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Sensor and Actuator Scheduling in Bilinear Dynamical Networks

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***Abstract*— In this paper, we investigate the problem of finding a sparse sensor and actuator (S/A) schedule that minimizes the approximation error between the input-output behavior of the fully sensed/actuated bilinear system and the system with the scheduling. The quality of this approximation is measured by an $\mathcal{H}_2$-like metric, which is defined for a bilinear (time-varying) system with S/A scheduling based on the discrete Laplace transform of its Volterra kernels. First, we discuss the difficulties of designing S/A schedules for bilinear systems, which prevented us from finding a polynomial time algorithm for solving the problem. We then propose a polynomial-time S/A scheduling heuristic that selects a fraction of sensors and node actuators at each time step while maintaining a small approximation error between the input-output behavior of the fully sensed/actuated system and the one with S/A scheduling in this $\mathcal{H}_2$-based sense. Numerical experiments illustrate the good approximation quality of our proposed methods.**

I. Introduction

Over the past few years, the controllability and observability properties of complex networks have been subjects of intense study in both the control and network science communities [1]–[16]. Moreover, the desire to perform control/estimation using a sparse set of actuators/sensors spans various application domains, ranging from infrastructure networks (e.g., sanitation and power networks) to real-time decision support systems (e.g., pandemic mitigation and soft target protection). This interest stems from the need to steer or observe the state of large-scale interconnected networks *with as few actuators/sensors as possible*, due to issues related to data deluge, cost, and energy depletion. For example, estimating the whole state of a complex network using fewer measurement units will reduce the cost of monitoring the network for systemic failures.

Previous studies have been mainly focused on solving sensor/actuator (S/A) scheduling problems for linear time-invariant networks. In [7], the authors propose an actuator scheduling with bounded performance loss for a generic set of performance metrics. The paper presents performance guarantees for their scheduling by bounding the difference between the fully actuated system and the one with scheduling. In [8], the authors approach the joint S/A scheduling problem and present a separation result for designing S/A schedules.

Going beyond linear systems. In several applications, from biology to engineering systems, linear models are inadequate to explain the behavior of large-scale dynamical networks. On the other hand, due to the complexity of nonlinear dynamical networks, the tools available to study them are not as well developed as those for linear counterparts. In this context, bilinear dynamical networks represent an important class of nonlinear systems: rich enough to describe practical systems, while also having substantial theory developed for analyzing them. Examples include several problems in electrical networks, surface vehicles, and immunology (see [17]–[19]).

In our previous work [20], [21], we use the $\mathcal{H}_2$-norm to propose a robustness metric for bilinear networks. In [20], we look into protecting a network subject to attacks on its interconnections. Specifically, we use the $\mathcal{H}_2$-norm to measure how much a given edge disturbance affects the performance of the network and to protect the edges with the largest influence. In [21], we investigate how the presence of multiplicative disturbances changes the behavior of $\mathcal{H}_2$-based node centralities in a bilinear network. In that work, we show that there is a magnitude range for multiplicative disturbances where the bilinear network is still stable but behaves qualitatively differently from any linear approximation.

The $\mathcal{H}_2$-norm for bilinear systems is a particularly popular performance metric for applications ranging from complexity reduction to robustness analysis [20], [22]–[25], since it holds many of the useful properties observed in the linear case. In particular, in [22] the author shows how the $\mathcal{H}_2$-norm relates the L_2 norm of the inputs to the L_∞ norm of the outputs of a given bilinear system.

Our contributions. We propose a heuristic for designing a S/A scheduling for bilinear networks. We formulate an alternating convex optimization heuristic for sensor and node actuator scheduling for arbitrary time horizons and for average active sensors and actuators per time step. The proposed algorithm presents good simulation results while also being tractable in polynomial time. We start Section II by properly defining our network's dynamics and the optimization problem of finding an optimal schedule. We then look at how to use the Volterra kernels of our system to define an $\mathcal{H}_2$-like metric for our network with scheduling. In Section III, we go through the characteristics that make this problem hard to solve and propose a simplified, more tractable version. Then, in Section IV we simulate a 5-node bilinear network and compare our results to a brute-force complexity reduction solution and greedy algorithm.

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II. Preliminary Definitions

A. Notations and Assumptions

Throughout the paper, the set of natural numbers less than or equal to n is denoted by $\mathbb{N}_{\leq n}$. Given a finite discrete set $\mathcal{S}$, the operator $|\mathcal{S}|$ is the number of elements in that set. The set $\{0, 1\}$ is the binary set and $\{0, 1\}^n$ is the set of all vectors of dimension n whose elements are in the binary set. The vector and matrix of all ones of dimensions n and $n \times n$ are given by $\mathbb{1}_n$ and $\mathbb{J}_n$, respectively, where the dimension might be omitted if it is clear from the context. For an arbitrary matrix or vector A let $\|A\|_0$ be the ℓ_0 quasi norm, that is, the number of non-zero elements in that matrix or vector.

