Remarks on Control Lyapunov Functions for Discontinuous Stabilizing Feedback

G.A. Lafferriere[1]
Department of Mathematics, Portland State University, Portland, OR 97207-0751
Eduardo D. Sontag[2]
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

Abstract

We present a formula for a stabilizing feedback law under the assumption that a piecewise smooth control-Lyapunov function exists. The resulting feedback is continuous at the origin and smooth everywhere except on a hypersurface of codimension 1. We provide an explicit and "universal" formula. Finally, we mention a general result connecting asymptotic controllability and the existence of control-Lyapunov functions in the sense of nonsmooth optimization.

1. Introduction

In this work we want to focus on nonsmooth Lyapunov functions that can be obtained by "pasting together" smooth ones (see [6] for other approaches). For these functions the verification of the Lyapunov property can be carried out using gradients, as in the smooth case.

A result by Artstein [1] guarantees the existence of a globally stabilizing feedback law provided the system has a smooth *control Lyapunov function.* Sontag [5] gave a constructive proof providing a formula for the feedback law in terms of Lie derivatives of the Lyapunov function and which is in a sense "universal". Sontag and Lin [8] then used a similar idea to allow for certain types of control constraints. Here we generalize that idea to certain cases when we only have a *piecewise smooth* Lyapunov function.

Non-holonomic systems of the form $\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$ (which appear naturally in the study of mechanical systems) cannot be stabilized by continuous static feedback at the origin since they do not satisfy Brockett's necessary conditions (see e.g. [7], Section 4.8). Stabilizing feedback laws have been found for several such systems using either time-varying feedback or dynamic feedback; see [3] and references there. Recently, however, piecewise continuous feedback laws were presented for two such examples in [2] and [4].

The approach presented here permits the construction of a piecewise continuous globally stabilizing feedback law from a piecewise smooth control Lyapunov function. We illustrate the theorem with an example.

Finally, we mention a general result connecting asymptotic controllability and the existence of control-Lyapunov functions in the sense of nonsmooth optimization. This result relies on recent work on viscocity solutions.

2. Stability

Notation. We denote by $\overline{M}$ the closure of the set M. If g is a vector field on $\mathbb{R}^n$ and $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth function then we denote by $L_g V$ the Lie derivative of V with respect to g (i.e. $L_g V(x) = \nabla V(x) \cdot g(x)$).

Definition 2.1 We say that a subset Γ of $\mathbb{R}^n$ is a *separating hypersurface* if Γ is an embedded, oriented, connected, $(n-1)$-dimensional sub-manifold and $\mathbb{R}^n \setminus \Gamma$ has two connected components. If Γ is a separating hypersurface let C^i, $i = 1, 2$ be the two connected components of $\mathbb{R}^n \setminus \Gamma$ and let $\mathbf{n}(x)$ be a unit normal vector on Γ which defines the orientation such that C^1 is on the "positive side". That is, if $p \in \Gamma$ and we choose local coordinates (W, φ) centered at p in which $W \cap \Gamma$ is the set $\{(x_1, \ldots, x_n) \in W : x_1 = \ldots = x_{n-1} = 0\}$ and $\mathbf{n}(x) = [0, \ldots, 0, 1]^T$ then $C^1 \cap W = \{(x_1, \ldots, x_n) \in W : x_n > 0\}$. We say that a vector field f *points towards* C^1 *on* Γ if $\forall x \in \Gamma \ \mathbf{n}^T(x) f(x) > 0$.

Definition 2.2 Let M be a connected open subset of $\mathbb{R}^n$ such that $0 \in \overline{M}$. Let f be a (Lipschitz) vector field on $M \cup \{0\}$, with $f(0) = 0$. Let U be an open subset of $\mathbb{R}^n$ such that $0 \in \overline{U} \subset M \cup \{0\}$. We say that $\overline{U}$ is f-invariant if every trajectory of f starting at $x \in \overline{U}$ remains in $\overline{U}$ for all $t > 0$. A function $V : \overline{M} \to \mathbb{R}$ is called a *Lyapunov function for* f *on* U if

(i) $\overline{U}$ is f-invariant

(ii) V is continuous on $\overline{M}$, and of class $\mathcal{C}^1$ on M

(iii) V is positive definite and proper on $\overline{M}$

(iv) $\forall x \in \overline{U} \ \ x \neq 0, L_f V(x) < 0$.

