On the Length of Inputs Necessary in Order to Identify a Deterministic Linear System

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Abstract—The family of m-input, n-dimensional linear systems can be globally identified with a generic input sequence of length $2mn$. This bound is the best possible. A best bound is provided also for a corresponding local identification problem.

I. INTRODUCTION

A result of the author, valid for a large class of discrete-time systems ([4]; see also [5] for the continuous-time analog) states that, given a parameterized family of systems, a generic long-enough input sequence is sufficient for I/O behavior identification, the length depending on the general form of the family. Although fairly constructive, this general result does not give useful bounds for the lengths of the needed sequences. For linear systems, we give such bounds in this note. The techniques used here are completely different from those used in order to prove the abstract result.

The results presented below give the smallest lengths needed for identification experiments. These results complement those presented by other authors—see, for example, [1]–[3]—who construct minimal realizations once the corresponding experiment has been performed.

II. DEFINITIONS

An (n, m, p)-*system* S (over the real numbers) is an n-dimensional, m-input, p-output, discrete-time, constant linear system; S is uniquely determined by a triple (F, G, H) of matrices (F is n by n, G is n by m, H is p by n). The input sequence $w = u_1 \cdots u_r$ (u_i in $\mathbf{R}^m$) *identifies* (n, m, p)-*systems* iff the following property holds for any two such systems S_i.

1) If S_1 and S_2 give the same zero-state output sequence $y_1, \cdots, y_r$, when w is applied, then S_1 and S_2 have the same zero-state I/O behavior.

When 1) is valid for any two *canonical* (= reachable and observable)

(n, m, p)-systems, w *identifies canonical* (n, m, p)-*systems*. The number $a(n, m, p)$ [respectively, $c(n, m, p)$] will denote the smallest integer r for which there is a sequence of length r which identifies (n, m, p)-systems (respectively, canonical (n, m, p)-systems). The number $a'(n, m, p)$ [respectively, $c'(n, m, p)$] will be the smallest integer r such that a *generic* input sequence w of length r satisfies 1) for all (m, n, p)-systems S_1, S_2 (respectively, canonical such systems). (In other words, such that there is an open dense set D in $(R^m)^r$ with the property that 1) is valid using any w in D.)

The above refer to global identification problems in the sense that a single ("universal") input sequence is to identify all (canonical) systems of a given dimension. A more "local" type of question deals with the following situation. Given a class as above of systems to be identified, find open dense sets of systems T_i, and an input sequence w_i for each such set, such that w_i distinguishes between the systems in the corresponding T_i. The latter deals with those situations in which the *a priori* information used for identification is not just of a discrete nature (bounds on dimensions of realizations), but also includes knowing that the system (or I/O map) in question belongs to a given generic set. Thus, almost all systems—except those in a "thin" set—will be identifiable by the action of the input sequence w_i.

An example of such a local question is posed below. The class of those m-input, p-output linear time-invariant I/O maps $f = (f_1, \cdots, f_p)'$ for which f_i has minimal realization of dimension n_i is denoted by $C(n_1, \cdots, n_p; m)$—this class is naturally viewed as a topological space; for concreteness, it will be seen as a subspace of the set of sequences of Markov matrices $(A_1, \cdots, A_{2n})$, n = maximum of the n_i. Each I/O map gives rise to such a unique sequence, and conversely, it follows from elementary realization theory that each sequence of this length gives rise to at most one map in $C(n_1, \cdots, n_p; m)$. The topology considered will then be the natural one inherited from the space of all matrix sequences, i.e., R^{2nmp}. For each set $n_1, \cdots, n_p$, $b(n_1, \cdots, n_p; m)$ will denote the smallest integer r such that the following property is true.

2) There is a family (T_i, w_i), each T_i open dense in $C(n_1, \cdots, n_p; m)$, and each w_i of length r, such that: a) the T_i cover $C(n_1, \cdots, n_p; m)$, and b) for each (T, w) and each S_1, S_2 in T, 1) holds.

III. RESULTS

Before stating the results, we consider the following problem for fixed n, m. To find those r and those inputs sequences w of length r such that if S is an $(n, m, 1)$-system giving zero output when w is applied, then S has zero I/O behavior. The I/O behavior of such a system is uniquely determined by a transfer matrix $W = (P_1/Q, \cdots, P_m/Q)$ where Q is a monic polynomial of degree n and $P_1, \cdots, P_m$ are polynomials of degree at most $n - 1$. (If S is not canonical, there may, of course, be more than one such representation of degree n.) The zero-output condition establishes a set of r linear equations $K_{r,n}(w)x = 0$ in the nm coefficients of the P_i. When r is less than nm, there are then nonzero solutions in the P_i. Choosing Q arbitrary, we conclude that if r is less than nm, there is, for *any* given w, some S with nonzero transfer matrix, but giving zero output for w. Moreover, S can be taken to be *canonical* and with Q a *product of linear factors*. Indeed, given any solution $P_1, \cdots, P_m$ of the above equations, it is enough to take for W the transfer matrix obtained by dividing by a Q which has all roots real and different from the roots of the P_i.

