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Stabilization with Respect to Noncompact Sets: Lyapunov Characterizations and Effect of Bounded Inputs*

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Abstract.
This paper deals with the global smooth stabilization of nonlinear systems with respect to not necessarily compact sets. We prove a converse Lyapunov theorem, and present a result on stability under input perturbations.
Key words: Converse Lyapunov theorems, stabilization to sets, BIBO stability

1. Introduction

This paper deals with issues related to the stabilization of nonlinear systems with respect to not necessarily compact sets. We are motivated by potential applications to a wide variety of areas.

As an illustration, consider problems of output feedback. In [3], the author poses a definition of "detectability" which involves the existence of an observer for which the error satisfies Lyapunov estimates which depend only on the difference $\|x(t) - z(t)\|$, where $x(t)$ is the state of the plant and $z(t)$ is the state of the observer. For the joint plant/observer system, detectability becomes stabilization with respect to the set $\mathcal{A} := \{(x, z) | x = z\}$.

In many applications, one is interested in stabilization of an output variable (as opposed to the complete state). Consider the following two-dimensional system:

$$\dot{x} = x\,, \tag{1}$$
$$\dot{y} = -y + ux\,, \tag{2}$$

with the variable y taken as the output. Observe that when $u \equiv 0$ the y variable converges exponentially to zero, uniformly on the initial state $(x(0), y(0))$. However, for nonzero u, no matter how small, the output diverges if $x(0) \neq 0$. Indeed, if $u \equiv \varepsilon$ and $x(0) = x_0$, one has $y(t) = y_0 e^{-t} + \varepsilon x_0 \sinh t \to \infty$. This is in marked contrast to the case of state-space stability, where at least for small controls and small initial states, bounded states result, if the system was asymptotically stable for $u \equiv 0$. Later in the paper we show how to construct a feedback stabilizer so that the closed-loop system obtained for this example after applying that feedback law does have suitable stability properties even for nonzero u.

Another example arises in adaptive control, where one does not usually obtain convergence of parameters but only of states; that is, one has to study in effect stability of the adaptive control system with respect to the set $\mathcal{A} := \{(0, \lambda), \lambda \in \Lambda\}$ where λ is the vector of unknown parameters and "0" stands for the zero state. As yet another motivation, many problems involving tracking and regulation can be expressed as partial stabilization problems (of an error signal). Finally, systems in which derivatives of controls appear can be reduced, adding integrators, to systems in which such derivatives do not appear, but at the cost of extra state variables which are not to be controlled.

We will not deal with the above applications in this paper, but will instead concentrate on some basic questions related to set stability. We give a converse Lyapunov theorem that does not assume compactness of the attracting set. This result was motivated, and follows to some extent the outline, of the converse Lyapunov result in [4], but with some major differences. First of all, we want a global rather than a local result, and several technical issues appear in that case. Second, we have not been able to follow many details of the proof in [4], especially those (critical ones) concerning Lipschitz properties, or those dealing with global smoothness. Thus we give a detailed self-contained proof. (We do use the material from [4] dealing with smooth approximations of functions on manifolds.) After establishing the converse theorem, we give a result on input to state stabilization that generalizes that available in the case of stablization to equilibrium points.

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2. Set Stability

We first review some standard concepts from stability theory. A function $\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a *$\mathcal{K}$-function* if it is continuous, strictly increasing and satisfies $\gamma(0) = 0$; it is a *$\mathcal{K}_\infty$-function* if in addition $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta(s, t) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a *$\mathcal{KL}$-function* if for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geq 0$ it is decreasing to zero as $t \to \infty$. A function $\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is *positive definite* if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$.

We will use the following notation throughout the paper: for each nonempty subset A of a metric space $\mathcal{X}$, and each $\xi \in \mathcal{X}$, denote $|\xi|_A := d(\xi, A) = \inf_{\eta \in A} d(\xi, \eta)$, the common point-to-set distance. If $\mathcal{X} = \mathbb{R}^n$, we let $|\xi|$ be the usual Enclidean norm, that is, $|\xi| = |\xi|_{\{0\}}$.

Consider the following system:

$$\dot{x} = f(x)\,, \quad x \in \mathbb{R}^n\,, \tag{3}$$

where f is assumed to be smooth (i.e., infinitely differentiable). We will assume that the system is complete, and denote by $x(t, x_0)$ (and sometimes simply by $x(t)$ if there is no ambiguity from the context) the solution at time t of (3) with $x(0) = x_0$. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a closed, invariant set of (3). We emphasize that we do not require $\mathcal{A}$ to be compact. We will also assume throughout the paper that

$$\sup_{\xi \in \mathbb{R}^n} \{|\xi|_{\mathcal{A}}\} = \infty\,. \tag{4}$$

Definition 2.1 We say that (3) is *uniformly globally asymptotically stable (UGAS) with respect to the set $\mathcal{A}$* if the following two properties hold:

1. *Uniform Stability.* There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$,
$$|x(t, x_0)|_{\mathcal{A}} < \varepsilon \text{ whenever } |x_0|_{\mathcal{A}} < \delta(\varepsilon) \text{ and } t \geq 0\,. \tag{5}$$
2. *Uniform Attraction.* For any $r, \varepsilon > 0$, there is a $T > 0$, such that
$$|x(t, x_0)|_{\mathcal{A}} < \varepsilon \tag{6}$$
whenever $|x_0|_{\mathcal{A}} < r$ and $t \geq T$. □

Remark 2.2 This definition differs from that in [4] where *uniform attraction* means there exist a fixed $r_0 > 0$ and $T : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ continuous, $(T(\mathbb{R}_{>0}) = \mathbb{R}_{>0})$, such that for any $\varepsilon > 0$,

$$|x(t, x_0)|_{\mathcal{A}} < \varepsilon \text{ whenever } |x_0|_{\mathcal{A}} < r_0 \text{ and } t \geq T(\varepsilon)\,. \quad □$$

Lemma 2.3 The uniform attraction property defined in Definition 2.1 is equivalent to the following: There exists a family of mappings $\{T_r\}_{r>0}$ with

- for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and strictly decreasing;
- for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is increasing as r increases and $\lim_{r\to\infty} T_r(\varepsilon) = \infty$;

such that

$$|x(t,\, x_0)|_{\mathcal{A}} < \varepsilon \;\text{ whenever }\; |x_0|_{\mathcal{A}} < r \text{ and } t \geq T_r(\varepsilon)\,.\square$$

We omit the proof here for lack of space.