Definition 2.3 Let N be an open subset of $\mathbb{R}^n$ with $0 \in \overline{N}$. A vector field f on N is *asymptotically stable on* N if for each neigborhood $\mathcal{V}$ of 0 there is a neighborhood $\mathcal{W}$ of 0 such that $\forall x \in \mathcal{W} \cap N$ the integral curve of f starting at x, $\phi(x,t)$, is defined for all $t > 0$, $\phi(x,t) \in \mathcal{V} \cap N$, and $\lim_{t\to\infty} \phi(x,t) = 0$.

The following two lemmas can be proved with classical arguments together with the fact that trajectories do not leave the appropriate regions.

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Lemma 2.1 *Let M, U and f as in definition (2.2). Let V be a Lyapunov function for f on U. Then f is asymptotically stable on U.*

Lemma 2.2 *Let M^1, M^2 be two open connected subsets of $\mathbb{R}^n$ such that $M^1 \cup M^2 = \mathbb{R}^n \setminus \{0\}$. Let $f^i : M^i \longrightarrow \mathbb{R}^n$, $i = 1,2$ be two vector fields. Assume also, that there exists a separating hypersurface Γ with $0 \in \Gamma$ and $\Gamma \setminus \{0\} \subset M^1 \cap M^2$. Let C^1, C^2 be the two connected components of $\mathbb{R}^n \setminus \Gamma$ and assume that $C^i \subset M^i$ and that f^i points towards C^i on Γ for $i = 1,2$. Finally assume that f^1, f^2 are asymptotically stable on M^1, M^2. Then, the vector field $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by*

$$f(x) = \begin{cases} f^1(x) & \text{if } x \in (\Gamma \setminus \{0\}) \cup C^1 \\ f^2(x) & \text{if } x \in C^2 \\ 0 & \text{if } x = 0 \end{cases}$$

is globally asymptotically stable.

3. Control-Lyapunov Functions

Definition 3.4 Let M be an open connected subset of $\mathbb{R}^n$. Given the system

$$(\Sigma) \qquad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

a function $V : \overline{M} \longrightarrow \mathbb{R}$ is called a *control Lyapunov function (clf)* for Σ on M if the following hold:

(i) V is continuous on $\overline{M}$, and of class $\mathcal{C}^1$ on M

(ii) V is positive definite and proper on $\overline{M}$

(iii) $\inf_{u \in \mathbb{R}^m} \{L_f V(x) + u_1 L_{g_1} V(x) + \cdots + u_m L_{g_m} V(x)\} < 0$ for each $x \in M$.

The following theorem presents the construction of a piecewise continuous stabilizing feedback law.

Theorem 3.1 *Let M^j, $j = 1,2$, be connected open subsets of $\mathbb{R}^n$ such that $M^1 \cup M^2 = \mathbb{R}^n \setminus \{0\}$. Consider the system (Σ) as above. Suppose there exists a separating hypersurface Γ with $0 \in \Gamma$, $\Gamma \setminus \{0\} \subset M^1 \cap M^2$. Let C^1, C^2 be the two connected components of $\mathbb{R}^n \setminus \Gamma$, with $C^j \subset M^j$, for $j = 1,2$. Let $V^j : M^j \longrightarrow \mathbb{R}^n$ be control Lyapunov functions for (Σ) on M^j.*

Assume the following transversality conditions hold:

(i) $f(x)$ is tangent to Γ for all $x \in \Gamma$,

(ii) $-L_{g_i} V^j(x) \cdot g_i(x)$ points to C^j on Γ for $j = 1,2$, $i = 1, \ldots, m$.

Then there exists a globally stabilizing feedback law which is smooth on $C^1 \cup C^2$.

Proof. Consider the set $S = \{(a,b) \in \mathbb{R}^2 : b > 0 \text{ or } a < 0\}$. Define

$$\psi(a,b) = \begin{cases} \frac{a + \sqrt{a^2 + b^2}}{b} & b \neq 0 \\ 0 & b = 0 \end{cases}$$

Then ψ is analytic on S. (The function $p = \psi(a,b)$ is a solution of $bp^2 - 2ap - b = 0$ for $(a,b) \in S$.) Define for $j = 1,2$, $i = 1, \ldots, m$, $x \in M^j$,the functions $\alpha^j(x) = L_f V^j(x)$, $b_i^j(x) = L_{g_i} V^j(x)$, and $\beta^j(x) = \sum_{i=1}^m (b_i^j(x))^2$.

