ACCESSIBILITY UNDER SAMPLING

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ABSTRACT

This note addresses the following problem: Find conditions under which a continuous-time (nonlinear) system gives rise, under constant rate sampling, to a discrete-time system which satisfies the accessibility property.

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1. INTRODUCTION

We consider systems of the form

(1.1) $(dx/dt)(t) = f(x(t),u(t))$.

States x(t) evolve in an analytic (paracompact) n-dimensional manifold M. Control values u(t) belong to a subset U of an analytic manifold P, such that int(U) is connected and $U \subseteq$ clos(int(U)) --for instance, U = a convex subset of $\mathbf{R}^m$ and P is $\mathbf{R}^m$ or any open set containing U. Further, we assume that f: $M \times P \rightarrow TM$ is analytic (this can be weakened for many of the results) and that (1.1) is complete, in the sense that solutions are defined for all t, for any x(0) and any piecewise constant u(.) with values in P (again, this assumption could be dropped for many of the arguments below).

Recall that (1.1) is said to satify the *accessibility* [resp., *strong accessibility*] *property from* x [4] iff the (positive time) reachable set A(x) [resp., the set $A_t(x)$ of states reachable from x in time exactly t, for some t>0] has a nonempty interior. Consider the orbit O(x) of a state x under the group *G* generated by the actions of piecewise constant controls, i.e., under the group of transformations generated by the diffeomorphisms

(1.2) $\exp(tf(.,u))$, u in U.

It is well known that (1.1) satisfies the accessibility property from x iff O(x) is a ngbd of x, or equivalently, iff $\dim L(x) = n$, where *L* is the Lie algebra generated by all the vector fields of the form f(.,u). Similarly, strong accessibility from x is equivalent both to (a) $\dim O_o(x) = n$ and to (b) $\dim L_o(x) = n$, where $O_o(x)$ is the "zero time orbit" of x --z is zero time reachable from x iff

(1.3) $z = [\exp(t_1 f(.,u_1))_o \ldots_o \exp(t_r f(.,u_r))](x)$,

with $t_1+\ldots+t_r = 0$,-- and where L_o is the ideal of *L* generated by all the vector fields of the form f(.,u) - f(.,v) for u and v in U.

Fix now a ("sampling period") $\lambda>0$, and let $A(x,\lambda)$ be the set of states of (1.1) reachable from x using controls which are constant on intervals of the form $[k\lambda,(k+1)\lambda)$. In analogy to the continuous time case (which can be thought of as the case $\lambda=0$,) we ask: When does there exist a positive λ such that $A(x,\lambda)$ has an open interior? This property may be called "sampled accessibility from x". It turns out that this question is equivalent to the following. Let $O(x,\lambda)$ be the orbit of x under the group G_λ generated by all the diffeomorphisms $\exp(\lambda f(.,u))$, for u in U. Then (1.1) will be sampled accessible from x iff $O(x,\lambda)$ is a ngbd of x, for some $\lambda>0$.

Denote also

(1.4) $\phi(x,u_1...u_r;\alpha_1...\alpha_r):=$

$[\exp(\alpha_r\lambda f(.,u_r))_o..._o\exp(\alpha_1\lambda f(.,u_1))]$ (x),

where each α_i is either +1 or -1. (Note: $u_1...u_r$ denotes the concatenation of the corresponding controls.) For simplicity, we shall often drop the α_i, and "some u_i" will implicitely mean "some u_i and some α_i". We shall add the subscript λ when λ is to be emphasized. Consider the number

(1.5) $\sup\{\text{rank } d_w\phi(x,u_1...u_r;\alpha_1...\alpha_r)\}$,

the sup taken over all r>=1 and all possible $\alpha = \alpha_1,...,\alpha_r$, and all $u_1,...,u_r$, and where d_w indicates the differential of the function $\phi(x,.;\alpha)$ with respect to the variables $w = u_1...u_r$. It follows from the theorem in the Appendix (see also [1]) that $O(x,\lambda)$ is an (immersed) submanifold of M, of dimension equal to (1.5). Furthermore, if y is *any* given state in the orbit $O(x,\lambda)$, then the sup is achieved among those w such that $\phi(x,w) = y$. In connection with the results in [3], we point out in the Appendix that the connected component of $O(x,\lambda)$ which contains x is equal to the orbit of x under the normal subgroup of G_λ generated by all the transformations of the form

(1.6) $\phi(.,u,v;\alpha,-\alpha)$.

The main problem we address is, then, that of characterizing those systems (1.1) for which the sup in (1.5) is equal to n for some positive λ. The characterization turns out to be surprisingly simple: If and only if $\dim L_o(x) = n$. (For smooth but nonanalytic systems, the same proof shows sufficiency of this condition.) We give a proof in section 2, showing also that if the rank condition holds then there is a ngbd of x which is contained in $O(x,\lambda)$ for *all* λ sufficiently small. This generalizes a classical result for linear systems ([2]). Section 3 includes a global study of the one-dimensional case. It is shown there, in particular, that in general the "bad" frequencies --at which global accessibility is lost,-- constitute a discrete set, but that this set may be rather pathological. An Appendix sketches an "orbit theorem" which applies to both continuous and sampled systems.