When $\mathcal{A}$ consists just of an equilibrium point, the above reduces to the usual notion of global asymptotic stability. The following result is well-known in that special case:

Proposition 2.4 The system (3) is UGAS with respect to a closed, invariant set $\mathcal{A} \subseteq \mathbb{R}^n$ if and only if there exists a $\mathcal{KL}$-function $\beta(s,\, t) : \mathbb{R}_{[0,\,\infty)} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$, such that, given any initial state x_0, the solution $x(t,\, x_0)$ satisfies

$$|x(t,\, x_0)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}},\, t)\,, \quad \text{any } t \geq 0\,. \tag{7}$$

Proof. [$\Longleftarrow$] Assume that there exists a $\mathcal{KL}$-function β such that $|x(t,\, x_0)|_{\mathcal{A}} \leq \beta\big(|x_0|_{\mathcal{A}},\, t\big)$, any $x_0 \in \mathbb{R}^n$, any $t \geq 0$. Let $c_1 := \sup \beta(\cdot,\, 0) \leq \infty$, and choose $\delta(\cdot)$ to be any $\mathcal{K}_\infty$-function with $\delta(\varepsilon) \leq \bar{\beta}^{-1}(\varepsilon)$, any $0 \leq \varepsilon < c_1$, where $\bar{\beta}^{-1}$ denotes the inverse function of $\bar{\beta}(\cdot) := \beta(\cdot,\, 0)$. (If $c_1 = \infty$, we can simply choose $\delta(\varepsilon) := \bar{\beta}^{-1}(\varepsilon)$.) Now for any $\varepsilon > 0$, and any $x_0 \in \mathbb{R}^n$ satisfying $|x_0|_{\mathcal{A}} < \delta(\varepsilon)$, the ensuing trajectory $x(\cdot,\, x_0)$ satisfies $|x(t,\, x_0)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}},\, t) \leq \beta(|x_0|_{\mathcal{A}},\, 0) < \beta(\delta(\varepsilon),\, 0) \leq \varepsilon$, if $\varepsilon < c_1$ or $|x(t,\, x_0)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}},\, t) \leq \beta(|x_0|_{\mathcal{A}},\, 0) < c_1 \leq \varepsilon$, if $\varepsilon \geq c_1$. This establishes the stability property.

Now for each $r \in (0,\, \infty)$, let $\tilde{\beta}_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ be any continuous, *strictly* decreasing function satisfying (1) $\tilde{\beta}_r(t) \geq \beta(r,\, t)$ and (2) for any fixed t, $\tilde{\beta}_r(t)$ increases as r increases, and $\lim_{r\to\infty} \tilde{\beta}_r(t) = \infty$. (For example, one of such a function can be defined as $\tilde{\beta}_r(t) := \dfrac{r}{t} + \beta(r,\, t)$.) Define $T_r(\varepsilon) := \tilde{\beta}_r^{-1}(\varepsilon)$. Then for any $r \in (0,\, \infty)$, $T_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and strictly decreasing, and for any fixed t, $T_{(\cdot)}(t)$ increases to ∞. It follows that for any given $\varepsilon > 0$, any $r \in (0,\, \infty)$ and any $x_0 \in \mathbb{R}^n$, if $|x_0|_{\mathcal{A}} < r$ and $t \geq T_r(\varepsilon)$, we have $|x(t,\, x_0)|_{\mathcal{A}} \leq \beta\big(|x_0|_{\mathcal{A}},\, t\big) < \beta(r,\, t) \leq \tilde{\beta}_r(t) \leq \tilde{\beta}_r(T_r(\varepsilon)) = \varepsilon$.

[$\Longrightarrow$] Assume that (3) is UGAS with respect to the closed set $\mathcal{A}$, and let δ be as in the definition. Let $\varphi(\cdot)$ be the $\mathcal{K}$-function $\delta^{-1}(\cdot)$.

Claim: For any $x_0 \in \mathbb{R}^n$ and any $t \geq 0$, $|x(t,\, x_0)|_{\mathcal{A}} \leq \varphi(|x_0|_{\mathcal{A}})$.

Proof: Otherwise, there exist $t_0 \geq 0$ and $\bar{x}_0$ such that

$$|x(t_0,\, \bar{x}_0)|_{\mathcal{A}} > \varphi(|\bar{x}_0|_{\mathcal{A}})\,. \tag{8}$$

Pick any positive number $\tau < |x(t_0,\, \bar{x}_0)|_{\mathcal{A}} - \varphi(|\bar{x}_0|_{\mathcal{A}})$ and choose $\varepsilon := \delta^{-1}(|\bar{x}_0|_{\mathcal{A}}) + \tau$. Then $|\bar{x}_0|_{\mathcal{A}} = \delta\big(\delta^{-1}(|\bar{x}_0|_{\mathcal{A}})\big) < \delta\big(\delta^{-1}(|\bar{x}_0|_{\mathcal{A}}) + \tau\big) = \delta(\varepsilon)$, so it follows from (5), applied with $t = t_0$, that

$$|x(t_0,\, \bar{x}_0)|_{\mathcal{A}} < \varepsilon = \varphi(|\bar{x}_0|_{\mathcal{A}}) + \tau < |x(t_0,\, \bar{x}_0)|_{\mathcal{A}}\,, \tag{9}$$

a contradiction.

Let $\{T_r\}_{r\in(0,\,\infty)}$ be as in Lemma 2.3, and for each $r \in (0,\, \infty)$ denote $\psi_r := T_r^{-1}$. Then, for each $r \in (0,\, \infty)$, $\psi_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and strictly decreasing. We also write $\psi_r(0) = +\infty$, which is consistent with that fact that $\lim_{t\to 0^+} \psi_r(t) = +\infty$. (Note: The property that $T_{(\cdot)}(t)$ increases to ∞ is not needed here.)

Claim: For any $|x_0|_{\mathcal{A}} < r$ and any $t \geq 0$, $|x(t,\, x_0)|_{\mathcal{A}} \leq \psi_r(t)$.

Proof: Otherwise, there exist $t_0 \geq 0$, $r_0 \in (0,\, \infty)$, $\bar{x}_0 \in \mathbb{R}^n$, $|\bar{x}_0|_{\mathcal{A}} < r_0$, such that $|x(t_0,\, \bar{x}_0)|_{\mathcal{A}} > \psi_{r_0}(t_0)$. Pick any positive number $\tau_1 < |x(t_0,\, \bar{x}_0)|_{\mathcal{A}} - \psi_{r_0}(t_0)$, and let $\varepsilon_0 := T_{r_0}^{-1}(t_0) + \tau_1 = \psi_{r_0}(t_0) + \tau_1$. Then since $t_0 = T_{r_0}\big(T_{r_0}^{-1}(t_0)\big) > T_{r_0}\big(T_{r_0}^{-1}(t_0) + \tau_1\big) = T_{r_0}(\varepsilon_0)$, we have the contradiction $|x(t_0,\, \bar{x}_0)|_{\mathcal{A}} < \varepsilon_0 = \psi_{r_0}(t_0) + \tau_1 < |x(t_0,\, \bar{x}_0)|_{\mathcal{A}}$.

