480 IEEE TRANSACTIONS ON ACOUSTICS, SPEECH, AND SIGNAL PROCESSING, VOL. ASSP-26, NO. 5, OCTOBER 1978

On First-Order Equations for Multidimensional Filters

EDUARDO D. SONTAG

Abstract—A construction is given to obtain first-order equation representations of a multidimensional filter, whose dimension is of the order of the degree of the transfer function.

I. Introduction

Let k be any field, m, t, integers. Let $k[z_1, z_2]$ [respectively, $k(z_1, z_2)$] denote the ring of polynomials (respectively, field of rational functions) in two indeterminates z_1, z_2. Let the degree of a polynomial $p = p(z_1, z_2)$ in z_i be denoted by $d_i p$. The *degree of* p is then $dp := (d_1 p, d_2 p)$. A rational function $w(z_1, z_2) = p(z_1, z_2)\, q(z_1, z_2)^{-1}$ is *causal* iff $d_i p \leqslant d_i q$, $i = 1,2$ and $q = z_1^{d_1 q} z_2^{d_2 q} + \cdots$. Denote then $d_i w := d_i q$, $dw := (d_1 w, d_2 w)$. A *transfer function* is a matrix

$$W(z_1, z_2) = (w_{ij}(z_1, z_2)) \in k(z_1, z_2)^{t \times m}$$

where each w_{ij} is causal.

A *system* Σ *of dimension* (n_1, n_2) is given by four matrices (F, G, H, J) where $F \in k^{n \times n}$, $G \in k^{n \times m}$, $H \in k^{t \times n}$, $J \in k^{t \times t}$ $(n := n_1 + n_2)$.

Introduce the polynomial matrix

$$A_{n_1, n_2} := \begin{bmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{bmatrix}$$

where I_r is the identity matrix of dimension r. Then, for each Σ of dimension (n_1, n_2), let

$$W_\Sigma(z_1, z_2) := H(A_{n_1, n_2} - F)^{-1} G.$$

It is easy to verify that W_Σ is a transfer function. When $m = t = 1$, $d_i W_\Sigma \leqslant n_i$. The *realization problem* that we wish to consider is: "given a transfer function W, find Σ with $W_\Sigma = W$." The motivation for this problem is the following. A transfer function describes a recursive "northeast causal" two-dimensional filter, and a system realization Σ corresponds to a set of first-order equations realizing the corresponding filter; specifically, denoting

$$F := \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = (H_1, H_2), \tag{1.1}$$

then Σ corresponds to a system of equations

$$\begin{aligned} x_1(h+1, k) &= F_{11} x_1(h, k) + F_{12} x_2(h, k) + G_1 u(h, k) \\ x_2(h, k+1) &= F_{21} x_1(h, k) + F_{22} x_2(h, k) + G_2 u(h, k) \\ y(h, k) &= H_1 x_1(h, k) + H_2 x_2(h, k) + J u(h, k) \end{aligned} \tag{1.2}$$

where $x_i(\cdot, \cdot): Z \times Z \to k^{n_i}$.

Generalizations to transfer functions in three or more variables $z_1, \cdots, z_r$ are straightforward and will not be discussed here.

Models of the above type were independently suggested by Roesser [8] and Sontag [9]; the latter reference also explains how the same mathematical problem appears in modeling neutral delay-differential systems.

In the present paper we expand in detail the realization method outlined in [9, disgression (5.6)]. The results were presented at the Amherst Workshop on Algebraic System Theory, held at the University of Massachusetts, June 14–19, 1976.

II. Main Results

Theorem (2.1): Every $W(z_1, z_2)$ *has a realization.*

Theorem (2.2): Let $m = t = 1$, $dW = (n_1, n_2)$. *Then*

a) there is a realization Σ of dimension $(n_1, 2n_2)$;

b) *if* $q(z_1, z_2) = q_1(z_1)\, q_2(z_2)$, *there is a realization* Σ *of dimension* (n_1, n_2).

Analogous conclusions hold, of course, by reversing the roles of z_1 and z_2.

Note that b) deals with what are usually called "separable" transfer functions. We shall prove both of the above theorems as corollaries of a general construction, outlined in the rest of this section. It is this *construction*, rather than the theorems themselves, which we consider to be the main contribution of this work. In the next section we shall present a conjecture for the case when k is algebraically closed, and in the last

Manuscript received March 16, 1977; revised July 25, 1977. This work was supported in part by U.S. Army Grant DA-ARO-D-31-124-72-G114 through the Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611.