B. Theoretical Background

We define a bilinear network as a tuple $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \omega, \mathcal{V}_a, \mathcal{V}_o, \mathcal{E}_a\}$ where $\mathcal{V} = \mathbb{N}_{\leq n}$ is the set of n nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges between nodes in $\mathcal{V}$, and $\omega : \mathcal{E} \to \mathbb{R}_+$ is an edge weight function. The subset of actuated nodes is denoted by $\mathcal{V}_a \subseteq \mathcal{V}$, and $\mathcal{V}_o \subseteq \mathcal{V}$ is the subset of observed (measured) nodes. Similarly, $\mathcal{E}_a \subseteq \mathcal{E}$ denotes the subset of actuated edges. If the weight function ω is unspecified, then the network is unweighted ($w(e) = 1$ for all $e \in \mathcal{E}$). In this paper, we assume $|\mathcal{V}| = n$, $|\mathcal{V}_a| = m_v$, $|\mathcal{V}_o| = o$, and $|\mathcal{E}_a| = m_e$. We consider a class of stable distributed networks that consist of a group of n subsystems/nodes (i.e., $\mathcal{V}$) whose state variables x_i, node inputs v_i, edge inputs w_ℓ, and node outputs y_i are scalar. The nodes' discrete-time dynamics with time step Δt evolve with time according to

$$\begin{aligned} x_i[k+1] &= \big(1 - \Delta t\, \gamma_i\big)\, x_i[k] + \\ & \quad \Delta t \sum_{e=(i,j)\in\mathcal{E}} \omega(e)\big(x_j[k] - x_i[k]\big) + \\ & \quad \Delta t \sum_{\ell=(i,j)\in\mathcal{E}_a} w_\ell[k]\, x_j[k] \,+\, \Delta t\, b_i v[k], \end{aligned} \tag{1}$$

where $w_\ell[k]$ and $v_i[k]$ are edge and node inputs, respectively, and $v = [v_1, \ldots, v_{m_e}]$. The discretization time of our system is given by small $\Delta t > 0$, and the range for the damping ratio γ_i is $0 < \gamma_i < 1/\Delta t$. The row-vector $b_i \in \mathbb{R}^{1\times m_v}$ selects which inputs affect a given node i and by how much. We now can rewrite the dynamics of the whole network as

$$\Sigma : \begin{cases} x[k+1] = N_0 x[k] + \sum_{\ell\in\mathcal{E}_a} w_\ell[k] N_\ell\, x[k] + Bv[k] \\ y[k] = Cx[k] \end{cases} \tag{2}$$

where $N_0 = I - \Delta t(L + \mathrm{diag}([\gamma_1, \cdots, \gamma_n]))$ is stable[1], and L is the Laplacian of the underlying linear graph defined by $\bar{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}, w\}$. The rows of matrix $B \in \mathbb{R}^{n\times m_v}$ are the vectors $\Delta t\, b_i$'s. Input matrix B captures the additive input structure in the network, and $C \in \mathbb{R}^{o\times n}$ captures the output structure (i.e., which node to measure) in the network. In (2), $\ell \in \mathcal{E}_a$ is an abuse of notation signifying that for each $\ell = (i, j) \in \mathcal{E}_a$ we define $N_\ell = N_{ij} := \Delta t\big(e_i e_j^\top + e_j e_i^\top\big)$ where e_i is the i-th column of the identity matrix.

[1]We consider $\Delta t > 0$ small enough to make N_0 stable. An alternative, albeit slightly less intuitive, way to enforce the stability of the consensus dynamics is to study the system with a grounded Laplacian, where we do not assume self-loops but take $N_0 = I - \Delta t(L + (1/n)\mathbb{J})$. For our simulations, we adopt grounded Laplacian instead of self-loops; however, the analysis is the same, the only difference being a design choice for the network.

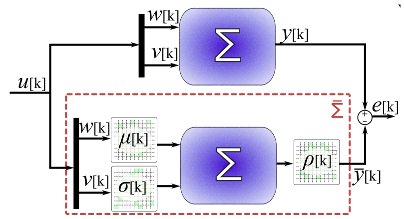

Fig. 1: Block diagram of the error system for Optimization Problem (4).

