag{22}$$

if $d(\pi_z) \leq \delta_j$ and $|d_2(t)| \leq \chi_j$ for some positive numbers δ_j, χ_j. Moreover, there exists moments $T_j > 0$ such that, for any π_z-trajectory as above with $z(0) \in G_{j+1}$, we have the relation

$$z(t') \in G_{j-1}, \ \ |z(t)| > R_{j-2} \quad \text{for all} \quad t \in [0, t'] \tag{23}$$

at some moment $t' \leq T_j$.

Remark 2.7 It can be shown that there exist continuous functions $\delta_z : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\chi_z : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which are positive outside 0 and such that for any disturbance $d_2(\cdot)$ satisfying

$$d_2(t) \leq \chi(z(t)) \qquad \text{for all} \quad t \geq 0 \tag{24}$$

and π_z-trajectory of (13) with sampling partition π_z determined by sampling rule (16), relations (22) and (23) hold. □

We have for the sets

$$H_j := G_{j+1} \setminus G_j$$

that

$$\mathbb{R}^n \setminus \{0\} = \cup_{j=-\infty}^{+\infty} H_j.$$

Now we can define *re-initialization rules* for determining sequential moments $t'_k \in \pi_z$, $k = 0, 1, \ldots$, of re-initialization (14) of the internal model (13).

Let $t'_0 = 0$, and assume that the moment t'_k has already been determined. Take an index j_k that satisfies

$$x'(t'_k) \in H_{j_k}. \tag{25}$$

Then the next moment t'_{k+1} of the re-initialization is defined as first moment $t' > t'_k$ from π_z such that one of the following two events occur:

$$x'(t') \in H_{j_{k+1}} \quad \text{for some} \quad j_{k+1} \leq j_k - 1 \tag{26}$$

$$x'(t') \notin G_{j_k+2} \tag{27}$$

2.2. Tracking Lemma

It was mentioned before that the tracking controller (10) can track trajectories of control system (12) by using only measured estimates $x'(t)$ (9) of state vector $x(t)$.

The next variant of Krasovskii-Subbotin Tracking Lemma [10] makes this statement precise.

Let us assume that function $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ satisfies the following growth and Lipschitz conditions:

$$|f(x, u)| \leq m \qquad \text{on} \quad R^n \times \mathbb{U} \tag{28}$$

$$|f(x_1, u) - f(x_2, u)| \leq l|x_1 - x_2| \text{ for all } \ x_1, x_2 \in \mathbb{R}^n, u \in \mathbb{U} \tag{29}$$

tive T, η, any measured estimate $x'(\cdot)$ and disturbances $d_1(\cdot)$, $d_2(\cdot)$ satisfying (19), (20), and any trajectory $z(\cdot)$ of (12) defined on $[0,T]$, an arbitrary π_x-trajectory of (10) with

$$d(\pi_x) \leq \eta^2 \tag{30}$$

and initial conditions

$$|x(0) - z(0)| \leq \eta \tag{31}$$

is defined on $[0,T]$ and satisfies

$$|x(t) - z(t)| \leq \gamma(\eta) \quad \text{for all} \quad t \in [0,T], \tag{32}$$

where

$$\gamma(\eta) := e^{2lT}(1 + l(1+m\eta)^2 + 4m)^{\frac{1}{2}}\eta \tag{33}$$

Remark 2.9 Since $\lim_{\eta\to+0}\gamma(\eta) = 0$, it follows from Lemma 2.8 that, on finite time intervals, the trajectories of (11) driven by the controller k_0 (10) approximate arbitrarily close (that is, they track) the trajectories of (12) if measurement errors, disturbances, diameter of the sampling partition are small and the initial conditions $x(0)$ and $z(0)$ are close enough.

It follows from the proof of Lemma 2.8 that if a trajectory $z(\cdot)$ stays in some compact set G, then, for sufficiently small η, constants m, l in (28), (29) can be defined as an upper bound and Lipschitz constant of f on the set $G' \times \mathbb{U}$ where G' is some compact neighborhood of G. This means that we can replace the global growth and Lipschitz conditions (28), (29) by assumptions of continuity and local Lipschitzness of f. □

3. Proof of Main Theorem

Let m_j and l_j denote respectively the upper bound for $|f(x,u)|$ and Lipschitz constant of f on the set $B_{R_{j+3}} \times \mathbb{U}$ for every integer j.