The author was with the Department of Mathematics, Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611. He is now with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

0096-3518/78/1000-0480$00.75 © 1978 IEEE

section we indicate the connection between certain realizations of "separable" transfer functions and the theory of "recognizable" power series as developed by Fliess [3].

Let $k[(z_2)]$ denote the set of causal transfer functions in the variable z_2 only, i.e., the set of all $p(z_2)q(z_2)^{-1}$ with deg $q \geqslant$ deg p. We introduce the notion of a

system R over $k[(z_2)]$ of dimension r:

this is just a 4-tuple of matrices

$$R = (A(z_2), B(z_2), C(z_2), D(z_2)),$$

where A, B, C, and D are matrices over $k[(z_2)]$ of dimensions r by r, r by m, t by r, and t by m, respectively. (This is a particular case of a *system over a commutative ring*; the general concept is studied in [10].)

A system R as above induces a transfer function

$$W_R(z_1, z_2) := C(z_2)(z_1 I_r - A(z_2))^{-1}B(z_2) + D(z_2).$$

Conversely, each $W(z_1, z_2)$ has an R-realization, i.e., is of the form W_R for some R. Indeed, given W there is some r such that we may write

$$\begin{aligned} W(z_1, z_2) &= (P_r(z_2)z_1^r + \cdots + P_0(z_2)) \\ &\quad \times (q_r(z_2)z_1^r + \cdots + q_0(z_2))^{-1} \\ &= P_r(z_2)q_r(z_2)^{-1} + (T_{r-1}(z_2)z_1^{r-1} + \cdots + T_0(z_2)) \\ &\quad \times (z_1^r + v_{r-1}(z_2)z_1^{r-1} + \cdots + v_0(z_2))^{-1} \end{aligned}$$

where the P_j are polynomial matrices, q_j are polynomials, and

$$v_j(z_2) := q_j(z_2)q_r(z_2)^{-1}$$

$$T_j(z_2) := (P_j(z_2) - P_r(z_2)v_j(z_2))q_r(z_2)^{-1}.$$

Then $W = W_R$ where $R = (A, B, C, D)$ is the rt-dimensional system over $k[(z_2)]$ given by

$$A := \begin{bmatrix} 0 & 0 \cdots 0 & -v_0 I_t \\ I_t & 0 \cdots 0 & -v_1 I_t \\ 0 & I_t \cdots 0 & -v_2 I_t \\ \vdots & \vdots & \vdots \\ 0 \quad 0 & 0 & 0 \\ 0 \quad 0 & I_t & -v_{r-1} I_t \end{bmatrix}, \qquad B := \begin{bmatrix} T_0 \\ \vdots \\ T_{r-1} \end{bmatrix}, \tag{2.3}$$

$$C := [0 \cdots 0 I_t], \qquad D := P_r q_r^{-1}.$$

The proof of Theorems (2.1) and (2.2) will be completed upon finding, for each R, a suitable Σ with $W_\Sigma = W_R$. We study then the possible passages from W_R to W_Σ. To study this, we note that for any $\Sigma = (F, G, H)$, $W_\Sigma = W_R$ when R is defined by [using the notations (1.1)]

$$A(z_2) = F_{11} + F_{12}(z_2 I_{n_1} - F_{22})^{-1}F_{21}$$

$$B(z_2) = G_1 + F_{12}(z_2 I_{n_1} - F_{22})^{-1}G_2$$

$$C(z_2) = H_1 + H_2(z_2 I_{n_1} - F_{22})^{-1}F_{21}$$

$$D(z_2) = J + H_2(z_2 I_{n_1} - F_{22})^{-1}G_2.$$

Equivalently, denoting

$$\hat{W}_R(z_2) = \begin{bmatrix} F_{12} \\ H_2 \end{bmatrix} (z_2 I_{n_1} - F_{22})^{-1}(F_{21}, G_2),$$

we see that

$$\begin{bmatrix} A(z_2) & B(z_2) \\ C(z_2) & D(z_2) \end{bmatrix} = \begin{bmatrix} F_{11} & G_1 \\ H_1 & J \end{bmatrix} + \hat{W}_R(z_2). \tag{2.4}$$

