REMARKS ON INPUT/OUTPUT LINEARIZATION*

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Abstract

In the context of realization theory, conditions are given for the possibility of simulating a given discrete time system, using immersion and/or feedback, by linear or state-affine systems.

1. Introduction

A number of papers have been written recently on the topic of system simulation by simpler types of systems, like linear and state-affine systems. In particular, we shall be concerned here with the notion of *immersion* introduced by M.Fliess, and more specifically, in regard to *discrete-time* nonlinear systems (see [1,2,3]).

In this note we remark that the setup in [4], which deals with nonlinear realization, generalizes in an almost immediate manner to deal with questions of immersion, resulting in conceptually simple statements and proofs. The approach resulting from this generalization should be compared to that in [1,2,3], which relies on a theory for real-analytic (in fact, 'entire',) systems and bases all proofs on appropiate power series expansions.

2. Responses and Systems

We shall need a number of definitions. Except for the generalization from single responses to families of responses, these are precisely as in [4].

A set **U** (input-value set) and a finite-dimensional vector space **Y** (output-value set) over a fixed field **k** will be assumed fixed. A *response* is a mapping $f:\mathbf{U}^*\to\mathbf{Y}$, where $\mathbf{U}^*$ denotes the free monoid over **U**. A *family (of responses)* Φ is any set of responses, which we write in a parametrized way as $\Phi = \{f^\lambda, \lambda\in\Lambda\}$. The response f is *of type J*, where $J = \{\delta_0,\cdots,\delta_m\}$ is a set of linearly independent functions $\mathbf{U}\to\mathbf{k}$ with $\delta_0\equiv 1$, iff for each $t\geq 1$ there are (finitely many) vectors a_α in Y such that

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$$f(\omega) = \sum \delta_\alpha(\omega) a_\alpha,$$

where the summation runs over J^t and, if $\alpha=(\alpha_1,\cdots,\alpha_t)$ and $\omega=(u_1,\cdots,u_t)$, then $\delta_\alpha(\omega)$ is the ('tensor') product $\delta_{\alpha_1}(u_1)\cdots\delta_{\alpha_t}(u_t)$. The *family* Φ is of *type J* iff each f^λ in Φ is. The family Φ is *affine* iff $\mathbf{U}=\mathbf{k}^m$, Φ is of *affine* type $J_0 := \{\delta_0,\cdots,\delta_m\}$ with δ_i = projection of $\mathbf{U}$ onto i-th factor, and the restrictions $f^\lambda|\mathbf{U}^t$ are all affine; Φ is *uniformly affine* if it is affine and the maps

$$f^\lambda_{lin}(\omega) := f^\lambda(\omega) - f^\lambda(\underline{0})$$

are all the same (i.e. independent of λ), say f, ($\underline{0}$ denotes here a zero input of length equal to length(ω),) and f is a (time-invariant) linear i/o response, i.e., the restrictions of f to all U^t are linear, and $f(\underline{0},\omega) = f(\omega)$, for any input sequence ω and any constantly 0 input sequence. Such families appear when dealing with affine systems (see below).

A *system* $\Sigma = (X,P,Q)$ consists of a set X and maps $P: X\times\mathbf{U}\to X$, $Q: X\to\mathbf{Y}$. The system Σ is *state-affine* if X is a **k**-vector space, and Q as well as each map $P(\cdot,u)$, $u\in\mathbf{U}$ are affine. It is *affine* if $\mathbf{U}=\mathbf{k}^m$ and P and Q are affine maps (i.e., the system is basically *linear* in the usual sense, except that we do not a priori impose any equilibrium state requirements), and is *of type J* (J as above) if there exist affine maps P_i such that $P(x,u) = \sum \delta_i(u) P_i(x)$. Note that a *finite-dimensional* state-affine system is always of type J, for some (finite) J. We use the notation P^* for the recursive extension of P to input sequences: $P^*(x,\varepsilon) := x$ (ε=empty sequence) and $P^*(x,\omega u) := P(P^*(x,\omega),u)$.