Define

$$\eta_j := \max\{\eta > 0 \ : \ \gamma_j(\eta) + 2\eta \leq R_{j-2} - R_{j-3}\},$$

where the function γ_j is given by (33) for $T = T_j$, $m = m_j$, $l = l_j$. Let

$$\tilde{\rho}(x) := \min\{\eta_i \ : j-3 \leq i \leq j+3\} \ \text{ for } \ R_j < |x| \leq R_{j+1},$$

$$\tilde{\delta}(x) := \min\{\eta_i^2 \ : j-3 \leq i \leq j+3\} \ \text{ for } \ R_j < |x| \leq R_{j+1},$$

and

$$\rho(x) := \min\{\tilde{\rho}(y) + \frac{1}{2}|y - x| \ : y \in \mathbb{R}^n\}$$

$$\delta_x(x) := \min\{\tilde{\delta}(y) + \frac{1}{2}|y - x| \ : y \in \mathbb{R}^n\}.$$

Note that functions ρ and δ_x are positive for $x \neq 0$, Lipschitz with constant $\frac{1}{2}$ and

$$\rho(x) \leq \eta_i, \quad \delta_x(x) \leq \eta_i^2 \tag{34}$$

for any $j-3 \leq i \leq j+3$ and $R_j < |x| \leq R_{j+1}$.

function ρ satisfies the inequality $\rho \leq \chi$.

By using the Lipschitz condition for the function ρ, we obtain that for any x, x satisfying $|x - x') \leq \rho(x)$, the following inequality holds:

$$|x - x'| \leq 2\rho(x'). \tag{35}$$

To verify that the functions ρ and δ_x satisfy the assertion of Theorem 3, let us consider two sequential moments t'_k, t'_{k+1} of the re-initialization of the internal model (13).

Let $t^* = \min\{t'_{k+1}, t'_k + T_{j_k}$ and denote by $\tilde{t}$ the maximal $t' \in [t'_k, t^*]$ such that

$$R_{j_k-3} < |x(t)| < R_{j_k+3} \tag{36}$$

$$R_{j_k-3} < |x'(t)| < 2R_{j_k+2} \tag{37}$$

for all $t \in [t'_k, t']$.

It follows from (17),(34) and (35) and that

$$|x(t'_k) - x'(t'_k)| \leq 2\rho(x'(t'_k)) \leq 2\eta_{j_k},$$

which implies that (36), (37) hold for $t = t'_k$ and $\tilde{t} > t'_k$.

Then we have from (34) that

$$\rho(x(t)) \leq \eta_{j_k} \ \ \delta_x(x'(t)) \leq \eta_{j_k}^2 \ \text{ for all } \ t \in [t'_k, \tilde{t}].$$

This implies due to (17) that we can apply Lemma 2.8 to obtain that the estimate

$$|x(t) - z(t)| \leq \gamma_j(\eta_j) \tag{38}$$

is valid for all $t \in [t'_k, \tilde{t}]$.

Then in view of (22), (23) and the definition of η_j we obtain that (36), (37) hold for $t = \tilde{t}$, which implies that both these relations are valid on the entire interval $[t'_k, t^*]$.

To show that

$$t'_{k+1} < t'_k + T_{j_k},$$

where the moment t'_{k+1} is defined by the relation (26), let us assume that this is not the case. Then (38) holds and

$$x'(t) \notin G_{j_k}$$

for all $t \in [t'_k, t'_k + T_{j_k}]$. Recall that in this case, for the trajectory $z(\cdot)$ of the internal model there exists $t' \in (t'_k, t'_k + T_{j_k}]$ such that the first inclusion in (23) holds. This implies that

$$\begin{array}{rcl} |x'(t')| & \leq & |z(t')| + \gamma_j(\eta_j) + \eta_j \\ & < & R_{j_k-1} + R_{j_k-2} - R_{j_k-3} < 2R_{j_k-1}. \end{array}$$

Because of (21) we obtain that $x'(t') \in G_{j_k}$. This contradiction proves that the relation (26) determines the re-initialization moment t'_{k+1} and

$$t'_{k+1} - t'_k < T_{j_k}. \tag{39}$$

Thus, we have proved the following Lemma.

$$x'(t) \in B_{2R_{j_k+2}} \subset G_{j_k+3} \quad \text{for all} \;\; t \in [t'_k, t'_{k+1} \tag{40}$$

$$x'(t'_{k+1}) \in H_{j_{k+1}} \subset G_{j_k}. \tag{41}$$

It is obvious that by using this Lemma we can prove that the dynamic hybrid controller provides the uniform convergence of $x'(t)$ to the origin in $\mathbb{R}^n$. Then it follows from (17) and (35) that $x(t)$ uniformly converges to the origin too.

To realize this plan, let us consider any $0 < r < R$ and initial point $x_0 \in B_R$ for the π_x-trajectory of (11) with $x(0) = x_0$. Then we have from (17) and definition η_j that there exists an integer $N = N(R)$ which does not depend upon x_0 such that $x'(0) \in G_N$ and $\lim_{R \to +0} N(R) = -\infty$. Define

$$M(R) := 4R_{N(R)+2},$$

and note that it satisfies (6). In accordance with Lemma 3.10 we have that

$$|x'(t)| \leq 2R_{N(R)+2},$$

which implies (5) because of (35). Define maximal the $K(r)$ such that $4R_{K(r)+2} < r$. By applying Lemma 3.10, we obtain that there is a moment t' such that

$$t' \leq T := \sum_{i=K(r)}^{N(R)} T_i$$

$x'(t') \in G_{K(r)}$ which implies due to the same Lemma that

$$|x(t)| \leq 4R_{K(r)+2} \quad \text{for all} \;\; t \leq T.$$

Then $x(\cdot)$ satisfies (4). This completes the proof of Theorem 3. ∎

Proof of Corollary 2.6. For arbitrary $0 < r < R$, integers $K = K(r)$ and $N = N(R)$ were defined in the proof of Theorem 3. Let us define

$$\eta := \min\{\rho(x) : R_{K-1} \leq |x| \leq M(R)\},$$

$$\delta := \min\{\frac{R_K}{m_{K+2}}, \min\{\delta_x(x) : R_{K-1} \leq |x| \leq M(R)\}\}.$$