Now for any $s \geq 0$ and $t \geq 0$, let

$$\bar{\psi}(s,\, t) := \min\left\{\inf{}_{r\in(s,\,\infty)} \psi_r(t),\, \varphi(s)\right\}\,. \tag{10}$$

Because of the last two claims, we have

$$|x(t,\, x_0)|_{\mathcal{A}} \leq \bar{\psi}(|x_0|_{\mathcal{A}},\, t)\,. \tag{11}$$

By its definition, for any fixed t, $\bar{\psi}(\cdot,\, t)$ is an increasing function (not necessarily strictly). Also because for any fixed $r \in (0,\, \infty)$, $\psi_r(t)$ decreases to 0 (this follows from the fact that $\psi_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and strictly decreasing), it follows that for any fixed s, $\bar{\psi}(s,\, t)$ decreases to 0 as $t \to \infty$.

Pick any function $\tilde{\psi} : \mathbb{R}_{[0,\,\infty)} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ with the following properties:

- for any fixed $t \geq 0$, $\tilde{\psi}(\cdot,\, t)$ is continuous and strictly increasing;
- for any fixed $s \geq 0$, $\tilde{\psi}(s,\, t)$ decreases to 0 as $t \to \infty$;
- $\tilde{\psi}(s,\, t) \geq \bar{\psi}(s,\, t)$.

Such a function $\tilde{\psi}$ always exists; for instance, it can be constructed as follows. Define first

$$\hat{\psi}(s,\, t) := \int_s^{s+1} \bar{\psi}(\varsigma,\, t)\, d\varsigma\,. \tag{12}$$

Then $\hat{\psi}(\cdot,\, t)$ is an absolutely continuous function on every compact subset of $\mathbb{R}_{\geq 0}$, and it satisfies $\hat{\psi}(s,\, t) \geq \bar{\psi}(s,\, t) \int_s^{s+1} d\varsigma = \bar{\psi}(s,\, t)$. It follows that $\frac{\partial \hat{\psi}(s,\, t)}{\partial s} = \bar{\psi}(s + 1,\, t) - \bar{\psi}(s,\, t) \geq 0$, a.e., and hence $\hat{\psi}(\cdot,\, t)$ is increasing. Also since for any fixed s, $\bar{\psi}(s,\, \cdot)$ decreases, so does $\hat{\psi}(s,\, \cdot)$. Note that

$$\begin{aligned} \bar{\psi}(s,\, t) \leq \bar{\psi}(s,\, 0) \;&=\; \min\left\{\inf{}_{r\in(s,\,\infty)} \psi_r(0),\, \varphi(s)\right\} \\ &=\; \varphi(s)\,, \end{aligned}$$

(recall that $\psi_r(0) = +\infty$), so by the Lebesgue dominated convergence theorem, for any fixed $s \geq 0$, $\lim_{t\to\infty} \hat{\psi}(s,\, t) = \int_s^{s+1} \lim_{t\to\infty} \bar{\psi}(\varsigma,\, t)\, d\varsigma = 0$. Now we see that the function $\hat{\psi}(s,\, t)$ satisfies all of the requirements for $\tilde{\psi}(s,\, t)$ except possibly for the strictly increasing property. We define $\tilde{\psi}$ as follows: $\tilde{\psi}(s,\, t) := \hat{\psi}(s,\, t) + \frac{s}{(s+1)(t+1)}$. Clearly it satisfies all the desired properties.

Finally, define $\beta(s,\, t) := \sqrt{\varphi(s)}\,\sqrt{\tilde{\psi}(s,\, t)}$. Then it follows that $\beta(s,\, t)$ is a $\mathcal{KL}$-function, and $|x(t,\, x_0)|_{\mathcal{A}} \leq \sqrt{\varphi(|x_0|_{\mathcal{A}})}\,\sqrt{\bar{\psi}(|x_0|_{\mathcal{A}},\, t)} \leq \beta(|x_0|_{\mathcal{A}},\, t)$, which concludes the proof of the Proposition. ∎

2.1. Lyapunov Functions

For smooth functions $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ and vector fields f, we use the standard Lie derivative notation $L_f V(\xi) := \dfrac{\partial V}{\partial x}(\xi) \cdot f(\xi)$.

Definition 2.5 A *Lyapunov function* for the system (3) with respect to a nonempty, closed, invariant set $\mathcal{A} \subseteq \mathbb{R}^n$ is a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that V is smooth on $\mathbb{R}^n \backslash \mathcal{A}$ and satisfies

1. there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that for any $\xi \in \mathbb{R}^n$,
$$\alpha_1(|\xi|_\mathcal{A}) \le V(\xi) \le \alpha_2(|\xi|_\mathcal{A}); \tag{13}$$
2. there exists a continuous, positive definite function α_3 such that for any $\xi \in \mathbb{R}^n \backslash \mathcal{A}$,
$$L_f V(\xi) \le -\alpha_3(|\xi|_\mathcal{A}). \tag{14}$$

A *smooth* Lyapunov function is one which is smooth on all of $\mathbb{R}^n$. □

Remark 2.6 One may assume, in the above definition, that all of $\alpha_1, \alpha_2, \alpha_3$ are smooth and of class $\mathcal{K}_\infty$. For α_1 and α_2, this is proved simply by finding two smooth functions $\tilde{\alpha}_1, \tilde{\alpha}_2$ in $\mathcal{K}_\infty$ so that $\tilde{\alpha}_1 \le \alpha_1 \le \alpha_2 \le \tilde{\alpha}_2$ for all s. For α_3, a new Lyapunov function W and a function $\tilde{\alpha}_3$ which satisfies (14) with respect to W, but is smooth and in $\mathcal{K}_\infty$, can be constructed as follows. First, pick $\tilde{\alpha}_3$ to be any smooth $\mathcal{K}_\infty$-function such that $\tilde{\alpha}_3(s) \le s\alpha_3(s)$ for all $s \in [0, \alpha_1^{-1}(1)]$. This is possible since α_3 is positive definite. Then let $\gamma : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}_{\ge 0}$ be a smooth $\mathcal{K}_\infty$-function such that

- $\gamma(r) \ge \alpha_1^{-1}(r)$ for all $r \in [0, 1]$;
- $\gamma(r) > \dfrac{\tilde{\alpha}_3(\alpha_1^{-1}(r))}{\alpha_3(\alpha_1^{-1}(r))}$ for all $r > 1$.