For any state $x\in X$, f^x is the response obtained when starting Σ at initial state x, i.e., $f^x(\omega) := Q(P^*(x,\omega))$. The *(response) family* Φ_Σ of Σ is the family $\{f^x, x\in X\}$. When $\Phi\subseteq\Phi_\Sigma$ we say that Σ *realizes* the family Φ. Note that if Σ is (state-affine) of type J, then Φ_Σ is also of type J. And if Σ is affine, then Φ_Σ is also uniformly affine.

Finally, we need the notion of the *observation space* L^Σ associated to Σ. This is the space of mappings from X into **Y** obtained in the following way. For each input sequence $\omega\in\mathbf{U}^*$ (possiby empty) let $Q^\omega(x) := Q(P^*(x,\omega))$ = last output that results from applying input sequence ω to initial state x. The set of all maps $[X\to\mathbf{Y}]$ is endowed with a natural **k**-linear structure (pointwise operations); then L^Σ is defined as the linear span of $\{Q^\omega, \omega\in U^*\}$.

3. Immersions and Morphisms

Let Σ and Σ be two systems, with response families Φ and $\bar{\Phi}$ respectively. The system Σ is *immersed* into the system Σ iff $\Phi\subseteq\bar{\Phi}$, i.e., Σ realizes Φ. Another way of saying this is that there is a mapping $\phi: X\to\bar{X}$ such that $\phi(x)$ and x give rise to the same response. We call any such mapping an *immersion*, and denote $\phi: \Sigma\to\Sigma$. (The definition used in [1,2,3] for analytic systems requires that there be an *analytic* such mapping, for a class of analytic systems. We prefer the more abstract definition, together with the remarks given below which insure that an immersion will necesarily have certain properties like analiticity provided that Σ and Σ satisfy appropiate conditions.)

Immersions are of course closely related to morphisms in the sense usual in system theory. A *morphism* ϕ: $\Sigma \to \Sigma$ is a mapping $\phi: X \to \bar{X}$ such that

$$\phi(P(x,u)) = \bar{P}(\phi(x),u) \text{ and } Q(x) = \bar{Q}(\phi(x)) \tag{1}$$

for each $x \in X$ and $u \in \mathbf{U}$. Then also $\phi(P^*(x,u)) = \bar{P}^*(x,u)$ for all x,u. Thus a morphism is necessarily an immersion. A partial converse is given below.

A system Σ is *observable* iff $f^x \neq f^z$ for each pair of states x,z. Equivalently, for each x,z in X there is a $\omega \in \mathbf{U}^*$ such that $Q^\omega(x) \neq Q^\omega(z)$. It is *linearly observable* iff X is a **k**-vector space and there are an integer r, input sequences $\omega^1, \cdots, \omega^r$ in $\mathbf{U}^*$, and a *affine* map π: $\mathbf{Y}^r \to X$ such that

$$\pi(Q^{\omega^1}(x), \cdots, Q^{\omega^r}(x)) = x \tag{2}$$

for each $x \in X$. This is a particular instance of the notion of "algebraic observability" introduced in [5] for a more general class of systems. With a more general notion, relativized to any given subcategory of systems (e.g. analytic,) some of the results to follow could be extended in a natural manner. Note that linear observability implies observability.

Proposition 1: Let ϕ: $\Sigma \to \Sigma$ be an immersion and assume that Σ is observable. Then ϕ is a morphism. If ψ: $\Sigma \to \Sigma$ is any other immersion, $\phi = \psi$.

Proof: Pick any x,u, and let $z := \phi(x)$. Consider first Q(x). This is $f^x(\varepsilon)$, which coincides with $\bar{f}^z(\varepsilon)$, and hence with $\bar{Q}(z)$, as desired. Let $\xi := P(x,u)$ and $\zeta := \bar{P}(z,u)$. We need to prove that $\bar{P}(z,u) = \zeta$, or equivalently, by observability, that for any $\omega \in \mathbf{U}^*$, $\bar{Q}^\omega(\zeta) = \bar{Q}^{\omega u}(z)$. But $\bar{Q}^\omega(\zeta) = \bar{f}^\zeta(\omega) = f^\xi(\omega) = Q^\omega(\xi) = Q^{\omega u}(x) = f^x(\omega u) = \bar{f}^z(\omega u) = \bar{Q}^{\omega u}(z)$, as desired. The uniqueness part is even easier: for all x, $f^{\phi(x)} = f^x = f^{\psi(x)}$ implies by observability that $\phi(x) = \psi(x)$.n