Theorem 1: $a(n, m, p) = a'(n, m, p) = c(n, m, p) = c'(n, m, p) = 2nm$.

Theorem 2: $b(n_1, \cdots, n_p; m) = (m+1)\max\{n_1, \cdots, n_p\}$.

To prove Theorem 1, we note first that the quantities are independent of p, so that it is enough to prove the case $p = 1$; indeed, each system with p outputs gives rise to p single-output systems, and conversely, a single-output system gives rise to a p-output system by simply repeating the same output p times, so that dealing with a p-output problem becomes equivalent to treating p simultaneous identification scalar problems. Further, it is clearly sufficient to show now that: 1) $a'(n, m, 1) \leq 2nm$, and 2) $2nm \leq c(n, m, 1)$. If 2) is false, there is a sequence w of length r less than $2nm$ identifying canonical $(n, m, 1)$-systems; thus, for

every two such systems S_1, S_2, their *difference* $S: = S_1 - S_2$ (parallel connection, subtracting outputs) is a $(2n, m, 1)$-system for which outputs being zero for w implies zero I/O behavior. This gives a contradiction: by the remarks in the previous paragraph, there is some S canonical such that the latter property is not satisfied, and if Q is a product of linear factors, this S can be written as $S_1 - S_2$ for some canonical $(n, m, 1)$-systems S_i (partial fractions expansion). To show 1), and in fact to obtain an *explicit description* of the set of identifying inputs, it is enough to note that $\det(K_{2nm,2n}(w))$ is not identically zero (this is easily seen using $w = e_1 0' e_2 0' \cdots e_m 0'$, $t = 2n - 1$, $e_i = i$th canonical vector in R^m).

To prove Theorem 2, we note first that $b(n; m) \leq b(n_1, \cdots, n_p; m)$ for $n = \max\{n_1, \cdots, n_p\}$ by an argument as above. Further, $n(m + 1) \leq b(n; m)$: up to isomorphism (= change of basis in the state space), canonical $(n, m, 1)$-systems admit a global parameterization (given by the observability canonical form) with exactly $n(m + 1)$ parameters, and an input sequence of length r maps these parameters polynomially into a sequence of r output values; this map being one-to-one in an open set forces the inequality. Consider now the set T of those observable $(n, m, 1)$-systems (F, G, H) for which (F, g) is reachable, where g = first column of G. The input sequence $w = e_1 0^{2n-1} e_2 0^{n-1} \cdots e_m 0^{n-1}$ identifies systems in T up to isomorphism. (Briefly, using the observable canonical form, the first $2n$ outputs permit determining F, g, and the other $(m-1)n$ outputs give the remaining columns of G.) Consider now the general case $p = 1$. Let S be a canonical system; observability then forces F to be cyclic. Thus (see [8, p. 42]), there is some u in R^m such that (F, Gu) is reachable; in fact, *almost every* u satisfies this property. Given such a u, a change of coordinates in R^m reduces this to the previous case $(F, g) = $ reachable. Thus, the theorem holds for $p = 1$ where the (T, w) in 2) of Section II are constructed as above (in terms of canonical realizations, T_u: = all those (F, G, H) for which (F, Gu) is reachable for each u). The case of arbitrary p now follows easily: given $f = (f_1, \cdots, f_p)$, there is a *common u* as above such that all f_i are reachable with respect to u, and the corresponding w (with n the maximum of the n_i) identifies every f_i simultaneously.

IV. REMARKS

The first of the above proofs rests upon an understanding of the rather interesting mapping of pairs of systems (S_1, S_2) into their difference $S_1 - S_2$. In terms of varieties of systems, this map is dominating in the sense of algebraic geometry, although it is a map between varieties of the same dimension $2n(m + p)$ and the fiber over the zero system has dimension $n(m + p)$.

It should be noted that the set of pairs (S, w) such that the input sequence w uniquely determines S among systems of the given dimension, i.e., such that the corresponding I/O pair (w, y) is identifiable in the sense of [1], [3], etc., is generic in the set of pairs [(n, m, p)-system, sequence of length r] precisely when r is at least $n(m + 1)$. Thus, in a rather precise sense it is true that almost any sequence of length at least $n(m + 1)$ identifies almost all systems. It is rather surprising, however, that $c(n, m, 1)$ is greater than $n(m + 1)$ in view of the fact that canonical $(n, m, 1)$-systems admit a global canonical form with $n(m + 1)$ parameters. This shows that the difficulties that one encounters here do not lie merely in the nonexistence of canonical forms.

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