Now define $\beta(s) := \int_0^s \gamma(r)\, dr$. Note that β is a smooth $\mathcal{K}_\infty$-function. Let $W(\xi) := \beta(V(\xi))$. This is smooth on $\mathbb{R}^n \backslash \mathcal{A}$, and $\beta \circ \alpha_1, \beta \circ \alpha_2$ bound W as in equation (13). Moreover, $\beta'(V(\xi)) = \gamma(V(\xi)) \ge \gamma(\alpha_1(|\xi|_\mathcal{A}))$, so

$$L_f W(\xi) = \beta'(V(\xi)) L_f V(\xi) \le -\gamma(\alpha_1(|\xi|_\mathcal{A}))\alpha_3(|\xi|_\mathcal{A}). \tag{15}$$

We claim that this is bounded by $-\tilde{\alpha}_3(|\xi|_\mathcal{A})$. Indeed, if $s := |\xi|_\mathcal{A} \le \alpha_1^{-1}(1)$, then from the first item above and the definition of $\tilde{\alpha}_3$,

$$\gamma(\alpha_1(s)) \ge s \ge \frac{\tilde{\alpha}_3(s)}{\alpha_3(s)};$$

if instead $s > \alpha_1^{-1}(1)$, then from the second item, also

$$\gamma(\alpha_1(s)) \ge \frac{\tilde{\alpha}_3(s)}{\alpha_3(s)}.$$

In either case, $\gamma(\alpha_1(s))\alpha_3(s) \ge \tilde{\alpha}_3(s)$, as desired. From now on, whenever necessary, we assume that α_1, α_2, α_3 are smooth $\mathcal{K}_\infty$-functions. □

Remark 2.7 The first condition in Definition 2.5 implies that V is continuous on all of $\mathbb{R}^n$, $V(x) = 0 \iff x \in \mathcal{A}$, and $V : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}_{\ge 0}$ (recall the assumption in equation (4)). □

For reasons of space, we omit the proof of the following result; it relies on constructing a smooth function of the form $W = \beta \circ V$, where $\beta : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}_{\ge 0}$ is built using a partition of unity.

Proposition 2.8 If there is a Lyapunov function for (3) with respect to $\mathcal{A}$, then there is also a smooth such Lyapunov function. □

The following lemma from [1] will be needed below.

Lemma 2.9 For each $\mathcal{K}$-function α of class C^1, there exists a $\mathcal{KL}$-function $\beta_\alpha(s, t)$ with the following property: if $y(\cdot)$ is any (locally) absolutely continuous function defined for $t \ge 0$ and with $y(t) \ge 0$ for all t, and satisfies the differential inequality

$$\dot{y}(t) \le -\alpha(y(t)), \ \ y(0) = y_0 \ge 0, \tag{16}$$

then it holds that $y(t) \le \beta_\alpha(y_0, t)$ for all $t \ge 0$. □

Remark 2.10 We only state the existence part of the Lemma here. Interested readers can consult the original paper [1] to see the concrete construction of the $\mathcal{KL}$-function β_α from the $\mathcal{K}$-function α. □

Since we are making a smoothness assumption on the vector field $f(x)$ in the system (3) (hence in particular, $f(x)$ is locally Lipschitz), and the system is assumed to be complete, by Gronwall's Lemma (see for instance, [2] for further details), $x(t, x_0)$ is locally Lipschitz. For later reference, we summarize this as follows:

Lemma 2.11 For any $T > 0$ and any compact $K \subseteq \mathbb{R}^n$, there is a constant $C > 0$ which only depends on the set K and the Lipschitz constant of $f(x)$ on K, such that for the trajectories $x(\cdot, \cdot)$ of the system (3), $|x(t, \xi) - x(t, \eta)| \le Ce^T|\xi - \eta|$ for any $\xi, \eta \in K$ and any $|t| \le T$. □

We are now ready to state a converse Lyapunov theorem in a form much useful later. It can be seen as a global version of the result given in [4]. The idea of the proof that we give is based on that in [4], but it differs substantially at various points. Again we make the assumption that $\mathcal{A} \subseteq \mathbb{R}^n$ is a nonempty, closed invariant set of (3).

Theorem 1 *The system (3) is UGAS with respect to $\mathcal{A}$ if and only if there exists a Lyapunov function V with respect to the set $\mathcal{A}$.*

Proof. [$\Longleftarrow$] Pick any trajectory $x(\cdot)$; then, as long as $x(t) \notin \mathcal{A}$, $\frac{dV(x(t))}{dt} \le -\alpha_3(|x(t)|_\mathcal{A}) \le -\alpha(V(x(t)))$, where α is the $\mathcal{K}_\infty$-function defined by $\alpha(\cdot) := \alpha_3(\alpha_2^{-1}(\cdot))$. Now let β_α be the $\mathcal{KL}$-function as in Lemma 2.9 with respect to α, and define

$$\beta(s, t) := \alpha_1^{-1}\big(\beta_\alpha(\alpha_2(s), t)\big). \tag{17}$$

Then β is a $\mathcal{KL}$-function, since both α_1 and α_2 are $\mathcal{K}_\infty$-functions. By Lemma 2.9,

$$V(x(t)) \le \beta_\alpha\big(V(x_0), t\big), \ \ \text{any } t \ge 0.$$

Hence $|x(t)|_\mathcal{A} \le \beta(|x_0|_\mathcal{A}, t)$. If $x(t) \in \mathcal{A}$ for all $t \ge T$, then $|x(t)|_\mathcal{A} \equiv 0$, and the same estimate holds. Therefore the system (3) is UGAS with respect to $\mathcal{A}$, by Proposition 2.4.

[$\Longrightarrow$] Assume now that the system is UGAS with respect to the set $\mathcal{A}$. Let δ and T_r be as in Definition 2.1 and Lemma 2.3.

Define $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(\xi) := \inf{}_{t \le 0}\{|x(t, \xi)|_\mathcal{A}\}. \tag{18}$$

By uniqueness of solutions, $g(x(t, \xi)) \le g(\xi)$, for each $t \ge 0$. Also

$$\delta(|\xi|_\mathcal{A}) \le g(\xi) \le |\xi|_\mathcal{A}. \tag{19}$$

The second half of (19) is obvious from $x(0, \xi) = \xi$. On the other hand, if the first half were not true, i.e., if there is a $t_0 \le 0$ such that $\delta(|\xi|_\mathcal{A}) > |x(t_0, \xi)|_\mathcal{A}$, then because of the definition of δ, applied with $\varepsilon = |\xi|_\mathcal{A}$, $t = -t_0$, and $x_0 = x(t_0, \xi)$, $|\xi|_\mathcal{A} = |x(-t_0, x(t_0, \xi))|_\mathcal{A} < |\xi|_\mathcal{A}$, which is a contradiction.

For any $0 < \varepsilon < r$, define $K_{\varepsilon,r} := \{\xi \in \mathbb{R}^n \mid \varepsilon \le |\xi|_{\mathcal{A}} < r\}$.

Fact 1: For all ε and r with $0 < \varepsilon < r$, there exists $q_{\varepsilon,r} \le 0$, such that: $\xi \in K_{\varepsilon,r}$ and $t < q_{\varepsilon,r} \implies |x(t,\xi)|_{\mathcal{A}} \ge r$.