When Σ is a finite-dimensional *state-affine* system, L^Σ is *finite-dimensional*, since in that case all the generators Q^ω are affine maps on a finite-dimensional space. Further, for such systems the two notions of observability coincide. Indeed, a dimensionality argument implies that, if Σ is observable, then there are finitely many sequences $\omega^1, \cdots, \omega^r$ such that the affine map

$$x \to (Q^{\omega^1}(x), \cdots, Q^{\omega^r}(x))$$

is one-to-one, and hence admits a left inverse π as in 2. If Σ is not observable, we can always find an observable system Σ, state-affine of the same type, and a morphism (hence an immersion) ϕ: $\Sigma \to \Sigma$; this is discussed for instance in [4]. Further, if Σ happens to be an affine system, also Σ is affine under this reduction. Thus every state-affine [resp., affine] system can be immersed into a linearly observable state-affine [resp., affine] system. Since the immersion relation is transitive, we conclude that if a system Σ can be immersed into a system Σ of one of these forms, we may assume without loss of generality that Σ is linearly observable.

In fact, the immersion itself is then analytic for systems defined by analytic equations, continuous for continuous systems, and so forth. To make this precise, let $L^{\Sigma\#}$ be the smallest space of maps X→k including the constants and all compositions of an element of L^{Σ} and a linear map **Y**→k. This is basically the set of all coordinates of observables. Call an immersion φ: Σ→Σ, where $\bar{X}$ is a vector space, *nice* iff for all linear θ: $\bar{X}$→k the composition θ°φ is in $L^{\Sigma\#}$. That is, the coordinates of φ are given by linear combinations of functions appearing as coordinates of observables (and constants). Accordingly, Σ is *nicely immersed* in Σ iff there is nice φ: Σ→Σ.

Proposition 2: If Σ can be immersed in a finite-dimensional state-affine [resp., affine] system Σ, then it can also be nicely immersed in such a system.

Proof: As remarked above, we may assume without loss that Σ is linearly observable. Let π be as in 2, for Σ, i.e. $\pi(\bar{Q}^{\omega^1}(x),\cdots,\bar{Q}^{\omega^r}(\bar{x})) = \tilde{x}$ for each $\tilde{x}\in\bar{X}$. Because φ is an immersion, it holds that $(Q^{\omega^1}(x),\cdots,Q^{\omega^r}(x)) = (\bar{Q}^{\omega^1}(\phi(x)),\cdots,\bar{Q}^{\omega^r}(\phi(x)))$. Now apply π to both sides. Then, for each x∈X, $\phi(x) = \pi(Q^{\omega^1}(x),\cdots,Q^{\omega^r}(x))$. Since π is affine, φ is nice.n

In relation to the dimensionality of L^{Σ} we note the following observation, which will be useful later:

Proposition 3: If Σ is immersed in Σ then there is a surjective mapping from L^{Σ} onto L^{Σ}. In particular, if Σ is immersed in a finite-dimensional state-affine system then L^{Σ} is finite-dimensional.

Proof: Let ϕ^*: [X'→**Y**] → [X→**Y**] be the dual operator defined by the composition $\phi^*(T) := T\circ\phi$. This is a linear map, and its restriction to L^{Σ} maps into (and in fact, onto) L^{Σ}.n

4. Canonical Realizations

Let $\Phi = \{f^{\lambda}, \lambda\in\Lambda\}$ be a fixed family as in the previous section. We associate to Φ the "Nerode space" L_{Φ} defined as follows. For each f^{λ} and each $\omega\in U^*$ let f^{λ}_{ω} be the response defined by $f^{\lambda}_{\omega}(\nu) := f^{\lambda}(\omega\nu)$ (i.e., 'preset' by input ω). The set of all responses is naturally a **k**-vector space under pointwise operations; L_{Φ} is then defined as the *affine hull* of

$$\{f^{\lambda}_{\omega},\ \omega\in U^*,\ \lambda\in\Lambda\}.$$

(We could use here instead the *linear span*; basically the same theory would result, except that state-*linear* systems would appear at various points instead of state-affine ones.)