2. A CHARACTERIZATION

Consider a given system as in (1.1), and associate to it, for a fixed $\lambda>0$, the "sampled" discrete time system

(2.1) $x(t+1) = f_\lambda(x(t),u(t))$, $t=0,1,...$

where $f_\lambda(x,v) := \phi(x,v;1)$. Note that (2.1) defines an "invertible" system ([1]), in the sense that each $f_\lambda(.,v)$ is a diffeomorphism (with inverse $f_\lambda(.,v;-1)$). We shall say that a pair (x,λ) is *normal* iff $O(x,\lambda)$ is open (i.e., has dimension n); this has an obvious interpretation in terms of "weak controllability" from x for (2.1). We shall use the notation N for the set of normal pairs (x,λ), and $N(x)$, $N(\lambda)$ for the sets of those λ and x respectively such that (x,λ) is in N. Since normality is characterized by the possibility of achieving full rank in (1.5), it follows that the complement of N is an analytic subset of $M\times\mathbf{R}_+$. Thus N is open, and analogous conclusions hold for each $N(x)$ and $N(\lambda)$.

Assume now that (x,λ) is normal. Thus the rank in (1.5) is n for suitable r, α_i, u_i. As a function of the u_i, then, the image of (1.4) contains an open subset V of M. In terms of (1.1), then, $A_t(x)$ contains V, where $t = \lambda(\sigma\alpha_i)$. Thus (1.1) is strongly accessible, and $\dim L_o(x) = n$. This suggests the following result:

(2.2) THEOREM. The following statements are equivalent, for any given state x:

(a) $\dim L_o(x) = n$;

(b) $N(x)$ is nonempty;

(c) $A(x,\lambda)$ has a nonempty interior for some $\lambda>0$;

(d) there exist an open set V and a $\Lambda>0$ such that $V \subseteq A(x,\lambda)$ for each $0<\lambda<=\Lambda$ --in particular, $(0,\Lambda] \subseteq N(x)$.

PROOF. The previous discussion shows that (b) implies (a). Since $O(x,\lambda)$ is a submanifold and $A(x,\lambda)$ is included in it, it follows that (c) implies (b). It only remains to establish that (a) implies (d).

Let $\dim L_o(x) = n$, so (1.1) is strongly accessible from x. From the results in [4] --or as a corollary to the theorem in the Appendix,-- it follows that there exist a state y, positive numbers $\underline{\tau}_1$, ..., $\underline{\tau}_r$, T, and a sequence of control values $u_1...u_r$ (in int(U), if desired,) such that

(2.3) $\mathbf{F}: \mathbf{R}_r(T) \to M$

has full rank differential at $\underline{\tau} = (\underline{\tau}_1,...,\underline{\tau}_r)$, where $\mathbf{R}_r(T) = \{(t_1,...,t_r) \mid \sigma t_i = T\}$, and where

(2.4) $\mathbf{F}(t_1,...,t_r)$:=

$[\exp(t_r f(.,u_r))_o ..._o \exp(t_1 f(.,u_1))]$ (x)

and $\mathbf{F}(\underline{\tau}) = y$. Thus there are ngbds W and B of $\underline{\tau}$ and y respectively such that $\mathbf{F}(W) = B$. Furthermore, there is a smooth map $\mathbf{G}$: B $\rightarrow$ W such that $\mathbf{F}_o\mathbf{G}$ = identity and $\mathbf{G}(y) = \underline{\tau}$. We shall prove that there are a ngbd $V \subseteq B$ of y and a $\Lambda>0$ such that $V \subseteq A(x,\lambda)$ for all $0<\lambda<=\Lambda$. Without loss of generality, take B to be (diffeomorphic to) a (closed) ball in $\mathbf{R}^n$, centered at y and included in the interior of another such ball B'. Pick a $\lambda_o>0$ which is less than all the $\underline{\tau}_i$; we may then assume that all t in W satisfy $\lambda_o < t_i$ for all i.

We shall now construct a family of continuous maps

(2.5) H_λ: B $\rightarrow$ B, $0<\lambda<\lambda_o$

such that (i) H_λ converges uniformly to the identity as $\lambda \rightarrow 0$, and (ii) $H_\lambda(B)\cap int(B) \subseteq A(x,\lambda)$ for all λ. It follows from (i) by a standard homotopy argument that there is an open ngbd V of y in B and a $\Lambda>0$ such that the image of H_λ includes V for all λ less than Λ, proving the theorem.