To perform the S/A scheduling in this network, we need to consider the possibility of switching a given S/A on or off in the system dynamics. For this purpose, we introduce three new sets of time-varying parameters, $\rho : \mathbb{N}_{\leq T} \to \{0, 1\}^o$, $\sigma : \mathbb{N}_{\leq T} \to \{0, 1\}^{m_v}$ and $\mu : \mathbb{N}_{\leq T} \to \{0, 1\}^{m_e}$ which indicate, respectively which sensors, node inputs and edge inputs are active at a given time-step k. Let $R[k] = \mathrm{diag}(\rho[k])$ and $S[k] = \mathrm{diag}(\sigma[k])$, then the system dynamics with time-varying S/A schedule is given by

$$\bar{\Sigma} : \begin{cases} x[k+1] = N_0 x[k] + \sum_{\ell\in\mathcal{E}_a} \mu_\ell[k] N_\ell x[k] w_\ell[k] + BS[k]v[k] \\ y[k] = R[k]Cx[k] \end{cases} \tag{3}$$

As can be seen, the dynamics of a bilinear network with scheduling is given by a specific type of time-varying bilinear system with a constant drift matrix.

C. Preliminary Formulation of the Scheduling Problem

In our S/A scheduling problem, we are interested in finding a scheduling (ρ, σ, μ) for a given time horizon T such that the network with scheduling $\bar{\Sigma}$ remains close to the fully sensed and actuated one (Σ) according to some metric. The problem under consideration is depicted in Fig. 1 and the optimization formulation of the S/A scheduling problem for bilinear networks can be written as:

$$\begin{aligned} \min_{\rho,\ \sigma,\ \mu} \quad & f(\Sigma - \bar{\Sigma}(\rho,\ \sigma,\ \mu)) \\ \text{s.t.} \quad & \|\rho\|_0 \leq \bar{\rho} \\ & \|\sigma\|_0 \leq \bar{\sigma} \\ & \|\mu\|_0 \leq \bar{\mu} \\ \rho \in \{0,1\}^{o\times T}&,\ \sigma \in \{0,1\}^{m_v\times T},\ \mu \in \{0,1\}^{m_e\times T} \end{aligned} \tag{4}$$

where $f(\cdot)$ is some performance metric for the system, and $\bar{\rho}$, $\bar{\sigma}$ and $\bar{\mu}$ are respectively the desired average number of sensors, node actuators and edge actuators per time step. For the simulation, we convert the functions ρ, σ and μ to matrices whose rows correspond to a given sensor/actuator and whose columns correspond to a specific time step.

The problem as formulated here is very hard to solve even before picking a performance metric $f(\cdot)$ due to its combinatorial nature. For the rest of this section, we will focus on defining a performance metric analogous to the $\mathcal{H}_2$-norm that is applicable to network (3). In Section III, we reformulate problem (4). We then propose an approximation algorithm to obtain an efficient S/A scheduling for a bilinear network.

D. Volterra Expansion of Discrete-Time Bilinear Systems

To extend the notions of Gramians and the $\mathcal{H}_2$-norm from time-invariant bilinear systems to the time-variant case given by (3), we first look into its general solution for zero initial conditions. We start by defining $\bar{N}[k] = [N_1\mu_1[k],\ N_2\mu_2[k],\ \ldots,\ N_{m_e}\mu_{m_e}[k]]$ and

$$
\begin{aligned}
\bar{x}_1[k+1] &= N_0\bar{x}_1[k] + BS[k]v[k]\\
\bar{x}_2[k+1] &= N_0\bar{x}_2[k] + \bar{N}[k]w[k]\otimes\bar{x}_1[k]\\
&\cdots\\
\bar{x}_i[k+1] &= N_0\bar{x}_i[k] + \bar{N}[k]w[k]\otimes\bar{x}_{i-1}[k]\\
&\cdots
\end{aligned}
\tag{5}
$$

These systems relate to the original one as follows:

$$
\sum_{i=1}^{\infty}\bar{x}_i[k+1] = N_0\sum_{i=1}^{\infty}\bar{x}_i[k] + \bar{N}[k]w[k]\otimes\sum_{i=1}^{\infty}\bar{x}_i[k] + BS[k]v[k], \tag{6}
$$

assuming the infinite sum $\bar{x}[k] = \sum_{i=1}^{\infty}\bar{x}_i[k]$ converges. We can then write the solution of each of the kernels as

$$
\begin{aligned}
\bar{x}_1[k] &= \sum_{\ell_1=0}^{k-1} N_0^{k-1-\ell_1}BS[\ell_1]v[\ell_1]\\
\bar{x}_2[k] &= \sum_{\ell_2=0}^{k-1} N_0^{k-1-\ell_2}\bar{N}[\ell_2]w[\ell_2]\otimes\bar{x}_1[\ell_2]\\
&\cdots\\
\bar{x}_i[k] &= \sum_{\ell_i=0}^{k-1} N_0^{k-1-\ell_i}\bar{N}[\ell_i]w[\ell_i]\otimes\bar{x}_{i-1}[\ell_i]\\
&\cdots
\end{aligned}
\tag{7}
$$