If Σ is a realization of Φ, there is a natural duality between L_{Φ} and the observation space L^{Σ} defined earlier. Consider the pairing

$$(\ ,\ ) : L_{\Phi}\times L^{\Sigma} \to \mathbf{Y}.$$

obtained by bilinearly extending the operation on generators

$$(f^{\lambda}_{\omega}, Q^{\nu}) := f^{\lambda}(\omega\nu).$$

It is necessary to see that this extension is well defined. Assume then that

$$\sum \alpha_{\omega\lambda} f^{\lambda}_{\omega} = \sum \beta_{\omega\lambda} f^{\lambda}_{\omega} \quad \text{and}$$

$$\sum \gamma_{\nu} Q^{\nu} = \sum \delta_{\nu} Q^{\nu}$$

where the first pair of sums are over a (finite) set of pairs (ω,λ) and the second over some (finite) set of inputs ν. Then, $\sum\gamma_{\nu}\sum\alpha_{\omega\lambda}f^{\lambda}(\omega\nu) = \sum\gamma_{\nu}\sum\alpha_{\omega\lambda}f^{\lambda}_{\omega}(\nu) = \sum\gamma_{\nu}\sum\beta_{\omega\lambda}f^{\lambda}_{\omega}(\nu) = \sum\gamma_{\nu}\sum\beta_{\omega\lambda}f^{\lambda}(\omega\nu) = \sum\beta_{\omega\lambda}\sum\gamma_{\nu}f^{\lambda}(\omega\nu)$, and it is enough to prove that $\sum\gamma_{\nu}f^{\lambda}(\omega\nu) = \sum\delta_{\nu}f^{\lambda}(\omega\nu)$ for each *fixed* (ω,λ). But realizability of Φ by Σ means that there is an $x \in X$ such that $f^x = f^{\lambda}$. Let $z := P^*(x,\omega)$ for this x and the given ω. Thus, $f^{\lambda}(\omega\nu) = Q^{\nu}(z)$ for each ν, so $\sum\gamma_{\nu}f^{\lambda}(\omega\nu) = (\sum\gamma_{\nu}Q^{\nu})(z) = (\sum\delta_{\nu}Q^{\nu})(z) = \sum\delta_{\nu}f^{\lambda}(\omega\nu)$, as desired. Further, this pairing is nondegenerate in the first term, i.e., $(\sum\alpha_{\omega\lambda}f^{\lambda}_{\omega},\sum\gamma_{\nu}Q^{\nu}) = 0$ for all $\sum\gamma_{\nu}Q^{\nu}$ implies that $\sum\alpha_{\omega\lambda}f^{\lambda}_{\omega} = 0$. It will also be nondegenerate in the second term iff the condition "$\sum\gamma_{\nu}Q^{\nu} = 0$ on all elements of the form $P^*(x,\omega)$ such that ω is an input sequence and $f^x \in \Phi$" implies $\sum\gamma_{\nu}Q^{\nu} = 0$. This happens in various cases of interest, for instance if there is some kind of 'span-reachability' for the system as in [4], and in any case certainly when $\Phi = \Phi_{\Sigma}$. We conclude that:

Proposition 4: If $\Phi = \Phi_{\Sigma}$, then L_{Φ} is finite dimensional if and only if L^{Σ} if finite dimensional.

We now show how to construct a 'canonical' realization Σ_{Φ} of any given family Φ. This realization will have the property that it is a finite dimensional state-affine system whenever L_{Φ} is finite dimensional, and is affine whenever Φ is also uniformly affine.