Proof: If the statement is not true, then there would exist ε, r with $0 < \varepsilon < r$ and $\xi \in K_{\varepsilon,r}$, and a sequence $\{t_k\}$ with $\lim_{t\to\infty} t_k = -\infty$ such that $|x(t_k,\xi)|_{\mathcal{A}} < r$. Pick k large enough so that $-t_k > T_r(\varepsilon)$, then by the uniform attraction property, $|\xi|_{\mathcal{A}} = |x(-t_k, x(t_k,\xi))|_{\mathcal{A}} < \varepsilon$, which is a contradiction. This proves the fact.

Therefore, for any $\xi \in K_{\varepsilon,r}$, $g(\xi) = \min_{t\in[q_{\varepsilon,r},0]} |x(t,\xi)|_{\mathcal{A}}$. (Since for any $t < q_{\varepsilon,r}$, from Fact 1 and (19), $|x(t,\xi)|_{\mathcal{A}} \ge r > |\xi|_{\mathcal{A}} \ge g(\xi)$.)

Claim 1: $g(\xi)$ is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$.

Proof: For any $\xi \in \mathbb{R}^n\backslash\mathcal{A}$, let $\bar{B}(\xi, \frac{1}{2}|\xi|_{\mathcal{A}})$ denote the closed ball centered at ξ and with radius $\frac{1}{2}|\xi|_{\mathcal{A}}$. Then $\bar{B}(\xi, \frac{1}{2}|\xi|_{\mathcal{A}}) \subseteq K_{\varepsilon,r}$ for some $0 < \varepsilon < r$ (for instance, ε and r can be chosen as $\frac{1}{4}|\xi|_{\mathcal{A}}$ and $2|\xi|_{\mathcal{A}}$, respectively). Pick a constant C as in Lemma 2.11 with respect to this closed ball. Then for any $\zeta, \eta \in \bar{B}(\xi, \frac{1}{2}|\xi|_{\mathcal{A}})$, there are $t_\zeta, t_\eta \in [q_{\varepsilon,r}, 0]$ such that

$$\begin{aligned} g(\zeta) - g(\eta) &= |x(t_\zeta,\zeta)|_{\mathcal{A}} - |x(t_\eta,\eta)|_{\mathcal{A}} \\ &\le |x(t_\eta,\zeta)|_{\mathcal{A}} - |x(t_\eta,\eta)|_{\mathcal{A}} \\ &\le |x(t_\eta,\zeta) - x(t_\eta,\eta)| \le C|\zeta - \eta|. \end{aligned}$$

Similarly, $g(\eta) - g(\zeta) \le C|\zeta - \eta|$. This establishes the claim.

Note that g is continuous at each $\xi \in \mathcal{A}$, since $g(\xi) = 0$ so $|g(\eta) - g(\xi)| = |g(\eta)| \le |\eta|_{\mathcal{A}} \le |\eta - \xi|$; thus g is globally continuous. (We are not claiming that g is locally Lipschitz on $\mathbb{R}^n$, though.)

Now define $U : \mathbb{R}^n \longrightarrow \mathbb{R}_{\ge 0}$ by

$$U(\xi) := \sup_{t\ge 0}\Big(g(x(t,\xi))\,k(t)\Big), \tag{20}$$

where $k : R_{\ge 0} \longrightarrow \mathbb{R}_{>0}$ is any strictly increasing, smooth function that satisfies

- there are some constants $0 < c_1 < c_2 < \infty$, such that $k(t) \in [c_1, c_2]$, for any $t \ge 0$;
- there is a bounded positive decreasing continuous function $\tau(\cdot)$, such that $k'(t) \ge \tau(t)$ for all $t \ge 0$.

(For instance, $\dfrac{c_1 + c_2 t}{1+t}$ is one of such a function.) Observe that

$$U(\xi) \le \sup_{t\ge 0}\Big(g(\xi)\,k(t)\Big) \le c_2 g(\xi) \le c_2|\xi|_{\mathcal{A}}, \tag{21}$$

and

$$U(\xi) \ge g(x(t,\xi))\,k(t)\Big|_{t=0} \ge c_1 g(\xi) \ge c_1\delta(|\xi|_{\mathcal{A}}). \tag{22}$$

For any $\xi \in \mathbb{R}^n$, since $0 \le g(x(t,\xi)) \le |x(t,\xi)|_{\mathcal{A}}$ and $|x(t,\xi)|_{\mathcal{A}} \overset{t\longrightarrow+\infty}{\longrightarrow} 0$, it follows that $\lim_{t\longrightarrow+\infty} g(x(t,\xi)) = 0$, and hence (using continuity of g) the supremum in the definition of U in (20) is a maximum, i.e., there exists some $t_\xi^U \in [0,\infty)$ such that $U(\xi) = g(x(t_\xi^U,\xi))\,k(t_\xi^U)$. In fact, we can get the following explicit bound:

Fact 2: For any $|\xi|_{\mathcal{A}} < r$, $t_\xi^U < T_r\left(\frac{c_1}{c_2}\delta(|\xi|_{\mathcal{A}})\right)$.

Proof: If the statement is not true, i.e., if $t_\xi^U \ge T_r\left(\frac{c_1}{c_2}\delta(|\xi|_{\mathcal{A}})\right)$, then by the uniformly attractive property, $|x(t_\xi^U,\xi)|_{\mathcal{A}} < \frac{c_1}{c_2}\delta(|\xi|_{\mathcal{A}})$. So we have

$$\begin{aligned} \delta(|\xi|_{\mathcal{A}}) \le \frac{1}{c_1}U(\xi) &= \frac{1}{c_1}g(x(t_\xi^U,\xi))\,k(t_\xi^U) \\ &\le \frac{c_2}{c_1}g(x(t_\xi^U,\xi)) \\ &\le \frac{c_2}{c_1}|x(t_\xi^U,\xi)|_{\mathcal{A}} \\ &< \delta(|\xi|_{\mathcal{A}}), \end{aligned}$$

which is a contradiction.

For any compact set $K \subseteq \mathbb{R}^n$, let $t_K^U := \max_{\xi\in K} t_\xi^U < \infty$. (Finiteness follows from Fact 2, as $K \subseteq \{\xi \mid |\xi|_{\mathcal{A}} < r\}$ for some $r > 0$.)

Claim 2: The function $U(\cdot)$ defined by (20) is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$.

Proof: For any compact set $K \subseteq \mathbb{R}^n\backslash\mathcal{A}$, and any $\xi,\eta \in K$, it follows from the definition of $U(\cdot)$ and Lemma 2.11 that

$$\begin{aligned} U(\xi) - U(\eta) &= g(x(t_\xi^U,\xi))\,k(t_\xi^U) - U(\eta) \\ &\le k(t_\xi^U)\Big(g(x(t_\xi^U,\xi)) - g(x(t_\xi^U,\eta))\Big) \\ &\le c_2\,|g(x(t_\xi^U,\xi)) - g(x(t_\xi^U,\eta))| \\ &\le C|\xi - \eta|, \end{aligned}$$

for some constant C that depends on K, where the last inequality follows because g is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$. By symmetry, also $U(\eta) - U(\xi) \le C|\xi - \eta|$. Hence, U is locally Lipschitz, as desired. This finishes the proof of Claim 2.