With this, consider the following adaptation of Lemma 1 from [26] to our time-varying bilinear system (3):

Lemma 1. *For $k \geq 1$, the output $y[k]$ of* (3) *can be expressed as $y[k] = R[k]C\bar{x}[k] = R[k]C\sum_{i=1}^{k}\bar{x}_i[k]$ or, alternatively*

$$
\begin{aligned}
\bar{y}[k] = &\sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_2=1}^{\ell_3-1}\sum_{\ell_1=0}^{\ell_2-1}\sum_{j_i,\ldots,j_2=1}^{m_e} R[k]CN_0^{k-1-\ell_i}\times\\
&\times N_{j_i}\mu_{j_i}[\ell_i]\ldots N_0^{\ell_3-1-\ell_2}N_{j_2}\mu_{j_2}[\ell_2]N_0^{\ell_2-1-\ell_1}\times\\
&\times BS[\ell_1]u[\ell_1]u_{j_2}[\ell_2]\ldots u_{j_i}[\ell_i].
\end{aligned}
\tag{8}
$$

The proof of this lemma is analogous to the one presented in [26] and a natural consequence of results from the Volterra series for general time-variant nonlinear systems [27]. The Volterra series expansion for general time varying bilinear systems is also studied in the literature [28] and can be easily adapted for our specific case. With this result we can define the Volterra kernels of (3) as

$$
\begin{aligned}
h_i[k,\ell_i,\ldots,\ell_2,\ell_1] = &\sum_{j_i,\ldots,j_2=1}^{m_e} R[k]CN_0^{k-1-\ell_i}N_{j_i}\mu_{j_i}[\ell_i]\cdots\times\\
&\times N_0^{\ell_3-1-\ell_2}N_{j_2}\mu_{j_2}[\ell_2]N_0^{\ell_2-1-\ell_1}BS[\ell_1],
\end{aligned}
\tag{9}
$$

which allow us to extend the definitions of Gramians and $\mathcal{H}_2$-norm from time-invariant bilinear systems to the particular case of equation (3).

E. Reachability Gramian

In this subsection, we start with the following definition.

Definition 1 (Reachability Gramian)**.** *For a bilinear system as defined in* (3) *define*

$$
\begin{aligned}
\bar{P}_1[k,\ell_1] &= N_0^{k-1-\ell_1}BS[\ell_1]\\
\bar{P}_i[k,\ell_i,\ldots,\ell_1] &= N_0^{k-1-\ell_i}\big[\mu_1[\ell_i]N_1\bar{P}_{i-1}\ \cdots\\
&\quad\cdots\ \mu_{m_e}[\ell_i]N_m\bar{P}_{i-1}\big],
\end{aligned}
\tag{10}
$$

then its Reachability Gramian *is defined as*

$$
\bar{P}[k] = \sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_1=0}^{\ell_2-1}\bar{P}_i\bar{P}_i^{\top}. \tag{11}
$$

Note that, unlike the time-invariant case, there is no need to assure convergence of the given Gramians, since we do not work with the steady-state case and for any finite k, $\bar{P}[k]$ is well defined. Even so, we need to make sure the system is stable, meaning that $\limsup_{k\to\infty}\|\bar{P}[k]\|$ is finite. To do that we adapt the condition present in [26] for the proper definition of $\bar{P}$ defined for (3).

Theorem 2. *The reachability Gramian* (11) *is stable if*

- N_0 *is stable;*
- $\mu[k]\sqrt{\|\sum_{k=1}^{m}N_kN_k^{\top}\|} < \sqrt{1-\beta}/\alpha$, $\forall k \leq T$ *where* β *and* α *are such that* $\|N_0^i\| \leq \alpha\beta^i$

The proof of this theorem is extensive and just slightly different from the solution already presented in [26], so we omit it here due to space limitations.