The state space of Σ_{Φ} is by definition L_{Φ}, seen as a vector space. The maps P and Q are defined as follows. For each $u \in \mathbf{U}$, let $P(\cdot,u)$ be the (well-defined!) extension of $P(f^{\lambda}_{\omega},u) := f^{\lambda}_{\omega u}$, and let Q be the extension of the evaluation $Q(f^{\lambda}_{\omega}) := f^{\lambda}(\omega)$. *This system is state-affine*, by construction. Applying proposition 4 to $\Phi := \Phi_{\Sigma}$, and using proposition 3, we then conclude:

Theorem A. Σ can be immersed in a finite dimensional state-affine system if and only if its observation space is finite dimensional.

This result can be made more precise in a number of ways. Assume for instance that Φ is of type J. Then Σ_{Φ} *is also of type J*. Indeed, write, for any $\omega \in \mathbf{U}^*$ and $u \in \mathbf{U}$,

$$f^{\lambda}(\omega u) = \sum \delta_{\alpha}(\omega)\delta_i(u)a^{\lambda}_{\alpha i}.$$

Then, $P(x,u) = \sum\delta_i(u)P_i(x)$, where P_i is defined on generators by the formula

$$P_i(f^{\lambda}_{\omega}) := \sum \delta_{\alpha}(\omega)a^{\lambda}_{\alpha i}.$$

The P_i are well-defined (use linear independence of the $\delta_i(u)$ to verify this) and are affine. We then have:

Theorem B. The system Σ can be immersed into a [finite dimensional] system of type J if and only if Φ_{Σ} is of type J [and L^{Σ} is finite dimensional].

Now assume that Φ is uniformly affine, and consider its canonical realization Σ_Φ. We claim that this is an affine system. Since Σ_Φ has affine type J_0, we only need to establish that all P_i, $i = 1,\cdots,m$ are constant. Note that we may write

$$f^\lambda(\omega u) = \sum u_i P_i(f^\lambda_\omega) + P_0(f^\lambda_\omega), \tag{3}$$

$$f(\omega u) = f^\lambda(\omega u) - f^\lambda(\underline{0}0), \tag{4}$$

where f is linear and independent of λ, and satisfies $f(\underline{0},u) = f(u)$ as in the definition. It follows from 4 that $f^\lambda_\omega(u)-f^\lambda_\omega(0) = f(\omega,u)-f(\omega,0) = f(\underline{0},u) = f(u)$, while from 3 this expresion equals $\sum u_i P_i(f^\lambda_\omega)$. Comparing coefficients of each u_i, the desired constancy is deduced. Thus we conclude:

Theorem C. The system Σ can be immersed in a [finite dimensional] affine system if and only if Φ_Σ is uniformly affine [and L^Σ is finite dimensional].

This is, with somewhat different terminology, the characterization given in [1,2,3] for immersions of analytic Σ into finite dimensional systems. The only difference is that the finite dimensionality condition given there is not in terms of L^Σ -which is natural in the general context treated here- but in terms of L^Σ_0, defined as follows.

Assume that $U = \mathbf{k}^m$. The *0-input* observation space L^Σ_0 is the subspace of L^Σ generated by the Q^ω with ω = a sequence of the type $\underline{0}$ (all zero). *In case Φ_Σ is a uniformly affine family*, for any observable we have that

$$Q^\omega(x) = f^x(\omega) = f^x(0)+f(\omega) = Q^0(x)+a_\omega,$$

where a_ω is a constant (vector in $\mathbf{Y}$) depending only on ω. That is, L^Σ is in that case included in the sum of L^Σ_0 and a finite dimensional space (constant functions), so that

$$L^\Sigma \textit{ is finite dimensional iff } L^\Sigma_0 \textit{ is.} \tag{5}$$

5. Feedback

We give now some results on linearization under feedback. These results are basically just restatements of those in [1], proved in slightly more generality using a more abstract approach. Further work along these lines is in progress.

Fix a parameter set Λ; all families of responses will be parametrized by this set. A *system-like* family is one for which the following property holds: for each $\lambda\in\Lambda$ and $u\in\mathbf{U}$ there is a $\mu\in\Lambda$ such that $f^\lambda(u\omega) = f^\mu(\omega)$ for all $\omega\in\mathbf{U}^*$. Such families appear when considering the families Φ_Σ. We shall assume that all families in this section are system-like.