For each i=1,...,r-1, let γ_i: [0,1] $\rightarrow$ P be a path connecting u_{i+1} and u_i. Pick any t in W and any $\lambda<\lambda_o$. Let

(2.6) $k_i=k_i[t,\lambda]:= \max\{k \mid \lambda k <= t_i'\}$,

for i=1,...,r, where $t_i' := t_1+...+t_i$ for t in W. Denote k_o:=-1. Consider now the control $v=v[t,\lambda]$ which is equal to

(2.7) u_i on $[\lambda(k_{i-1}+1),\lambda k_i)$

for i=1,...,r, and

(2.8) $\gamma_i((t_i'/\lambda)-k_i)$ on $[\lambda k_i,\lambda(k_i+1))$,

for i=1,...,r-1. The control v is defined on $[0,\lambda k_r]$. Let v[t,0] be the control which assumes values u_i on the intervals $[t_{i-1}',t_i')$ and u_1 on $[0,t_1)$.

Note the following facts: (a) the measure of the set where $v[t,\lambda]$ differs from v[t,0], or is undefined, is less than $r\lambda$, and can therefore be made small uniformly on t, and further, the values of $v[t,\lambda]$ all belong to a fixed compact (union of the images of the γ_i), and (b) continuity of the γ_i implies that $v[t,\lambda]$ depends continuously on t, for fixed λ, provided that control functions are given a topology of uniform convergence (for any metric for P). Note that it is essential for (b) that the $v[t,\lambda]$ have all the same length, for any fixed λ (true because all the t are in $\mathbf{R}^r(T)$). Let $p[t,\lambda]$ be the state reached in (1.1) using control $v[t,\lambda]$.

It follows from (a) that $p[t,\lambda]$ converges uniformly to $p[t,0]$ = **F**(t) as $\lambda \to 0$. Since **F**(W) = B, we may assume (taking a smaller λ_o if necessary) that all the $p[t,\lambda]$ map into B'. From (b) we conclude that $p[t,\lambda]$ is continuous on t. Let q: B' $\to$ B be a retraction mapping B'-B into the boundary of B. The desired maps H_λ are then given by

(2.9) H_λ:= q(p[**G**(.),λ]).

This completes the proof. */

Since *N* is open, it also follows that, for any x for which dimL_o(x) = n, there are a Λ>0 and a ngbd V of x such that (z,λ) is normal for each 0<λ<=Λ and z in V. Thus (pick V connected) there are V and Λ such that V $\subseteq$ O(x,λ) for all such λ. We conclude that, for every connected compact K such that dimL_o(x) = n for all x in K, there is a Λ>0 such that K $\subseteq$ O(x,λ) for all x in K and 0<λ<=Λ (weak controllability on K).

3. THE ONE-DIMENSIONAL CASE

For this section, M = **R**. Although elementary, this case provides some feeling for the kinds of pathologies that may occur. We let *B*:= complement of *N* in M×$\mathbf{R}_+$.

Call a point z in M *invariant* if f(z,u) = 0 for all u (i.e., *L*(z) = {0}). In that case, both {x<z} and {x>z} are invariant under the dynamics (1.1), so each of them gives rise to a new system (1.1) with state space again (diffeomorphic to) **R**. Thus *B* is the union of the corresponding sets *B*', *B*" obtained from each of these, and of the set {(z,λ), λ>0}. We shall assume from now on, therefore, that (1.1) has no invariant points. Call *B* *trivial* if *B* is empty or it equals M×$\mathbf{R}_+$, and consider the λ-projection

(3.1) *C* = {λ | (x,λ)∈*B*, some x}.

These are the sampling periods for which (1.1) is not *globally* weakly controllable. We shall prove:

(3.2) THEOREM. (M=**R** and no invariant points.) If *B* is nontrivial, then *C* is a discrete subset of **R**.

In particular, the system is globally weakly controllable for all small enough sampling times (if nontrivial). Theorem (3.2) will follow from a more detailed study of the following sets. For any two (complete) vector fields X, Y, write

(3.3) *B*(X,Y):= {(x,λ) | exp(kλX)(x)=exp(kλY)(x), all integers k}.

Take two vector fields of the form X = f(.,u) and Y = f(.,u'), u,u' in U. Assume that (x,λ) is not in *B*(X,Y), so that ϕ_d(x,w;α) =/= ϕ_d(x,w',α) for some k>0, where w=u^k, w'=$(u')^k$, and α = sequence of k 1's or k (-1)'s. Since U is connected, the image of φ(x,.,α) contains a nontrivial interval. Thus dimO(x,λ) = 1, and x is not in *B*. Conversely, assume that (x,λ) belongs to all the *B*(X,Y) of the above form. Then O(x,λ) is included in the discrete set {exp(kλX)(x), k=integer}, for any fixed X, and so (x,λ) is in *B*. We conclude that

(3.4) *B* = ∩{*B*(X,Y), X=f(.,u),Y=f(.,v), u,v in U}.

It follows that it is sufficient to prove (3.2) for the sets of type *B*(X,Y).

(3.5) LEMMA. Assume that *B* is nontrivial. Then, for any X,Y as above, X(x)Y(x)>0 for all x.