F. An $\mathcal{H}_2$-like Metric for Time-Varying Bilinear Systems

The $\mathcal{H}_2$-norm of the time-invariant bilinear system (2) is defined as a function of the multivariate discrete Laplace transform of its Volterra Kernel (H_i) as [29]

$$
\begin{aligned}
\|\Sigma\|_{\mathcal{H}_2}^2 = &\Bigg(\sum_{i=1}^{\infty}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{1}{2\pi}\operatorname{trace}(H_i^*(e^{j\omega_1},\ldots,e^{j\omega_i})\\
&\times H_i(e^{j\omega_1},\ldots,e^{j\omega_i}))d\omega_1\ldots d\omega_i\Bigg)^{1/2},
\end{aligned}
\tag{12}
$$

which can be shown to relate to the Volterra kernels of the system through Palancherel's Theorem. In [29] the authors show the relationship between the $\mathcal{H}_2$-norm of a time-invariant bilinear system (2) and the steady state reachability Gramian (P) as $\|\Sigma\|_{\mathcal{H}_2}^2 = \operatorname{trace}(CPC^{\top})$. The $\mathcal{H}_2$-norm defined this way has been shown to share a lot of similarities with its linear counterpart. For example, for both cases, it measures the trace of the covariance matrix of the output when subject to white Gaussian noise inputs ($\mathcal{N}(0,I)$) [20], [22]. Furthermore, for both deterministic linear and bilinear systems the $\mathcal{H}_2$-norm relates the ℓ_2-norm of the input to the ℓ_∞-norm of the output, albeit in slightly different ways.

The $\mathcal{H}_2$-norm characterization for bilinear time-invariant systems does not hold immediately for (3) and for the reachability Gramian defined in (11) due to the time-variant parameters of the system. We still, however, use the relationship between the defined Volterra kernels (9) and the Reachability Gramian to motivate the following definition.

Definition 2 ($\mathcal{H}_2$-like Metric for Bilinear Networks with S/A Scheduling)**.** *For the bilinear network* (3) *with scheduling* *with time horizon T, we define the $\mathcal{H}_2$-like metric as follows*

$$\|\bar{\Sigma}\|^2_{\mathcal{H}_2} = \frac{1}{T}\operatorname{trace}\left(\sum_{k=1}^{T}\sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_1=0}^{\ell_2-1}\|h_i(k,\ell_i,\ldots,\ell_1)\|_F\right), \tag{13}$$

where h_i are given by (9).

For simplicity, we refer to the operator in Definition 2 as the $\mathcal{H}_2$-norm of the bilinear system with scheduling, since it is also based on the Volterra kernels of the system. Moreover, if the scheduling is constant (that is, not varying in time) it simplifies to the known definition of the $\mathcal{H}_2$-norm for the resulting time invariant bilinear system. However, we make the distinction here that all the previously known properties of the $\mathcal{H}_2$-norm for linear and bilinear time-invariant systems do not necessarily hold for this definition. Even so, with Definition 2, we can state the following theorem.

Theorem 3. *The $\mathcal{H}_2$-norm of a bilinear network with scheduling defined in* (13) *can be computed as*

$$\|\bar{\Sigma}\|^2_{\mathcal{H}_2} = \frac{1}{T}\operatorname{trace}\left(\sum_{k=1}^{T} R[k]C\bar{P}[k]C^\top R[k]\right), \tag{14}$$

where $\bar{P}[k]$ is given by (11).

Defined this way, the $\mathcal{H}_2$-norm is an average of the "instantaneous $\mathcal{H}_2$-norm" of the system in the time horizon. Notice that since we assume the conditions of Theorem 2 are satisfied, $\bar{P}[k]$ is stable, which means this average is finite for any finite horizon T. Under these conditions, the $\mathcal{H}_2$-norm, as defined in 2 measures an average lower bound on the controllability energy of the system. Although the previous properties of the $\mathcal{H}_2$-norm of time-invariant bilinear systems do not necessarily hold for this definition, in the simulation section, we investigate its empirical effectiveness as a performance metric.

III. The Scheduling Problem Reformulated

In this section, using the performance metric (13) given by Definition 2, we obtain a tractable formulation for problem (4). As stated in (12), the $\mathcal{H}_2$-norm can be computed as a function of the Reachability Gramian; making use of this fact, we can state the following theorem.

Theorem 4. *The cost function $f(\Sigma-\bar{\Sigma}(\rho,\ \sigma,\ \mu)) = \|\Sigma-\bar{\Sigma}(\rho,\ \sigma,\ \mu)\|^2_{\mathcal{H}_2}$ of a bilinear network under a scheduling with time horizon T can be written in the form below.*

$$\begin{aligned} f(\Sigma-\bar{\Sigma}(\rho,\ \sigma,\ \mu)) &= \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_1=0}^{\ell_2-1}\times \\ &\times\sum_{j_T=1}^{o}\sum_{j_i,\ldots,j_2}^{m_e}\sum_{j_1=1}^{m_v}\Big(1-\rho_{j_T}[k]\mu_{j_i}[\ell_i]\ldots \\ &\times\mu_{j_2}[\ell_2]\sigma_{j_1}[\ell_1]\Big)\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}) \\ &= f(\Sigma) - f\left(\bar{\Sigma}(\rho,\sigma,\mu)\right), \end{aligned} \tag{15}$$