If the measurement $x'(\cdot)$, the disturbances $d_1(\cdot)$, $d_2(\cdot)$ satisfy (19), (20), and the diameter of the sampling partition π_x is less than δ, then for any π_x-trajectory $x(\cdot)$ with $|x(0)| \leq R$, the relations (17),(18) hold, and the partition π_x satisfies the sampling rule (15) for all t such that $R_{K-1} \leq |x'(t)| \leq 2R_N$. For such t, Lemma 3.10 is valid, and $x'(t)$ and $x(t)$ attain respectively G_K and B_{4R_K} before some moment $T > 0$, and stay there till the moment when $x'(t)$ attains G_{K-1}. But the set G_{K-1} is not necessarily invariant with respect to $x'(t)$. This means that the re-initialization moment can be defined by (27). Thus, $x'(t)$ can jump at such a moment outside G_{K-1}. It is easy to estimate that due to the choice of $delta$, the vector $x'(t)$ can jump to B_{2R_K}, which means that $x'(t)$ will stay in $B_{R_{K+2}}$ till the moment it again attains G_{K-1}. It follows from (35) that for all $T \geq T$,

$$|x(t)| \leq 2R_{K+2} < r,$$

which proves that (4),(5) holds.

[1] Artstein, Z., "Stabilization with relaxed controls," *Nonl. Anal., TMA* **7**(1983): 1163-1173.

[2] Brockett, R.W., "Asymptotic stability and feedback stabilization," in *Differential Geometric Control theory* (R.W. Brockett, R.S. Millman, and H.J. Sussmann, eds.), Birkhauser, Boston, 1983, pp. 181-191.

[3] Clarke, F.H., *Methods of Dynamic and Nonsmooth Optimization.* Volume 57 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, S.I.A.M., Philadelphia, 1989.

[4] Clarke, F.H., Yu.S. Ledyaev, E.D. Sontag, and A.I.Subbotin, "Asymptotic stabilization implies feedback stabilization," *IEEE Trans. Automat. Control* (1997), to appear. (Preliminary version: "Asymptotic stabilization and feedback stabilization," *Proc. Conf. on Information Sciences and Systems (CISS 96),* Princeton, NJ, 1996, 1232-1237.)

[5] Clarke, F.H., Yu.S. Ledyaev, and A.I. Subbotin, "Universal feedback strategies for differential games of pursuit," *SIAM J. Control*, **35** (1997), 552 - 561.

[6] Coron, J.-M., and L. Rosier, "A relation between continuous time-varying and discontinuous feedback stabilization," *J.Math. Systems, Estimation, and Control* **4**(1994): 67-84.

[7] Filippov, A.F., "Differential equations with discontinuous right-hand side," *Matem. Sbornik* **5**(1960): 99-127. English trans. in *Amer. Math. Translations* **42**(1964): 199-231.

[8] Hermes, H., "Discontinuous vector fields and feedback control," in *Differential Equations and Dynamic Systems* (J.K. Hale and J.P. La Salle, eds.), Academic Press, New York, 1967.

[9] Krasovskii, N.N., *Stability of Motion*, Stanford University Press, Stanford, 1963.

[10] Krasovskii, N.N., and A.I. Subbotin, *Positional differential games*, Nauka, Moscow, 1974 [in Russian]. French translation *Jeux differentiels*, Editions Mir, Moscou, 1979. Revised English translation *Game-Theoretical Control Problems*, Springer-Verlag, New York, 1988.

[11] Ledyaev, Yu.S., and E.D.Sontag, "A Lyapunov characterization of robust stabilization" (1997) (in preparation).

[12] Ryan, E.P., "On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback," *SIAM J. Control Optim.* **32**(1994): 1597-1604.

[13] Sontag E.D., "A Lyapunov-like characterization of asymptotic controllability," *SIAM J. Control and Opt.* **21**(1983): 462-471.

[14] Sontag E.D., *Mathematical Control Theory, Deterministic Finite Dimensional Systems*, Springer-Verlag, New York, 1990.

[15] Sontag, E.D., and H.J. Sussmann, "Remarks on continuous feedback," in *Proc. IEEE Conf. Decision and Control, Albuquerque, Dec. 1980*, IEEE Publications, Piscataway, pp. 916-921.