PROOF. An x such that f(x,u)=0 for some u is an *equilibrium point*. Let x be any such point. Since x is invariant, f(x,v)=/=0 for some v in U. It follows that exp(λf(.,u))(x) = x =/= exp(λf(.,v)(x) for all λ>0, so (x,λ) is not in *B*, for any λ>0. We claim that there are no equilibrium points. Indeed, assume that f(x,u) = 0 for some (x,u), and replace U by a compact set which contains this u and is included in the closure of the original U. Pick any non-eq.pt. y<x in M, and let Z:= inf{z>y | z eq.pt.}. By compactness of U, z is itself an eq.pt., so z=/=y. Pick v,v' such that f(z,v)=0 and f(z,v')=/=0. By definition of z, f(a,v)=/=0 and f(a,v')=/=0 for all a in the interval [y,z). Compare the trajectories exp(tf(.,v))(y) and exp(tf(.,v'))(y). Assume first that f(y,v)>0. Then the v-trajectory converges to z, as $t \to \infty$, while the v'-trajectory does not. Same conclusion for f(y,v)<0 if one takes the limit as $t \to -\infty$ instead. It follows that, for every λ>0, (y,λ) is not in *B*(X,Y), for X=f(.,v) and Y=f(.,v'), and hence also for some v,v' in the original U. A similar argument holds if y>x. So the existence of an eq.pt. implies that *B* is empty, contradicting nontriviality. So f(x,u)=/=0 for each x and all u, and so (recall U is connected) the f(x,.) indeed have constant sign.

We are thus led to the study of the sets *B*(X,Y) with, say, X(x)>0 and Y(x)>0 for all x. Call such vector fields "positive". Conversely, any such pair {X,Y} gives rise to a system (1.1) with *B* = *B*(X,Y); this is a consequence of the following characterization, which is easy to obtain but very useful:

(3.6) LEMMA. Let X,Y be positive (analytic, complete) vector fields. There is then an analytic function g: $\mathbf{R} \to \mathbf{R}$, with derivative (dg/dt)(t)>-1 for all t and such that, for some diffeomorphism b(.),

(3.7) g(t+kλ)=g(t) for all integers k iff (b(t),λ)$\in$*B*(X,Y),

for any t in **R** and any λ>0. Further, g is constant iff X=Y. Conversely, given any analytic g with derivative bounded below, and any (strictly increasing) diffeomorphism b, there exists a system, and in particular there are positive X,Y, such that *B* = *B*(X,Y) and (3.7) holds.

PROOF. Let a(t):= exp(tX)(0), b(t):= exp(tY)(0), both analytic and strictly increasing. Let c:= a^{-1}, d(t):= c(b(t)). Define

(3.8) g(t):= d(t)-t.

Since c(.) and d(.) are increasing, g has derivative > -1. Let x be any state, and t_o:= b^{-1}(x). Note that exp(tX)(x) = a(c(x)+t), exp(tY)(x) = b(t_o+t). So these two trajectories are equal at t iff g(t_o+t) = g(t_o). Further, since g(0)=0, g is constant iff g=0, which happens iff a(t) = b(t) for all t. This proves the first part of the lemma. Conversely, assume given g and a diffeomorphism b. Multiplying g by a constant, we may assume that (dg/dt)(t) > -1/2 for all t. Let U = [0,1], and introduce for each u the function d_u(t) = ug(t)+t; note that the derivative of d_u is >1/2, for all u. Thus a_u(t):= b(d_u^{-1}(t)) is well defined (and analytic). We may then introduce f(x,u):= (da_u/dt)(a_u^{-1}(x)). Let X:= f(x,0), X_u:= f(x,u) for u>0, and Y=f(x,1). Reversing the previous argument shows that, for any u>0, $\exp(tX_u)(b(x)) = \exp(tX)(b(x))$ iff $g(x+t) = g(x)$ (independent of u). For this system, then, $B(X,X_u) = B(X,Y)$ for all u>0. Thus B = B(X,Y), and (3.7) holds.

Fix now a function g satisfying the properties in (3.6). We shall denote by *B*(g) the set of pairs (t,λ) with $\lambda>0$ such that $g(t+k\lambda) = g(t)$ for all integers k. Also, let *C*(g) be the projection of *B*(g) in the λ-coordinate.

(3.9) LEMMA. Let (t,λ), (t',λ') be in *B*(g). Then,

(3.10) $|g(t)-g(t')| <= |h\lambda+k\lambda'|$

for any integers h,k such that $h\lambda+k\lambda' =/= 0$.

PROOF. Consider any such h,k, and let $r:= |h\lambda+k\lambda'|$. For suitable integers a,b, $r = b\lambda'-a\lambda$. Without loss of generality, take $m:= g(t)-g(t')$ to be positive. Assume that r<m; there is then some integer s such that $t'-t-m < -sr < t'-t$. Let c:=as, d:=bs. We then have

(3.11) $0 < (t'+d\lambda')-(t+c\lambda) < m$,

and (by hypothesis)

(3.12) $g(t+c\lambda)-g(t'+d\lambda') = g(t)-g(t') = m$.