where $\mathbf{l_i} = [\ell_i,\ \ldots,\ \ell_1]$, $\mathbf{j_i} = [j_i,\ \ldots, j_1]$ and $\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}) \geq 0$ is given by

$$\begin{aligned}\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}) &= (c_{j_T}\otimes c_{j_T})(N_0^{k-1-\ell_i}\otimes N_0^{k-1-\ell_i})(N_{j_i}\otimes N_{j_i})\ldots \\ &\times(N_0^{k-1-\ell_2}\otimes N_0^{k-1-\ell_2})(N_{j_2}\otimes N_{j_2})\times \\ &\times(N_0^{k-1-\ell_1}\otimes N_0^{k-1-\ell_1})(b_{j_1}\otimes b_{j_1}).\end{aligned} \tag{16}$$

Due to space limitations, the proof is omitted, but we provide its general idea. The result of Theorem 4 can be proved by first writing the equations of the error system given in Fig. 1. After computing the controllability Gramian for the error system, we use the results of Theorem 3 to compute its $\mathcal{H}_2$-error. After some algebraic manipulation one can reach the result given by Theorem 4.

This is derived directly from our choice of performance metric and allows us to separate the metric for the difference of the systems as the difference between the metrics of each system. Using Theorem 4 and relaxing our integer constraints to a continuous interval between zero and one, the original optimization can be simplified to:

$$\begin{aligned} \max_{\rho,\ \sigma,\ \mu}\quad & \frac{1}{T}\sum\Big(\rho_{j_T}\mu_{j_i}\ldots\mu_{j_2}\sigma_{j_1}\Big)\kappa_i \\ \text{s.t.}\quad & \|\rho\|_0 \leq \bar{\rho} \\ & \|\sigma\|_0 \leq \bar{\sigma} \\ & \|\mu\|_0 \leq \bar{\mu} \\ & \rho\in[0,1]^{o\times T},\ \sigma\in[0,1]^{m_v\times T},\ \mu\in[0,1]^{m_e\times T} \end{aligned} \tag{17}$$

where the limits of the summations and function arguments from (15) are omitted for simplicity. It is easy to see that such problem consists in the maximization of a posynomial subject to posynomial constraints. This, however, is not a geometric programming problem and cannot be solved simply by convexifying the cost function and constraints. Furthermore, the exponentially increasing number of terms makes any greedy approach impractical.

Due to the hardness of the original problem, we look into the simplified decoupled case, specifically finding a scheduling for sensors and node actuators for a given fixed set of edge actuators.

A. Alternating Optimization Heuristic

We start by pointing out that for a fixed set of edge actuators, the cost of (17) becomes linear in the set of sensors for a given set of node actuators and vice-versa. This motivates the use of an alternating optimization algorithm for each scheduling independently. Formally, for a given set of node and edge actuators, we can write the (relaxed) problem of finding an optimal sensor schedule by solving the following optimization problem

$$\begin{aligned} \max_{\rho}\quad & \frac{1}{T}\sum_{k=1}^{T}\sum_{j_T=1}^{o}\rho_{j_T}[k]\kappa_{sen}(k,j_T) \\ \text{s.t.}\quad & \|\rho\|_0\leq\bar{\rho} \\ & \rho\in[0,1]^{o\times T} \end{aligned} \tag{18}$$

and similarly, for a given set of edge actuators and sensors, we can find the optimal set of node actuators by solving

$$\begin{array}{ll} \max\limits_{\sigma} & \frac{1}{T}\sum\limits_{\ell_1=0}^{T-1}\sum\limits_{j_1=1}^{m_v}\sigma_{j_1}[\ell_1]\kappa_{act}(\ell_1,j_1) \\ \text{s.t.} & \|\sigma\|_0 \le \bar{\sigma} \\ & \sigma \in [0,1]^{m_v\times T} \end{array} . \tag{19}$$

where

$$\kappa_{\text{sen}} = \sum_{\ell_1=0}^{T-1}\sum_{j_1=1}^{m_v}\sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_2=1}^{\ell_3-1}\sum_{j_i,\dots,j_2}^{m_e}\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}), \tag{20}$$

and

$$\kappa_{\text{act}} = \sum_{k=1}^{T}\sum_{j_T=1}^{o}\sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_2=1}^{\ell_3-1}\sum_{j_i,\dots,j_2}^{m_e}\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}), \tag{21}$$

To evaluate the effectiveness of this heuristic, we will compare it to a couple of other heuristics:

1) Time Invariant Selection: For this approach, we disregard the time-variant aspects of (3), performing, instead, an $\mathcal{H}_2$-based complexity reduction of the original system (2). This helps illustrate the advantages of considering the more complex time-variant case. Notice that if ρ and σ are constants in time and μ is given, we can further (17) to

$$\begin{array}{ll} \max\limits_{\rho,\ \sigma} & \sum\limits_{j_T=1}^{o}\sum\limits_{j_1=1}^{m_v}\rho_{j_T}\sigma_{j_1}\tilde{\kappa}(j_T,j_1) \\ \text{s.t.} & \|\rho\|_0 \le \bar{\rho} \\ & \|\sigma\|_0 \le \bar{\sigma} \\ & \rho\in[0,1]^o,\ \sigma\in[0,1]^{m_v} \end{array} \tag{22}$$

where

$$\tilde{\kappa}(j_T,j_1) = \sum_{k=1}^{T}\sum_{\ell_1=0}^{T-1}\sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_2=1}^{\ell_3-1}\sum_{j_i,\dots,j_2}^{m_e}\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}) \tag{23}$$

The relatively small number of parameters of problem (22) ($O(m_v o)$) allows us to obtain the solution through brute force assuming the problem is small enough.

2) Greedy Algorithm: Alternatively, we formulate a greedy algorithm for selecting a S/A scheduling as a secondary comparison basis to our heuristic. To design the greedy step, consider the problem of finding a sensor/actuator pair at given sensor/actuator times to add to a pre-existing schedule such as to maximize the $\mathcal{H}_2$-norm of the system subject to the scheduling. That is, given a scheduling ρ^- and σ^-, a candidate pair $\rho_i[\ell_s]$ and $\sigma_j[\ell_a]$ will increase the objective function by

$$\bar{\kappa}(\ell_s,i,\ell_a,j) + \sum_{\rho^-_{j_T}[k]\neq 0}\bar{\kappa}(k,j_T,\ell_a,j) + \sum_{\sigma^-_{j_1}[\ell_1]\neq 0}\bar{\kappa}(\ell_s,i,\ell_1,j_1),$$

where

$$\bar{\kappa}(k,j_T,\ell_1,j_1) = \sum_{i=1}^{k}\sum_{\ell_i=i-1}^{k-1}\cdots\sum_{\ell_2=1}^{\ell_3-1}\sum_{j_i,\dots,j_2}^{m_e}\kappa_i(k,j_T,\mathbf{l_i},\mathbf{j_i}). \tag{24}$$

As such, the greedy step consists in selecting the pair with the maximum increment to the cost function.

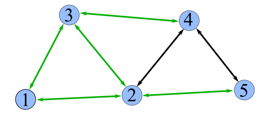

Fig. 2: Bilinear Network with 5 nodes used during our simulations in Section IV. All nodes are actuated/sensed and green edges are actuated.

$\frac{\|\bar{\Sigma}-\tilde{\Sigma}\|_{\mathcal{H}_2}}{\|\Sigma\|_{\mathcal{H}_2}} \times 100\%$	$\bar{\sigma}=1$	$\bar{\sigma}=2$	$\bar{\sigma}=3$	$\bar{\sigma}=4$	$\bar{\sigma}=5$
$\bar{\rho}$ = 1	00.00%	00.78%	00.63%	00.62%	00.00%
$\bar{\rho}$ = 2	00.78%	00.00%	00.43%	00.44%	00.00%
$\bar{\rho}$ = 3	00.62%	00.44%	00.00%	00.45%	00.00%
$\bar{\rho}$ = 4	00.62%	00.44%	00.46%	00.00%	00.00%
$\bar{\rho}$ = 5	00.00%	00.00%	00.00%	00.00%	00.00%

TABLE I: Percentage performance difference between the time-invariant independent S/A selection, and brute-force solutions with the fully sensed and actuated case as a baseline. Green means gain while red means loss.

IV. Simulations and Numerical Results

In this section, we present numerical examples to illustrate our heuristic approach. Our simulations are performed with a time horizon $T = 7$ steps and for the 5-node graph given in Fig. 2, where all nodes are sensed and actuated and the five green edges were randomly selected to be actuated. The discrete-time dynamics were derived from the grounded graph Laplacian and the simulation conducted for varying values of $\bar{\sigma}$ and $\bar{\rho}$, as evident in the tables.

A. Time-Invariant Cases and the Separation Principle

Our first set of simulations are conducted for the time-invariant case, that is, we look for the solution of the combinatorial optimization problem given by (22). The first approach is to compute the solution for S/A selections separately and then combine them. Alternatively, due to the small size of our problem, we can also compute the optimal solution through a brute-force method. In Table I we present the difference between the independent selection and the brute force solution in percentage gain with respect to the fully sensed and actuated system.