If $\eta \in \mathcal{A}$, then $U(\eta) = 0$, and hence for any ξ, $|U(\xi) - U(\eta)| = U(\xi) \le c_2\,|\xi|_{\mathcal{A}} \le c_2\,|\xi - \eta|$, so U is continuous at points of $\mathcal{A}$. Thus U is continuous everywhere.

Pick any $\xi \in \mathbb{R}^n\backslash\mathcal{A}$, and any $h > 0$. Let $\eta := x(h,\xi)$. Then there is some $t := t_\eta^U \ge 0$ such that

$$\begin{aligned} U(\eta) &= g(x(t,\eta))\,k(t) \\ &= g(x(t+h,\xi))\,k(t+h)\left(1 - \frac{k(t+h) - k(t)}{k(t+h)}\right) \\ &\le U(\xi)\left(1 - \frac{k(t+h) - k(t)}{k(t+h)}\right) \\ &\le U(\xi)\left(1 - \frac{k(t+h) - k(t)}{c_2}\right), \end{aligned} \tag{23}$$

if h is small enough. Still for this ξ and h, for any $r > |\xi|_{\mathcal{A}}$, define

$$T_{\xi,h}^r := \max_{0\le\bar{t}\le h} T_r\left(\frac{c_1}{c_2}\delta(|x(\bar{t},\xi)|_{\mathcal{A}})\right). \tag{24}$$

Fact 3: $t + h < T_{\xi,h}^r$.

Proof: If this were not true, then $t + h \ge T_{\xi,h}^r$, and hence in particular picking $\bar{t} = h$ in (24), we would have that $t + h \ge T_r(\frac{c_1}{c_2}\delta(|\eta|_{\mathcal{A}}))$, so by definition of T_r it is the case that $|x(t,\eta)|_{\mathcal{A}} = |x(t+h,\xi)|_{\mathcal{A}} < \frac{c_1}{c_2}\delta(|\eta|_{\mathcal{A}})$. Then, using (22), and recalling that $t = t_\eta^U$,

$$\begin{aligned} \delta(|\eta|_{\mathcal{A}}) \le \frac{1}{c_1}U(\eta) &= \frac{1}{c_1}g(x(t,\eta))\,k(t) \\ &\le \frac{c_2}{c_1}|x(t,\eta)|_{\mathcal{A}} < \delta(|\eta|_{\mathcal{A}}), \end{aligned}$$

which is a contradiction. This proves Fact 3. From (23), we have

$$\begin{aligned} \frac{U(x(h,\xi)) - U(\xi)}{h} &\le -\frac{U(\xi)\,(k(t+h) - k(t))}{c_2\,h} \\ &= -\frac{U(\xi)}{c_2}\,k'(t + \theta h), \end{aligned}$$

where the last inequality follows from the mean value theorem for some $0 \le \theta \le 1$. Hence, by the assumptions made on the function k, we have $\frac{U(x(h,\xi)) - U(\xi)}{h} \le$

$-\frac{U(\xi)}{c_2}\tau(t+\theta h) \le -\frac{U(\xi)}{c_2}\tau(T^r_{\xi,h})$. Since U is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$, it is differentiable almost everywhere in $\mathbb{R}^n\backslash\mathcal{A}$, and hence for any $r > |\xi|_{\mathcal{A}}$,

$$\begin{aligned} L_f U(\xi) &= \lim_{h\longrightarrow 0+} \frac{U(x(h,\xi)) - U(\xi)}{h} \\ &\le -\lim_{h\longrightarrow 0+} \frac{U(\xi)}{c_2}\tau(T^r_{\xi,h}) \\ &= -\frac{U(\xi)}{c_2}\tau\left(\lim_{h\longrightarrow 0+} T^r_{\xi,h}\right) \\ &= -\frac{U(\xi)}{c_2}\tau\left(T_r(\frac{c_1}{c_2}\delta(|\xi|_{\mathcal{A}}))\right) \\ &\le -\frac{c_1\delta(|\xi|_{\mathcal{A}})}{c_2}\tau\left(T_r(\frac{c_1}{c_2}\delta(|\xi|_{\mathcal{A}}))\right) \\ &= -\bar{\alpha}_r(|\xi|_{\mathcal{A}}), \quad \text{a.e.,} \end{aligned} \tag{25}$$

where

$$\bar{\alpha}_r(s) := \frac{c_1\delta(s)}{c_2}\tau\left(T_r(\frac{c_1}{c_2}\delta(s))\right). \tag{26}$$

Now define the function $\bar{\alpha}$ by $\bar{\alpha}(s) := \sup_{r>s}\bar{\alpha}_r(s)$. Note that $\bar{\alpha}_r(0) = 0$ for any $r > 0$, so $\bar{\alpha}(0) = 0$. Also, since $\tau(\cdot)$ decreases and $T_{(\cdot)}(s)$ increases, we have

$$\begin{aligned} \bar{\alpha}(s) &= \frac{c_1\delta(s)}{c_2}\sup_{r>s}\tau\left(T_r(\frac{c_1}{c_2}\delta(s))\right) \\ &= \frac{c_1\delta(s)}{c_2}\tau\left(\inf{}_{r>s}T_r(\frac{c_1}{c_2}\delta(s))\right) \\ &\ge \frac{c_1\delta(s)}{c_2}\tau\left(T_{2s}(\frac{c_1}{c_2}\delta(s))\right). \end{aligned}$$

In particular $\bar{\alpha}(s) > 0$ if $s > 0$. Hence $\bar{\alpha}$ is positive definite. Notice that (25) is true for any $r > |\xi|_{\mathcal{A}}$, so

$$L_f U(\xi) \le -\sup_{r>|\xi|_{\mathcal{A}}}(-\bar{\alpha}_r(|\xi|_{\mathcal{A}})) = -\bar{\alpha}(|\xi|_{\mathcal{A}}) \tag{27}$$

for almost all ξ. From Theorem 2.5 of [4], there exists a C^∞ function $V : R^n\backslash\mathcal{A} \longrightarrow \mathbb{R}_{\ge 0}$ such that, for almost all $\xi \in \mathbb{R}^n\backslash\mathcal{A}$,

$$|V(\xi) - U(\xi)| < \frac{1}{2}U(\xi) \quad\text{and}\quad L_f V(\xi) \le \frac{1}{2}L_f U(\xi). \tag{28}$$

Extend V to $\mathbb{R}^n$ by letting $V|_A = 0$ and again denote the extension by V. Note that V is continuous on $\mathbb{R}^n$, since $|V(\xi)| < \frac{3c_2}{2}|\xi|_{\mathcal{A}}$. So V is a Lyapunov function, as desired, with $\alpha_1(s) = \frac{c_1}{2}\delta(s)$, $\alpha_2(s) = \frac{3c_2}{2}s$ and $\alpha_3(s) = \frac{1}{2}\bar{\alpha}(s)$. ∎

3. Control Problem

For each $i = 1, \ldots, m$, let $\sigma_i : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth nondecreasing function satisfying $\sigma_i(0) = 0$ and a global Lipschitz condition: there is some constant c so that, for all $u, v \in \mathbb{R}$, $|\sigma_i(u) - \sigma_i(v)| \le c|u - v|$.