An *i/o family* will mean a family $\Gamma = \{\gamma^\lambda, \lambda\in\Lambda\}$ of maps γ^λ: $\mathbf{U}^*\to\mathbf{U}^*$ each of which is length preserving. Given a family of responses Φ, and a Γ like this, we define the new family of responses $\tilde\Phi := \Phi_\Gamma$ via the compositions $\tilde f^\lambda := f^\lambda\circ\gamma^\lambda$, and denote $\tilde\Phi \le \Phi$ if $\tilde\Phi$ can be obtained in this way from Φ. We also denote $\tilde\Phi \sim \Phi$ if both $\tilde\Phi\le\Phi$ and $\Phi\le\tilde\Phi$ hold. Consider the *degree* of the family Φ:

$$d=d(\Phi) := \max\{t \text{ s.t. } f^\lambda|\mathbf{U}^j \text{ constant if } j\le t \text{ and } \lambda\in\Lambda\},$$

and assume that $d<\infty$ for the systems here considered. For $j\le d$, $v_j(\lambda)$ denotes the common value of the restriction of f^λ to $\mathbf{U}^j$. Since the elements of Γ preserve length, it is clear that if $v_j(\lambda)$ is like this and $\Phi = \Phi_\Gamma$ then also

$$\tilde{f}^\lambda|\mathbf{U}^j = v_j(\lambda), \tag{6}$$

and so

$$\tilde{d} \ge d. \tag{7}$$

In particular, $\tilde{d}=d$ when $\Phi\sim\Phi$. Pick now any integer j and any $\nu\in\mathbf{U}^+$, $\lambda\in\Lambda$, and $\omega_1\in\mathbf{U}^j$. Then $\tilde{f}^\lambda(\nu\omega_1) = f^\lambda(\gamma^\lambda(\nu),\omega')$, for some ω' of length j. Since Φ is system-like, there is a μ, which depends only on ν, such that $f^\lambda(\gamma^\lambda(\nu),\omega') = f^\mu(\omega')$. If $j\le d(\Phi)$, this last expression equals $f^\mu(\omega_2)$ for *every other* ω_2 of length j, so that:

Lemma 5: Assume that $\Phi\le\Phi$. Let $j\le d(\Phi)$, $\nu\in U^+$, $\lambda\in\Lambda$, and take $\omega_i\in\mathbf{U}^j$, i=1,2. Then $\tilde{f}^\lambda(\nu\omega_1) = f^\lambda(\gamma^\lambda(\nu),\omega_2)$.

We shall be interested in a very particular class of linear systems. Call a family Φ *special* iff (it has finite degree and) for some $\kappa\le d(\Phi)$ there are scalars $\{a\ \}$ and a linear transformation D: $\mathbf{U}\to\mathbf{Y}$ such that

$$f^\lambda(u0^\kappa) = \sum a\ \ v\ \ (\lambda) + Du \tag{8}$$

for all $u\in\mathbf{U}$ (sum is over $=0,...,\kappa$) and all $\lambda\in\Lambda$. Note that, since Φ is system-like and $\kappa\le d(\Phi)$, 8 is also the form of $f^\lambda(u\omega)$ for all $\omega\in\mathbf{U}^\kappa$. It can be proved by induction that if this property holds with some $\kappa<d(\Phi)$ then it also holds with $\kappa=d(\Phi)$. Such a family is necessarily uniformly affine and finite-dimensional realizable; this can be seen directly by exhibiting an affine finite dimensional system realizing it, as done for the analogous situation in [1]. For example, the linear system with state space X:= $\mathbf{Y}^{\kappa+1}$ and equations (in $\mathbf{Y}$-block form) $x_i' = x_{i+1}$, $i=0,\cdots,\kappa-1$, and $x_\kappa' = \sum a\ \ x\ \ +Du$, and output $y = x_o$ realizes such a family; more precisely, f^λ is the response associated to the initial state $(v_0(\lambda),\cdots,v_\kappa(\lambda))$.