The third condition in the definition of clf, applied to V^j, M^j is equivalent to asking that $\beta^j(x) = 0 \Longrightarrow \alpha^j(x) < 0$. That is, $(\alpha^j(x), \beta^j(x)) \in S$ for $j = 1,2$.

We define feedback laws $k^j = (k_1^j, \ldots, k_m^j)$, $j = 1,2$ by

$$k_i^j(x) = \begin{cases} -b_i^j(x) \psi(\alpha^j(x), \beta^j(x)) & x \neq 0 \;\; x \in M^j \\ 0 & x = 0 \end{cases}$$

These functions are smooth on M^j. If f, g_i as well as V^j are real analytic then k^j is real analytic. On each M^j this is the same universal formula used in the smooth case (see [5]).

We now define vector fields h^j on M^j, for $j = 1,2$ by $h^j(x) = f(x) + \sum_{i=1}^m k_i^j(x) g_i(x)$. These vector fields are smooth on M^j. Since $h^j(x)$ points into C^j on Γ for $j = 1,2$, we conclude that $\overline{C^j}$ is h^j-invariant. Finally, the function V^j is a Lyapunov function for h^j on C^j, $j = 1,2$. Indeed

$$\begin{aligned} L_{h^j} V(x) &= L_{f^j} V^j(x) + k_1^j L_{g_1} V(x) + \cdots + k_m^j L_{g_m} V(x) \\ &= \alpha^j(x) - \beta^j(x) \psi(\alpha^j(x), \beta^j(x)) \\ &= -\sqrt{(\alpha^j(x))^2 + (\beta^j(x))^2} \qquad < \quad 0 \end{aligned}$$

By Lemma 2.1, the vector fields h^j are asymptotically stable. We can apply now Lemma 2.2 to conclude that the feedback law

$$k(x) = \begin{cases} k^1(x) & x \in \Gamma \cup C^1 \\ k^2(x) & x \in C^2 \end{cases}$$

results in a globally asymptotically stable system. ∎

4. Example

Consider the system on $\mathbb{R}^2$: $\dot{\xi} = g(\xi) u$ where $\xi = [x \;\; y]^T$, and $g(\xi) = \left[x^2 - y^2 \;\; 2xy\right]^T$. Here $m = 1$ and $f \equiv 0$. The orbits of g are in Figure 1 (a). It can be shown by a topological argument that this system is not locally smoothly stabilizable.

In this case a natural Lyapunov function is given by the following formula (see also Figure 1 (b))

$$V(x,y) = \begin{cases} \frac{x^2 + y^2}{\sqrt{x^2 + y^2} + |x|} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}$$

Instead of using explicitly this function, we may consider two partially defined functions. Let $\epsilon > 0$ and let K_ϵ^1, K_ϵ^2 be the cones with vertex at the origin given (in polar coordinates) by, $K^1_\epsilon = \{(r,\theta) : \pi - \epsilon \le \theta \le \pi + \epsilon\}$ and $K^2_\epsilon = \{(r,\theta) : -\epsilon \le \theta \le \epsilon\}$.

We can take $M^j = \mathbb{R}^2 \setminus K^j$ and consider V^j on M^j for $j = 1,2$ defined by $V^1(x,y) = \frac{x^2+y^2}{\sqrt{x^2+y^2}+x}$, $V^2(x,y) = \frac{x^2+y^2}{\sqrt{x^2+y^2}-x}$, and $V^1(0,0) = V^2(0,0) = 0$.

As separating hypersurface Γ we can take the y-axis oriented with the right half-plane as the positive side. Clearly, V^j is smooth on M^j and continuous on $\overline{M^j}$. The functions V^j are also proper on $\overline{M^j}$ since $V^j(x,y) \le k \Longrightarrow \sqrt{x^2+y^2} \le 2k$.