Consider the following smooth nonlinear system:

$$\dot{x} = f(x) + \sum_{i=1}^{m}\sigma_i(u_i)g_i(x) = f(x) + G(x)\tilde{\sigma}(u), \tag{29}$$

where $x(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}$ for all $t \ge 0$ and we denote $\tilde{\sigma}(u) := (\sigma_1(u_1), \ldots, \sigma_m(u_m))'$ for each $u = (u_1, \ldots, u_m)' \in \mathbb{R}^m$.

When all $\sigma_i(u) = u$, this is just a system affine in controls.

Assume that the system (3) is smoothly stabilizable with respect to a closed invariant set $\mathcal{A} \subseteq \mathbb{R}^n$ by means of a feedback law $u = k(x)$. The objective is to find a new control law of the feedback type:

$$u = k(x) + v, \tag{30}$$

possibly with different k, which has the property that the resulting closed-loop system

$$\dot{x} = f(x) + G(x)\tilde{\sigma}(k(x) + u) \tag{31}$$

is "bounded input bounded state" stable with respect to the set $\mathcal{A}$. This is a natural generalization of the case studied in [1], in which $\mathcal{A} = \{0\}$.

For any measurable function $u : [0, \infty) \longrightarrow \mathbb{R}^m$, we denote $\|u\| :=$ ess. sup. $\{|u(t)|, t \ge 0\}$. As earlier, $|\xi|_{\mathcal{A}}$ is the distance from the point ξ to the set $\mathcal{A}$.

For each $x_0 \in \mathbb{R}^n$ and each measurable locally essentially bounded u, we denote by $x(t, x_0, u)$ the trajectory of the system (29). This is defined on some maximal interval $[0, T_{x_0,u})$, with $T_{x_0,u} \le +\infty$.

Definition 3.12 The system (29) is *globally input-to-state stable (ISS) with respect to a closed set $\mathcal{A}$* if there exist a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}$, and a $\mathcal{K}$-function γ such that, for each measurable locally essentially bounded $u(\cdot)$ and each $x_0 \in \mathbb{R}^n$, the solution $x(t) = x(t, x_0, u)$ satisfies the estimate

$$|x(t)|_{\mathcal{A}} \le \beta(|x_0|_{\mathcal{A}}, t) + \gamma(\|u\|) \tag{32}$$

for each $t \in [0, T_{x_0,u})$. □

The above definition implies in particular that any bounded input results in a state trajectory that stays within a bounded distance — namely, $\beta(|x_0|_{\mathcal{A}}, 0) + \gamma(\|u\|)$ — from the set $\mathcal{A}$. We allow for the trajectory to stop being defined (case in which $T_{x_0,u} < \infty$), but it must remain near $\mathcal{A}$. (If $\mathcal{A}$ would be compact, this definition wuld imply $T_{x_0,u} = \infty$.)

Definition 3.13 The system (29) is *smoothly stabilizable with respect to a closed set $\mathcal{A}$* if there exists a smooth map $k : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that the system (31) with $u \equiv 0$ is UGAS with respect to the set $\mathcal{A}$. It is *smoothly input-to-state stabilizable with respect to a closed set $\mathcal{A}$* if there is such a k so that the system (31) becomes ISS with respect to $\mathcal{A}$. □

Prior to the stating the main result of the paper, we first state a lemma which will be needed in the proof of the main theorem.

Lemma 3.14 Consider the system (29). Assume that α is a C^1 $\mathcal{K}$-function and V is a C^1 function: $\mathbb{R}^n \longrightarrow \mathbb{R}_{\ge 0}$ such that for any initial value $x_0 \in \mathbb{R}^n$ and any measurable essentially bounded u, there exists a closed subset $S \subseteq \mathbb{R}^n$ such that

1. S is forward invariant;
2. for any $t_0 \ge 0$, if $x(t_0) \notin S$, then necessarily
$$\left.\frac{dV(x(t))}{dt}\right|_{t=t_0} \le -\alpha\big(V(x(t_0))\big);$$
3. V is positive definite and proper with respect to the set $\mathcal{A}$, i.e., there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that
$$\alpha_1(|\xi|_{\mathcal{A}}) \le V(\xi) \le \alpha_2(|\xi|_{\mathcal{A}}), \text{ any } \xi \in \mathbb{R}^n.$$

Then, there exists an $\mathcal{KL}$-function β (which depends only on α) so that, for each x_0 and each essentially bounded control u as above, the ensuing trajectory $x(\cdot)$ is so that, for each $t \ge 0$:

1. $|x(t)|_{\mathcal{A}} \le \beta(|x_0|_{\mathcal{A}}, t)$ or
2. $x(t) \in S$ for all $t \ge T$.

Proof. Proceeding as in the proof of Theorem 1, we let β_α be the $\mathcal{KL}$-function as in Lemma 2.9, with respect to the $\mathcal{K}$-function α, and define

$$\beta(s,\, t) := \alpha_1^{-1}\big(\beta_\alpha(\alpha_2(s),\, t)\big)\,. \tag{33}$$

Then β is a $\mathcal{KL}$-function.

Because S is forward invariant, it is only necessary to prove that if $x(t) \notin S$ for all t in some interval $[0,\, T)$, then the first case in the lemma must hold for such t. But then, by assumption, this means

$$\frac{dV(x(t))}{dt} \le -\alpha\big(V(x(t))\big)\,, \quad \text{any } t \in [0,\, T)\,. \tag{34}$$

By Lemma 2.9,

$$|x(t)|_{\mathcal{A}} \le \beta\big(|x_0|_{\mathcal{A}},\, t\big)\,, \quad \text{any } t \in [0,\, T)\,. \qquad \blacksquare$$

The main result of this paper is as follows.