Therefore, if we begin with $R = (A, B, C, D)$, finding Σ is equivalent to obtaining a decomposition (2.4). In other words, we must solve a "minimal realization problem" for the transfer matrix (in z_2) in the left-hand side of (2.4). It is well known from linear system theory (see, for instance, [2, p. 219]) that the minimal possible dimension n_2 is given by

$$n_2 = \text{M.d. } \hat{W}_R(z_2) = \text{McMillan degree of } \hat{W}_R(z_2)$$

where

M.d. $\hat{W}_R(z_2)$ = degree of the least common denominator of all minors of $\hat{W}_R(z_2)$.

The problem of finding Σ can thus be decomposed into two parts: first find R, for instance (but not necessarily!), via (2.3), obtaining n_1; then construct Σ with n_2 = M.d. $\hat{W}_R(z_2)$. There are well-known algorithms for such a construction [2, p. 235 ff.]. It must be noted that different R can lead to different $\hat{W}_R(z_2)$, so both n_1 and n_2 depend on the construction. Moreover, one may instead first "extract" z_2 and *then* z_1, obtaining a different result. Although the method can be expected to give realizations of rather low dimensions, it is clear that much more research is needed before the situation becomes well understood.

In any case, the bounds of Theorem (2.2) are easily proved from the above construction via (2.3) and (2.4). Indeed,

$$\hat{W}_R(z_2) = \begin{bmatrix} & \vdots & -v_0 & T_0 \\ & \vdots & \vdots & \vdots \\ \text{constants} & \vdots & -v_{r-1} & T_{r-1} \\ & \vdots & 1 & P_r q_r^{-1} \end{bmatrix};$$

since $-v_i T_j - T_i(-v_j) = P_i q_j / q_r^2$, it follows that q_r^2 is a common denominator for all minors, so

$$\text{M.d. } \hat{W}_R(z_2) \leqslant 2 \deg q_r = 2n_2,$$

proving a). And b) is clear since it means that all v_j are constant.

Example (2.5): Let

$$W(z_1, z_2) = \frac{z_1 + z_2}{z_1 z_2 - 1},$$

k = real numbers. Then an R-realization is

$$z_1 x = z_2^{-1}x + (1 + z_2^{-2})u$$

$$y = x + z_2^{-1}u$$

and a (1, 2)-dimensional Σ-realization is

$$x_1(h + 1, k) = x_2(h, k) + u(h, k)$$

$$x_2(h, k + 1) = x_1(h, k) + x_3(h, k)$$

$$x_3(h, k + 1) = u(h, k)$$

$$y = x_1 + x_3.$$

III. A Conjecture

Conjecture (3.1): Let k be algebraically closed (for instance, k = complex numbers). Let $W(z_1, z_2)$ be a scalar (i.e., $m = t = 1$) transfer function of degree (n_1, n_2). Then there exists a realization Σ of W such that dim $\Sigma = (n_1, n_2)$.

When n_1 (or n_2) = 1, this conjecture is true, since one may always factor $T_0 = (P_0 q_r^{-1}) \cdot (q_0 q_r^{-1})$ as a product of causal z_2-transfer functions, so $R = (-q_0 q_r^{-1}, P_0 q_r^{-1}, q_0 q_r^{-1}, P_r q_r^{-1})$ gives rise to a $\hat{W}_R(z_2)$ with McMillan degree n_2.

Example (3.2): Consider again the W in (2.5), but this time let k = complex numbers. A direct calculation shows that a Σ of dimension (1, 1) realizes W if and only if

$$x_1(h + 1, k) = a x_2(h, k) + b u(h, k)$$

$$x_2(h, k + 1) = a^{-1} x_1(h, k) + c u(h, k)$$

$$y = b^{-1} x_1 + c^{-1} x_2$$

and $(abc)^2 + 1 = 0$.

Remarks (3.3):

a) *There can be no rational procedure for finding realizations of minimal dimensions.* Indeed, such a procedure, involving only additions and multiplications, would give the same result independently of the field of definition. But Examples (2.5) and (3.2) show that minimality is field-dependent. This situation is completely at variance with the ordinary linear system case.

b) If the conjecture is true, one may expect also, by an algebraic-geometric argument, generically finitely many realizations under the group $GL(n_1) \times GL(n_2)$ acting in the obvious way.