By the mean value theorem, this contradicts dg/dt>-1.

(3.13) COROLLARY. If λ and λ' are rationally independent, and if (t,λ), (t',λ') are in *B*(g), then g(t) = g(t').

(3.14) COROLLARY. Assume that *C*(g) has a limit point in **R**. Pick (t',λ') and (t'',λ'') in *B*(g). Then g(t') = g(t'').

PROOF. We shall use the following observation twice: Assume that $\{a_i\}$ is a converging sequence of distinct real numbers, and let f be any nonzero real number. There are then (i) a subsequence $\{a_j\}$ of $\{a_i\}$, and (ii) sequences $\{b_j\}$, $\{c_j\}$ of integers, such that the numbers $e_j:= b_ja_j + c_jf$ are all nonzero and $\{e_j\}$ converges to zero. [Proof: assume that $a_i \to a$. Let b_i, c_i be integers such that b_i=/=0 and $|b_ia+c_if| < 1/i$ (if a=0 use just c_i=0, otherwise consider the group generated by a and f). Now pick any a_j, $j=j_i$, such that the inequality is still satisfied and e_j=/=0.] Assume that $\{(t_n,\lambda_n)\} \subseteq B(g)$, with all λ_n distinct and converging to λ (which may be zero). Applying the above observation with $f:= \lambda'$, we conclude --for a subsequence of the (t_n,λ_n)-- that the $b_i\lambda_i+c_i\lambda'$ are are all nonzero and converge to 0. By lemma (3.9), $|g(t_i)-g(t')|$ also converges to 0. Taking in turn a subsequence of the $\{\lambda_i\}$, and $f:= \lambda''$, we can also conclude that $|g(t_i)-g(t'')|$

converges to zero. so g(t')=g(t''), as desired.

(3.15) PROPOSITION. If g is nonconstant then *C*(g) is discrete as a subset of **R**.

PROOF. Assume that there are infinitely many distinct λ_i <= K, with (t_i,λ_i) in *B*(g). By (3.14), there is a constant c such that $g(t_i+k\lambda_i) = c$ for all i and all integers k. Let $t_i' = t_i \bmod(\lambda_i)$ such that $t_i' \in [0,K]$. Thus $g(t_i') = c$ and $\{t_i'\}$ is bounded. Since g is nonconstant and analytic, there are only finitely many t_i'. But then there are infinitely many $t_i'' := t_i'+\lambda_i$ --since there are infinitely many λ_i-- and these are also bounded, with $g(t_i'')=c$. This again contradicts nonconstancy of g.

Theorem (3.2) now follows from (3.15) and (3.6). Actually, we can prove somewhat more. Since *B* is analytic, each subset with constant λ also is, so *B* is the union of a discrete set and a union of lines $L_i := \{(x,\lambda_i), x \text{ in } M\}$. So g is periodic with period λ_i, for all i. Since periods form a subgroup, g nonconstant implies that the λ_i are integer multiples of some fixed $\lambda>0$. So the nondiscrete part of *B* is of the form

(3.16) $\{(x,k\lambda), x \text{ in } M, k = \text{integer}\}$.

The set *C*(g) may be rather complicated. Consider the following example. Take a sequence of numbers $\{a_n\}$ such that

(3.17) $\sigma\,(a_n)^{-1} < 1/\pi$, and

(3.18) $\cos(\pi x/a_n) > 1-2^{-n}$ if $x \in [-n,n]$.

Now let $g_n(x) := \cos(\pi x/a_n)$ and g:= (infinite) product of the g_n. This product is well defined because there is by (3.18) normal convergence on compacts, and g is indeed analytic. Further, consider its derivative

(3.19) $g' = \sigma\,(g/g_n).g_n'$.

Since $|g/g_n|<1$ and $|g_n'|<\pi/a_n$, also $|g'|<1$. The zeroes of g are those of its factors, i.e., the union of the sets

(3.20) $\{(t_n+ka_n), k = \text{integer}\}$,

where $t_n := a_n/2$. So all a_n are in *C*(g). If (t,λ) is in *B*(g) and λ is not rationally dependent with some a_n, then (3.13) says that g(t) = 0, so λ= some a_n, a contradiction. Thus *C*(g) contains all the a_n and no other rationally independent numbers. For constructing sequences $\{a_n\}$ as above, consider the following argument: Let $\{b_n\}$ be such that $\cos(\pi x/a) > 1-2^{-n}$ whenever x is in [-n,n] and $a>b_n$ (just let b_n be such that $\cos(\pi n/b_n) > 1-2^{-n}$). Now pick any sequence $\{a_n\}$ satisfying (3.17) and such that $a_n>b_n$ for all n. Note that, in particular, one could choose the a_n to be rationally independent.