Note that if Σ = (A,B,C) is a linear system in the usual sense, then Φ_Σ is special iff there is a κ such that CA B = 0 for all $= 0,\cdots,d-1$ and $CA^{\kappa+1}$ is a linear combination of the CA^j, $j\le\kappa$. A somewhat more general definition of 'special' could be given, in the style of [1], where the property is required of each output channel separately; the results are analogous but notations become somewhat more involved. Let $j\le d(\Phi)$. Consider the following property for Φ and Γ as above:

For some linear D, scalars $\{a\ \}$, $f^\lambda(\gamma^\lambda(u),0^\kappa) = \sum a\ \ f^\lambda(0\ \ ) + Du$, all $\lambda\in\Lambda$, $u\in\mathbf{U}$, (9)

the sum over $= 0,\cdots,\kappa$.

Proposition 6: If there is an i/o family Γ such that 9 is satisfied then Φ_Γ is special.

Proof: Let $\Phi := \Phi_\Gamma$. So $\tilde{d}\ge d\ge\kappa$. By lemma 5, $\tilde{f}^\lambda(u,0^\kappa) = f^\lambda(\gamma^\lambda(u),0^\kappa)$, which is as in 9. By 6, $\tilde{f}^\lambda(0\ \ ) = \tilde{v}\ \ (\lambda) = v\ \ (\lambda)$; thus Φ is special.n

Let $\Phi = \Phi_\Sigma$, for a given system Σ. We say that the i/o family Γ is *feedback-like* for Φ iff there exists a ('generating feedback') map K: **X**×**U** → **U** such that K(x,·) is invertible for each fixed x, and each γ^λ is defined as usual by the outputs of the system obtained from x'=P(x,K(x,u)) and y=K(x,u). If this K is of the form F(x)+G(x)u, with G(x) invertible, we say the feedback is *affine*. Note that the invertibility condition insures that Φ_Γ can be obtained, conversely, from [affine] feedback applied to Φ. We write $\Upsilon = \{\upsilon^\lambda\}$ for the inverse family to Γ; thus ϒ consists of length preserving maps with $\upsilon^\lambda \circ \gamma^\lambda$ = identity for each λ. (Somewhat less than invertibility is needed for the results to follow; also, a more general, input/output, definition can be given of feedback-like Γ.)

Corollary 7: The feedback-like family Γ is such that Φ_Γ is special iff condition 9 is satisfied.

Proof: Sufficiency is just Proposition 6. Conversely, assume that there is such a Γ. Then $\tilde{d} = d$ for the degrees of $\tilde{\Phi} = \Phi_\Gamma$ and Φ respectively (from 7). By lemma 5, with the roles of Φ and $\tilde{\Phi}$ reversed, $f^\lambda(\gamma^\lambda,0^\kappa) = \tilde{f}^\lambda(u0^\kappa)$, which by assumption is as in 9. Since, by 6, $\tilde{v}$ (λ) = v (λ), 9 holds with the same κ, {a }, and D.n

Assume now that there is an affine Γ with $\tilde{\Phi} := \Phi_\Gamma$ uniformly affine. Then the following property must be satisfied:

there exist s,R *such that* $f^\lambda(v0^d) = \tilde{f}^\lambda(\upsilon^\lambda(v),0^d) = s(\lambda)+R(\lambda)v$ for all $v \in U$. (10)

This is because $\Phi = \tilde{\Phi}_\Upsilon$, so we may apply lemma 5 to this pair. Since $\tilde{f}^\lambda(u0^d)$ and υ^λ are each affine, their composition also is.

Assume now that $f^\lambda(v0^d)$ satisfies 10. Then corollary 7 says that an affine Γ exists making Φ_Γ special iff there are F, G, and constants {a }, such that, (substituting the form $\gamma^\lambda(u) = F(\lambda)+G(\lambda)u$ into the left hand side of 9,)

$$R(\lambda)G(\lambda) = D = \text{constant},$$

and

$$s(\lambda) + R(\lambda)F(\lambda) = \sum a \;\; v \;\; (\lambda).$$

Thus one recovers the conditions in [1], since clearly $s(\lambda) = f^\lambda(0^{d+1})$ and $v_j(\lambda) = f^\lambda(0^j)$.

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