Although the results for both the brute-force and independent optimization are the same in most instances, sometimes they are different, as evident by the red entries of Table I. As expected, the brute-force solution is always better or equal to the independent optimization, but the fact that it is strictly better sometimes (for example $\bar{\rho} = 3$ and $\bar{\sigma} = 2$) indicates that the separation principle does not hold for this problem.

B. Alternating Optimization and Greedy Methods

Next we compare our heuristics with other potential solutions to the problem. Notice that for all time-varying scheduling simulations, the constraints on the number of sensors and actuators are enforced on average, meaning they can be disrespected at a given time-step if that is compensated at a different one. In Table II, we show the results for our heuristics for scheduling compared to the brute force solution of the time-invariant selection.

Table II illustrates the potential gain of a time-varying schedule versus the simpler time-invariant selection. Our

$\frac{\|\bar{\Sigma}-\tilde{\Sigma}\|_{\mathcal{H}_2}}{\|\Sigma\|_{\mathcal{H}_2}} \times 100\%$	$\bar{\sigma}=1$	$\bar{\sigma}=2$	$\bar{\sigma}=3$	$\bar{\sigma}=4$	$\bar{\sigma}=5$
$\bar{\rho}$ = 1	00.00%	03.76%	04.79%	05.50%	05.88%
$\bar{\rho}$ = 2	03.76%	00.00%	04.12%	06.74%	07.25%
$\bar{\rho}$ = 3	04.80%	04.11%	01.48%	05.94%	08.61%
$\bar{\rho}$ = 4	04.73%	06.06%	05.93%	01.30%	06.05%
$\bar{\rho}$ = 5	05.88%	07.24%	08.62%	06.04%	0.00%

TABLE II: Percentage performance difference between the time-variant alternating optimization heuristic versus the brute force solution for the time-invariant selection. Green means gain while red means loss.

$\frac{\|\bar{\Sigma}-\tilde{\Sigma}\|_{\mathcal{H}_2}}{\|\Sigma\|_{\mathcal{H}_2}} \times 100\%$	$\bar{\sigma}=1$	$\bar{\sigma}=2$	$\bar{\sigma}=3$	$\bar{\sigma}=4$	$\bar{\sigma}=5$
$\bar{\rho}$ = 1	00.00%	03.32%	04.73%	05.47%	05.88%
$\bar{\rho}$ = 2	03.31%	02.19%	05.86%	08.13%	10.36%
$\bar{\rho}$ = 3	04.74%	05.85%	02.29%	09.82%	13.74%
$\bar{\rho}$ = 4	05.47%	08.33%	09.83%	02.77%	11.17%
$\bar{\rho}$ = 5	05.88%	10.36%	13.75%	11.17%	0.00%

TABLE III: Percentage performance difference between the time-variant alternating optimization heuristic and the greedy solution. Green means gain while red means loss.

heuristic for S/A scheduling is almost always significantly better than the best possible S/A selection, being only slightly worse on two instances. Furthermore, our heuristic is based on solving polynomial-time algorithms while a brute-force solution grows exponentially on the complexity of the problem even for the simpler time-invariant S/A selection problem.

On Table III we can see that our heuristic outperforms the greedy algorithm in basically all instances. Even compared to the brute-force solution of the time-invariant case the greedy algorithm we proposed fails to produce good results, which indicates its inadequacy in solving this problem.

V. Conclusions

In this paper, we consider the problem of designing S/A scheduling for bilinear networks. We formulate our problem as a specific class of time-varying bilinear systems and adapt traditional definitions of Gramians and an $\mathcal{H}_2$-norm for these systems. After discussing the exponentially increasing complexity of the joint problem of S/A scheduling, we simplify it to a joint S/A scheduling, ignoring the edge actuators. We propose an alternating optimization heuristic to solve this problem and compare it with other approaches: the time-invariant case (complexity reduction) and a greedy algorithm for scheduling. Our simulations indicate that the separation principle does not hold even for the time-invariant selection of sensors and node actuators, motivating our search for a more complex algorithm. We then show that our heuristic successfully leverages the advantages of allowing for time variation on the choice of sensors and actuators, since it performs significantly better on average than the brute-force solution of the time-invariant case. Our algorithm also outperforms the greedy solution, despite both being able to consider time-varying scheduling, which indicates our approach might be more adequate for finding an approximate solution of this problem. Despite all that, our heuristics still perform less well than the brute-force solution of the time-invariant selection for two cases, which indicates the suboptimality of our solutions. Furthermore, tests on larger networks were made difficult due to the time-consuming process of computing all parameters of the problems (such as in κ_i in (17)), even if the final number of parameters for optimization is small.

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