APPENDIX

We provide here a (fairly straightforward) generalization of the theorem, given in [5] for continuous time systems (see also [1] for a discrete time version) establishing that orbits (weak reachability sets) are submanifolds in a natural way. The following objects are assumed given ("smooth" = infinitely differentiable or analytic, in all that follows):

(A.1) a smooth manifold M,

(A.2) a set A,

(A.3) an idempotent map $-: A \to A$, and

(A.4) for each a in A, (i) a manifold U_a, such that $U_a = U_{-a}$, (ii) an open subset D_a of $M \times U_a$, and (iii) a smooth $g_a: D_a \to M$, such that:

(A.5) $(g_a(x,u),u)$ is in D_{-a} if (x,u) is in D_a, and

(A.6) $g_{-a}(g_a(x,u),u) = x$ for all such (x,u).

The following examples motivate the above: (a) Continuous time (not necessarily complete) systems; here A is $U \times \{1,-1\}$ (U = control value set in (1.1)), "-" sends (u,α) to $(u,-\alpha)$, $U_a = \mathbf{R}_+$, and $g_{(u,\alpha)}(x,t) = \exp(\alpha t f(.,u))(x)$, with D_a = domain of g_a. (b) Invertible discrete time systems: $x(t+1) = f(x(t),u(t))$, $u(t)$ in a manifold, $f(.,u)$ invertible for each u; here $A = \{1,-1\}$, with obvious "-", all U_a = control value manifold, and $g_a(x,u) := f(x,u)$ for $a=1$ and $=[f^{-1}(.,u)](x)$ for $a=-1$. (c) Zero-time control for continuous time systems; here A is the set of all those sequences $(a_1,...,a_r)$ of elements of the A in (a) such that $\sigma\alpha_i = 0$, with $-(a_1,...,a_r) := (-a_r,...,-a_1)$, and the obvious choices in (A.4). (d) An analogous zero-time discrete example.

Let B be the free semigroup on A. If $b = (a_1,...,a_r)$, $-b$ is by definition the sequence $(-a_r,...,-a_1)$; U_b is the product of the corresponding U_a, $a=a_i$, and $g_b: M \times U_b \to M$ is the induced (partial) action. For the empty word #, $U_\#$ has one element and $g_\#$ is the identity. When b is clear from the context, we omit the corresponding subscript. We shall use a concatenation notation to exhibit sequences in U_b. The sets D_b are defined inductively as follows:

(A.7) $(x,uw) \in D_{ab}$ iff $(x,u) \in D_a$ and $(g_a(x,u),w) \in D_b$,

for u in U_a and w in U_b. These open sets are the domains of the maps g_b. For $w = u_1...u_r$ in U_b, let $\tilde{w} := u_r...u_1$ (in U_{-b}). Then $(g_b(x,w),\tilde{w})$ is in D_{-b} whenever (x,w) is in D_b, and

(A.8) $g_{-b}(g_b(x,w),\tilde{w}) = x$.

The main object of study is

(A.9) $O(x) := \{z \mid g_b(x,w) = z, \text{ some } b,w\}$.

We introduce the following notations for differentials. Let b be in B, w in U_b, b = (b',c,b") any factorization, and w = w'vw" a corresponding factorization for w. Then $d_c g_b(x,w)$ is by definition the differential of $g_b(x,w'(.)w")$ with respect to the variables in U_c, evaluated at the point v. When c=b, we often omit the subscript and write just $dg_b(x,w)$ or even dg(x,w). Differentials with respect to x will be written d_x. The main result is:

(A.10) THEOREM. Let x be in M. Then O(x) has a unique structure of smooth (immersed) submanifold of M such that (i) the (restricted) maps g_b: $(O(x)\times U_b)\cap D_b \to O(x)$ are all smooth for b in B, and (ii) for any y in O(x), the dimension of O(x) is equal to

(A.11) $r(x,y) = \sup \{\text{rank } dg_b(x,w)\}$,

where the sup is taken over all b and w such that (x,w) is in D_b and $g_b(x,w) = y$.

(A.12) REMARK. For the systems considered in part 2, the control set was not required to be a manifold, but the above theorem can still be applied to conclude that the orbits (denoted as $O(x,\lambda)$ there) are submanifolds. Indeed, note first that P may be assumed to be connected (because U is), and let $O'(x,\lambda)$ [resp., $O''(x,\lambda)$] be the orbit obtained when P [resp., int(U)] is used as the control value set. The above theorem gives that both of these orbits are submanifolds. Say that $O'(x,\lambda)$ has dimension k. Pick any z in $O'(x,\lambda)$. Since $O'(x,\lambda) = O'(z,\lambda)$, there is a control sequence over P such that the rank in (A.11) --i.e., that in (1.5),-- is k, for some $dg_b(z,w)$. By analyticity --and P being connected,-- there is also a control v with values in int(U) giving rank $dg_b(z,v) = k$. So $O''(z,\lambda)$ contains a ngbd (relative to $O'(z,\lambda)$) of z, say V (this uses part (i) of A.10). Now assume that z is in also in $O(x,\lambda)$. since $U \subseteq \text{clos}(\text{int}(U))$, z is also in the closure of $O''(x,\lambda)$ with respect to $O'(z,\lambda)$. Pick a V as above; then V intersects $O''(x,\lambda)$, and it follows that z is in the latter. In fact, the construction given below results in the same submanifold structures for both $O'(x,\lambda)$ and $O''(x,\lambda)$. We conclude that $O(x,\lambda) = O''(x,\lambda)$, so the former can be given the submanifold structure of $O''(x,\lambda)$.