IV. Recognizable Transfer Functions

A transfer function is *recognizable* iff every entry has the form $p(z_1, z_2)q(z_1)^{-1}q(z_2)^{-1}$. We now show how some results of [4] can be translated into the context of two-dimensional filters by looking at another type of first-order difference equations.

We consider in this section *systems* $\Gamma = (F_1, F_2, G, H)$ *of dimension n* given by equations of type

$$x(h, k) = F_1 x(h - 1, k) + F_2 x(h, k - 1) + Gu(h, k)$$

$$y = Hx$$

where $x(h, k)$ is in k^n and $F_1 F_2 = F_2 F_1$.

Theorem (4.1): $W(z_1, z_2)$ has a Γ-realization if and only if $W(z_1, z_2)$ is recognizable. Furthermore, expanding

$$W(z_1, z_2) = \sum_{i,j \geq 0} A_{ij} z_1^{-i} z_2^{-j},$$

the dimension of a minimal Γ-realization is equal to the rank n of the block Hankel matrix $H(W)$ which has rows and columns indexed by the pairs (i, j) and the (i, j)th block is

$$\binom{i+j}{i}^{-1} A_{ij}.$$

Any two realizations of this dimension n are equal except for a change of basis in k^n.

Proof: Easy consequence of [4]. The proof by Fliess provides an explicit Γ-realization of dimension n = rank $H(W)$.

Remarks (4.2):

a) Representations of "separable," i.e., recognizable, transfer functions by systems somewhat similar to our Γ were given by Attasi [1] and Fornasini and Marchesini [5], [6], adding a term of the form $-F_1 F_2 x(h - 1, k - 1)$ to the right side. This minor variation of the above can be treated also via the theory of recognizable series.

b) In general, there are Σ-realizations of dimension $(n_1 + n_2)$, much less than the dimensions of possible Γ-realizations. For instance, a straightforward modification of an example of [6] is $W(z_1, z_2) = (z_1 + z_2 + 2)z_1^{-1} z_2^{-1}$, which has a (1, 1)-dimensional Σ-realization [by Theorem (2.2)], but rank $H(W) = 4$.

V. Final Remarks

The bound given in Theorem (2.2a) improves considerably that given by [5], [6], in which the dimension of Σ is of the order of $n_1 \cdot n_2$ rather than $n_1 + n_2$.

In an interesting recent paper, Kung *et al.* [7] have independently arrived at Theorem (2.2).

Acknowledgment

The author wishes to express his gratitude to Dr. E. W. Kamen for introducing him to the operator study of delay-differential systems, and hence, indirectly, to the present topic, and to Prof. M.A. Arbib and Dr. E. G. Manes for the opportunity of presenting his results at the Amherst Workshop.

References

[1] S. Attasi, "Systèmes linéaires homogènes à deux indices," IRIA Lab. Rep. 31, 1973.

[2] C. T. Chen, *Introduction to Linear System Theory.* New York: Holt, Reinhart and Winston, 1970.

[3] M. Fliess, "Séries reconnaissables, rationnelles et algébriques," *Bull. Sci. Math.*, vol. 94, pp. 231-239, 1970.

[4] —, "Matrices de Hankel," *J. Math. Pures. Appl.*, vol. 53, pp. 197-224, 1974.

[5] E. Fornasini and G. Marchesini, "Algebraic realization theory of two-dimensional filters," in *Variable Structure Systems*, A. Ruberti and R. Mohler, Eds., Springer L.N. Econ. and Math. Systems.

[6] —, "State-space realization theory of two-dimensional filters," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 484-492, 1976.

[7] S. K. Kung, B. Levy, M. Morf, and T. Kailath, "New results in 2-D systems theory, 2-D state-space models–Realization and the notions of controllability, observability and minimality," in *Proc. IEEE (Special Issue on Multidimensional Systems)*, June 1977.

[8] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 1-10, 1975.

[9] E. D. Sontag, "On linear systems and noncommutative rings," *Math. Syst. Theory*, vol. 9, no. 4, pp. 327-344, 1975.

[10] —, "Linear systems over commutative rings: A survey," *Ricerche di Automatica*, vol. 7, no. 1, pp. 1-34, 1976.