Theorem 2 *Smoothly stabilizability with respect to a closed set $\mathcal{A}$ implies smooth input-to-state stabilizability with respect to the same closed set $\mathcal{A}$.*

Proof. Assume that k_1 is as in the definition of smooth stabilizability, i.e., that

$$\dot{x} = f(x) + G(x)\tilde{\sigma}\big(k_1(x)\big) := \tilde{f}(x) \tag{35}$$

is UGAS with respect to the closed invariant set $\mathcal{A}$. Applying Theorem 1 to (35) (cf Remark 2.6 and Proposition 2.8), we get a smooth $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\ge 0}$ and three smooth $\mathcal{K}_\infty$-functions α_1, α_2 and α_3, satisfying, for any $\xi \in \mathbb{R}^n$, $\alpha_1(|\xi|_{\mathcal{A}}) \le V(\xi) \le \alpha_2(|\xi|_{\mathcal{A}})$ and $a := L_{\tilde{f}}V(\xi) = \frac{\partial V}{\partial x}(\xi) \cdot \Big(f(\xi) + G(\xi)\tilde{\sigma}\big(k_1(\xi)\big)\Big) \le -\alpha_3(|\xi|_{\mathcal{A}})$. Define $k(\xi) := k_1(\xi) - b(\xi)$, where $b := (b_1, b_2, \dots, b_m)'$, and $b_i(\xi) := L_{g_i}V(\xi)$, $i = 1, \dots, m$. We will show that this k provides input-to-state stabilizability with respect to $\mathcal{A}$.

We now analyze the closed-loop system (31). Pick any measurable locally essentially bounded control $u(\cdot)$, and an $x_0 \in \mathbb{R}^n$. Let $x(\cdot) = x(\cdot, x_0, u)$. For almost all $t \in [0,\, T_{x_0,\,u})$,

$$\frac{dV(x(t))}{dt} = a(x(t)) + \sum_{i=1}^{m} d_i(t), \tag{36}$$

where $d_i(t) := b_i(x(t))\Big\{\sigma_i\big(k_1(x(t))_i + u_i(t) - b_i(x(t))\big) - \sigma_i\big(k_1(x(t))_i\big)\Big\}$. It is easy to see that, for each i, and each t, $d_i(t) \le \dfrac{cu_i^2(t)}{4}$. Indeed, fix an i and a t, and write $\mu := u_i(t)$, $b := b_i(x(t))$, $\sigma := \sigma_i$, and $\kappa := k_1(x(t))_i$. If either $b = 0$ or $\mu - b = 0$ then the claim is clear; thus from now on we assume that these are both nonzero. If $b < 0$ and $\mu - b < 0$, $\sigma(\kappa + \mu - b) - \sigma(\kappa) \ge -c|\mu - b| = c(\mu - b)$, so also

$$\begin{aligned} b\ \{\sigma(\kappa+\mu-b) - \sigma(\kappa)\} &\le cb(\mu - b) \\ &= -c(b - \frac{\mu}{2})^2 + c\frac{\mu^2}{4} \\ &\le c\frac{\mu^2}{4}\,, \end{aligned}$$

which is the desired inequality. The same argument applies if both $b > 0$ and $\mu - b > 0$. We are left with the case where these have different signs. If $b > 0$, then $\sigma(\kappa + \mu - b) - \sigma(\kappa) \le 0$, so $b\{\sigma(\kappa + \mu - b) - \sigma(\kappa)\} \le 0$, and again the desired inequality holds. Finally, if $b < 0$, then $\sigma(\kappa + \mu - b) - \sigma(\kappa) \ge 0$, and multiplying by b we also obtain the claimed inequality.

We conclude that (36) is bounded by $a(x(t)) + \sum_{i=1}^m c\frac{u_i^2(t)}{4} \le a(x(t)) + \frac{c}{4}\|u\|^2$. This expression is in turn bounded by $\frac{a(x(t))}{2}$ if the following inequality holds: $-a(x(t)) \ge \frac{c}{2}\|u\|^2$. Introducing the $\mathcal{K}_\infty$-function $\gamma(s) := (\sqrt{\alpha_3})^{-1}\left(\sqrt{\frac{c}{2}}\, s\right)$, where $(\sqrt{\alpha_3})^{-1}(\cdot)$ denotes the inverse function of $\sqrt{\alpha_3(\cdot)}$, we have that, for each $t \in [0,\, T)$:

$$|x(t)|_{\mathcal{A}} \ge \gamma(\|u\|) \implies \dot{V}(x(t)) \le -\frac{1}{2}\alpha_3(|x(t)|_{\mathcal{A}})\,. \tag{37}$$

Finally, let $\alpha(s) := \frac{1}{2}\alpha_3(\alpha_2^{-1}(s))$. Observe that α is also a smooth $\mathcal{K}_\infty$-function. We can summarize all of the above by:

$$|x(t)|_{\mathcal{A}} \ge \gamma(\|u\|) \implies \dot{V}(x(t)) \le -\alpha(V(x(t))) \tag{38}$$

for each $t \in [0,\, T_{x_0,\,u})$. Consider the following sublevel set of V: $S := \{\xi \in \mathbb{R}^n \mid V(\xi) \le \alpha_2(\gamma(\|u\|))\}$.

Claim: With respect to this trajectory, S is forward invariant.

Proof: Otherwise, there exist $\varepsilon > 0$ and $t_1 > t_0$ such that $V(x(t_1)) \ge c + \varepsilon$. Let t_1 be the smallest real number so that this inequality holds, for this fixed ε. It follows that $V(x(t)) > c$ in some neighborhood of t_1, so that also (38) holds for all t near t_1, and hence $V(x(t)) > V(x(t_1))$ for some $t \in (t_0, t_1)$, contradicting the minimality assumption on t_1. This establishes the claim.

Consider now the $\mathcal{K}_\infty$-function defined by $\zeta(s) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma(s)$, and observe that $|x(t)| \le \zeta(\|u\|)$ whenever $t \in [0, T)$ is such that $x(t) \in S$. Also note that for any $t_0 \in [0, T)$, if $x(t_0) \notin S$, then from $\alpha_2(|x(t_0)|_{\mathcal{A}}) \ge V(x(t_0)) > \alpha_2(\gamma(\|u\|))$, we have that (38) must hold for such t_0. Applying Lemma 3.14, we obtain a $\mathcal{KL}$-function β associated to α. The theorem follows, as $|x(t)|_{\mathcal{A}}$ is bounded by the sum of $\beta(|x_0|_{\mathcal{A}}, t)$ and $\zeta(\|u\|)$, for all $t \in [0, T)$. $\blacksquare$

As an illustration, consider the system (1) and (2) discussed in the introduction. This is GAS when $u = 0$. With $V(x, y) = y^2/2$, we compute the feedback law $k(x, y) = -xy$. Now the closed loop system is

$$\begin{aligned} \dot{x} &= -x\,, \qquad &(39) \\ \dot{y} &= -y - x^2y - ux\,. \qquad &(40) \end{aligned}$$

With for instance a constant control $u \equiv \varepsilon$ and initial states $x(0) = x_0$, $y(0) = y_0$, we compute the solution as $y(t) = -\frac{\varepsilon}{x_0}\exp(-t - \frac{x_0^2 e^{2t}}{2} + \frac{x_0^2}{2}) + \frac{\varepsilon}{x_0}\exp(-t) + y_0\exp(-t - \frac{x_0^2 e^{2t}}{2} + \frac{x_0^2}{2})$ which remains bounded (in fact, it converges to zero, for any x_0, y_0).

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