(A.13) REMARK. We prove now the statement in section 2 concerning the connected component $C_\lambda(x)$ of $O(x,\lambda)$ which contains x. Consider first the following more general situation, for any setup as in (A.1)-(A.6) for which the U_a are all equal, say to U, are connected, and all maps are total ($D_a = M\times U$). Let

A' be the set of all pairs c = (b,-b), for b in B, and define manifolds V_c as follows. Let b = da, with a in A. Then $V_c := U_d \times U^2$. For b=#, $V_\#$ has a single point. Now let $g_c(x,wuv) := g_{(b,-b)}(x,wuv\tilde{w})$, all total maps. A new "system" is obtained, which satisfies the assumptions (A.1)-(A.6); let O'(x) be the corresponding orbit of x. Note that O'(x) is connected, because the images of the maps g_c, c in A', are all connected and they all contain $g_c(x,wuu)$ = x, and the same holds for iterates of the g_c. So O'(x) is included in the connected component C(x) of O(x) at x. Further, both manifolds have the same dimension. Indeed, pick a b,w such that $g_b(x,w)$ = x and $dg_b(x,w)$ has full rank. Thus the tangent space to O(x) at x is generated by the image of $dg_b(x,w)$, i.e., by the images of the differentials $d_a g_b(x,w)$, for all factorizations b = (e',a,e) and corresponding factorizations w = v'uv, with a in A. For any such factorization, write c:= (-e,-a,a,e) --this belongs to A'-- and consider w':= $\tilde{v}$ uuv. Then, $d_a g_c(x,w')$ is equal to $d_a g(x,w)$. But the image of the former is in the tangent space of O'(x) at x, so O(x) and O'(x) have the same dimension. We may repeat the argument at each z in C(x), concluding that O'(z) is a ngbd of z in C(z) = C(x). A connectivity argument gives then that indeed O'(x) = C(x), as wanted --the normal subgroup generated by the transformations in (1.6) gives the transformations indexed by A'.

In order to prove the theorem, we shall need some more notation. For b in B, m_b (or just m) will be the map $g_b(x,.)$, with domain $L_b := \{w \mid (x,w) \in D_b\}$. We also make the convention that a statement like "$g_b(x,w)$ = y" will mean "(x,w) is in D_b and g(x,w) = y".

Fix an x in M, and let O = O(x). We establish first that r(x,y) = r(x,z) for any y,z in O. Pick b,c in B and w,w' such that $g_b(x,w)$ = y, $g_c(x,w)$ = z, and rank[dg(x,w)] = r(x,y). Introduce e:= (b,-b,c) and v:= $w\tilde{w}w'$. Since $g(x,w\tilde{w})$ = x, it follows that g(x,v) = z. So rank[dg(x,v)] <= r(x,z). Let F:= $g_{(-b,c)}(.,\tilde{w}w')$ --with domain the open set $\{x \mid (x,\tilde{w}w') \in D_{(-b,c)}\}$. Since $d_xF(p)$ is a linear isomorphism for all p in the domain of F, it follows that r(x,y) = rank[dg(x,w)] = rank[$d_xF(y)_o dg(x,w)$] = rank[$d_b g(x,v)$] <= rank[dg(x,v)] <= r(x,z). A symmetric argument concludes the equality. Let r be the common value of the r(x,y).

Consider now the set S of all triples s:= (b,Q,h), where b is in B and:

(A.14) Q is an r-dimensional embedded submanifold of L_b,

(A.15) $m_b|Q: Q \to M$ is injective and has differential of constant rank r,

(A.16) h: $Q \to \mathbf{R}^r$ is a diffeomorphism with an open subset h(Q).

Fix one such s, and consider the set m(Q); this is a subset of O. The bijection m|Q induces a canonical manifold structure on this set, for which both m|Q and $\phi := h_o(m|Q)^{-1}$ are diffeomorphisms (and such that ϕ is a chart). We now prove that, for this structure, (a) the inclusion i: m(Q) $\rightarrow$ M has injective differential at every point, and (b) for any smooth structure C for O for which the theorem holds, the subset m(Q) is open --relative to C,-- and the identity map provides a diffeomorphism between the two structures.