A direct calculation shows that the conditions of Theorem 3.1 are then easily checked. We conclude that the feedback law

$$k(x) = \begin{cases} -\dfrac{\left(\sqrt{x^2+y^2}\right)^3}{\sqrt{x^2+y^2}+x} & x > 0 \text{ or } (x = 0 \text{ and } y \neq 0) \\ -\dfrac{\left(\sqrt{x^2+y^2}\right)^3}{-\sqrt{x^2+y^2}+x} & x < 0 \\ 0 & x = y = 0 \end{cases}$$

makes the system globally asymptotically stable.

5. Further Remarks

In [6] a converse Lyapunov theorem for control Lyapunov functions was proved. More precisely, it was shown that asymptotic controllablility (defined below), which is a minimal condition for stabilizability, is sufficient to insure the existence of a control Lyapunov function V which is C° but not necessarily smooth (with derivatives of V replaced with Dini derivatives).

Definition 5.5 Consider the system $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, with $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$, $u \in U \subset \mathbb{R}^n$ bounded, $f(0,0) = 0$. We say that the system is *asymptotically controllable* if the following conditions hold:

1. $\forall x \in \mathbb{R}^n$, $\exists u(\cdot)$, measurable, locally essentially bounded, $u : [0,\infty) \longrightarrow U$ such that the solution corresponding to the control u, $\phi(t,x,u)$, is defined for all t and $\phi(t,x,u) \longrightarrow 0$ as $t \to \infty$.

2. $\forall \varepsilon > 0$, $\exists \delta > 0$, such that, if $\|x\| < \delta$ then $\exists u(\cdot)$ as in (1) such that $x(t) \longrightarrow 0$ as $t \to \infty$ and $\|x(t)\| \le \varepsilon$, for all $t \ge 0$.

The above mentioned control Lyapunov function was given by

$$V(x) = \inf_{u(\cdot)} \int_0^\infty N(\|\phi(t,x,u)\|)dt$$

where N is of class $\mathcal{K}$. The resulting function V is continuous, proper and positive definite, and satisfies a Dini condition (see [6] for more details). We now give a more geometric interpretation of such condition using the theory of subdifferentials.

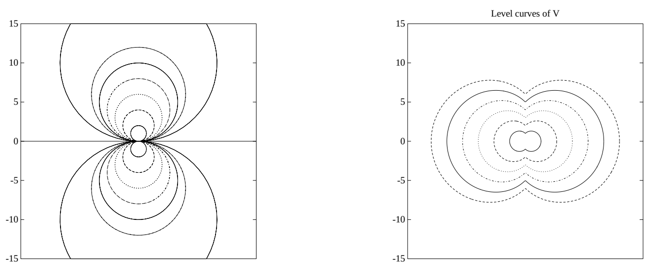

Figure 1: Orbits of g and level curves of V

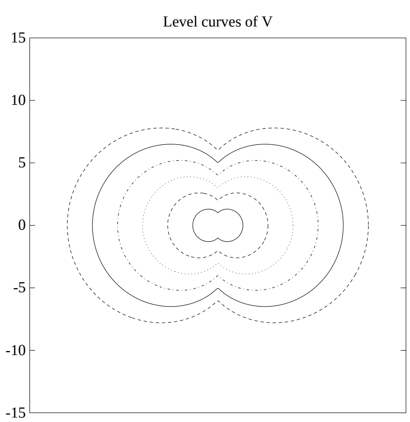

Figure 2: Piecewise smooth Lyapunov function

Definition 5.6 For a continuous function $V : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, Ω open, we define the subdifferential $D^- V(x)$ as the set

$$D^- V(x) = \{p : \liminf_{y \to x} \frac{V(y) - V(x) - p(y-x)}{|y-x|} \geq 0\}$$

Using a standard calculation, one can show that the function V is a viscosity supersolution of the Hamilton-Jacobi-Bellman equation. That is,

$$\forall x, \ \ \forall p \in D^- V(x) \quad \sup_{u \in U} \{-pf(x,u) - N(\|x\|)\} \geq 0$$

Therefore, $\inf_{u \in U}\{pf(x,u)\} \leq -N(\|x\|)$. Since for $x \neq 0$ $N(\|x\|)/2 > 0$, there exists u such that

$$pf(x,u) < \frac{N(\|x\|)}{2} < 0$$

Observe that this is exactly the condition that a standard Lyapunov function must satisfy with p replacing the gradient.

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