The inclusion i factors as $m_o j_o (m|Q)^{-1}$, where j is the embedding of Q in L_b. Property (a) follows from the corresponding properties for its factors (for m, the properties hold on Q, which is sufficient). We now prove (b). Consider m as a map from L_b into O (with structure C); this map is smooth (property (i) in theorem: m is a restriction of g). So m|Q is also smooth into (O,C). Since the latter is a submanifold of M, and rank[dm|Q] = r (constant) as a map into M, this rank is also r as a map into (O,C). But this submanifold has dimension r, by part (ii) of the theorem. Thus m(Q) is indeed open rel to C, and m|Q is a diffeomorphism between (m(Q),C) and Q, so (b) follows.

We now prove that the family of all such charts (m|Q,ϕ) defines a smooth (r-dimensional) manifold structure on O, and that property (i) holds. It will then follow from (a) above that this structure makes O into a submanifold of M, and the uniqueness statement follows from (b).

The sets m(Q) cover O: Pick any y in O and let b,w be such that $g_b(x,w) = y$ and dm(w) = dg(x,w) has rank r. Thus dm has maximal rank at w, so there is an r-dimensional embedded submanifold Q of L_b, containing w, such that (A.14), (A.15) are satisfied; replacing Q if necessary by an open subset of Q, a suitable h can be found for (A.16).

Compatibility: Pick any two charts (m(Q),ϕ) and (m'(P),β) corresponding to (b,Q,h) and (c,P,k) respectively. Let V:= m(Q)$\cap$m'(P). We need to establish (a) that $\phi(V)$ is open in $\phi(m(Q))$, and (b) that $\beta_o \phi^{-1}$ is smooth on V. Pick an arbitrary y in V; thus there are w,w' in Q,P with y = m(w) = m'(w'). Let e:= (b,-c,c) in B, and take v:= $w\tilde{w}'w'$. Note that rank[dm(v)] >= rank[$d_c g(x,v)$] = rank[dg(x,w')] = r. Since dm(v) always has rank at most r, it has maximal rank at this v. So there is an open subset Z of L_e which contains v and such that $m_e(Z)$ is an r-dimensional embedded submanifold of M. Introduce the open set W [resp., W'] consisting of those u in L_b [resp., L_c] such that $u\tilde{w}'w'$ [resp., $w\tilde{w}'u$] is in Z. Then w is in W and w' is in W'. Let P':= P$\cap$W', Q':= Q$\cap$W. Since Q is an embedded submanifold of L_b, and W is open in

L_b, also Q' is open in Q, and similarly for P,P'. Note that m|Q' maps into $m_e(Z)$, and is injective with differential of constant rank r. Thus m establishes a diffeomorphism between Q' and an open subset C of $m_e(Z)$. Similarly for m'|P' and an open D in $m_e(Z)$. Note that $C\cap D \subseteq V$. Also, w',w are in P', Q' respectively, so y is in $C\cap D$. Since m|Q is injective, $(m|Q)^{-1}(C\cap D) = (m|Q')^{-1}(C\cap D)$, which is then open in Q, because $C\cap D$ is open in C. So $\phi(C\cap D)$ is open in $h(Q) = \phi(m(Q))$. Thus $\phi(z)$ has a ngbd included in $\phi(m(Q))$, and (a) follows. To prove (b), note that ϕ maps $C\cap D$ (embedded submanifold of $m_e(Z)$) diffeomorphically onto $\phi(C\cap D)$, which is open in h(Q) and contains $\phi(y)$. A similar statement holds for β. So $\beta_o\phi^{-1}$ gives a diffeomorphism between $\phi(C\cap D)$ and $\beta(C\cap D)$, and (b) follows.

Property (i) of the theorem: We first establish that the maps m_b are smooth. Pick w in L_b, z = g(x,w). Since r(x,z) = r, there are a c and a w' in L_c with g(x,w') = z and dg(x,w') = r. Let e:= (b,-c,c) and v:= $w\tilde{w}$'w'. It will suffice to prove that m_e is smooth on some ngbd of v, because m_b is (in a suitable ngbd of w) a restriction of m_e. Note that r >= rank[m(v)] >= rank[d_cg(x,v)] = rank[dg(x,w')] = r (this uses that m($w\tilde{w}$') = x). So m achieves maximal rank at v. There is then a chart C of L_e, centered at v, and diffeomorphic to a cube in $\mathbf{R}^s\times\mathbf{R}^r$, such that, if Q is the embedded submanifold corresponding to the factor $\mathbf{R}^r$, then rank[dm(v)] is constantly r on Q and m_e is injective on Q. Let h give the corresponding diffeomorphism of Q with $\mathbf{R}^r$. Then (e,Q,h) gives rise to a chart (m(N),ϕ). So m_e|C is then the composition of the projection onto Q and of m|Q, and is therefore smooth. To prove now that g_c is smooth as a map into O, pick any (z,w) in D_c, z in O. Let (b,Q,h) give a chart around z. For (g,v) in a ngbd of (z,w) in $(O\times U_c)\cap D_c$,

(A.17) $g_c(y,v) = m_{(b,c)}((m|Q)^{-1}(y),v)$,

so g_c is indeed smooth. This completes the proof of the theorem.

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