European Journal of Control (1995) 1:24

European Journal of Control

On the Input-to-State Stability Property

Eduardo D. Sontag*

Department of Mathematics Rutgers University, New Brunswick, New Jersey, USA

The 'input to state stability' (ISS) property provides a natural framework in which to formulate notions of stability with respect to input perturbations. In this expository paper, we review various equivalent definitions expressed in stability, Lyapunov-theoretic, and dissipation terms. We sketch some applications to the stabilisation of cascades of systems and of linear systems subject to control saturation.

Keywords: Cascades; Dissipation; Lyapunuv functions; Saturation; Stability.

1. Introduction

There are two very conceptually different ways of formulating the notion of stability of control systems. One of them, which we may call the *input/output approach*, relies on operator-theoretic techniques. Among the main contributions to this area, one may cite the foundational work by Zames, Sandberg, Desoer, Safanov, Vidyasagar, and others. In this approach, a 'system' is a causal operator F between spaces of signals, and 'stability' is taken to mean that F maps bounded inputs into bounded outputs, or finite-energy outputs. More stringent typical requirements in this context are that the gain of F be finite (in more classical mathematical terms, that the operator be bounded), or that it have finite incremental gain (mathematically, that it be globally Lipschitz). The input/output approach has been extremely successful in the robustness analysis of linear systems subject to nonlinear feedback and mild nonlinear uncertainties, and in general in the area that revolves around the various versions of the small-gain theorem. Moreover, geometric characterizations of robustness (gap metric and the like) are elegantly carried out in this framework. Finally, i/o stability provides a natural setting in which to study the classification and parameterisation of dynamic controllers.

On the other hand, there is the model-based, or *state-space approach* to systems and stability, where the basic object is a forced dynamical system, typically described by differential or difference euqations. In this approach, there is a standard notion of stability, namely Lyapunov asymptotic stability of the unforced system. Associated to such a system, there is an operator F mapping inputs (forcing functions) into state trajectories (or into outputs, if partial measurements on states are of interest). It becomes of interest then to ask to what extent Lyapunov-like stability notions for a state-space system are related to the stability, in the senses discussed in the previous paragraph, of the associated operator F. It is well-known (see, e.g., [1]) that, in contrast to the case of linear systems, where there is – subject to mild technical assumptions – an equivalence between state-space and i/o stability, for nonlinear systems the two types of properties are not so closely related. Even for the very special and comparatively simple case of 'feedback linearisable' systems, this relation is far more subtle than it might appear at first sight: if one first linearises a system and then stabilises the equivalent linearisation, in terms of the original system one does not in general obtain a closed-loop system that is input/output stable in any reasonable sense. (However, it is always possible to make a choice of a – usually different – feedback law that

---

*Supported in part by US Air Force Grant F49620-95-1-0101.

Correspondence and offprint requests to: E.D. Sontag, Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA. E-mail: sontag@hilbert.rutgers.edu.

Received 17 January 1995; Accepted 14 April 1995
Recommended by C. Samson and A. Isidori

achieves such stability, in the linearisable case as well as for all other stabilisable systems, as will be discussed below.)

It is the purpose of this article to present a brief and informal survey of various links between the two alternative paradigms of stability, i/o and state-space, through the systematic use of the notion of 'input to state stability' (ISS). This notion differs fundamentally from the operator-theoretic ones that have been classically used in control theory, first of all because it takes account of initial states in a manner fully compatible with Lyapunov stability. Second, boundedness (finite gain) is far too strong a requirement for general nonlinear operators, and it must be replaced by 'nonlinear gain estimates', in which the norms of output signals are bounded by a nonlinear function of the norms of inputs; the definition of ISS incorporates such gains in a natural way. The ISS notion was originally introduced in [2] and has since been employed by several authors in deriving results on control of nonlinear systems. It can be stated in several equivalent manners, which indicates that it is at least a mathematically natural concept: dissipation, robustness margins, and classical Lyapunov-like definitions.

The dissipation characterisations are closely related to the pioneering work of Willems [3], who introduced an abstract concept of energy dissipation in order to unify i/o and state space stability, and in particular with the purpose of understanding conceptually the meaning of Kalman–Yakubovich positive-realness (passivity), and frequency-domain stability theorems in a general nonlinear context. His work was continued by many authors, most notably Hill and Moylan (see e.g. [4,5]). (However, although extremely close in spirit, technically our work does not make much contact with the existing dissipation literature. Mathematically it is grounded instead in more classical converse Lyapunov arguments in the style of Massera, Kurzweil and Zubov.)

The equivalences between different notions of input to state stability originate with the paper [2], but the definitive conclusions were obtained in recent work jointly carried out with Yuan Wang in [6], which in turn built upon research with Wang and Yuandan Lin in [7] and [8]; the input-saturated results are based on joints papers with Wensheng Liu and Yacine Chitour ([9]) as well as Sussmann and Yang ([10]). Some recent and very relevant results by Teel ([11]) and Jiang, Praly and Teel ([12]) are also mentioned. In the interest of exposition, the style of presentation in this survey is informal. The reader should consult the references for more details and, in some cases, for precise statements.

1.1. Preliminaries

This paper deals with continuous time systems of the standard form

$$\dot{x} = f(x, u), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. (Since global asymptotic stability will be of interest, and when such a property holds the state space must be Euclidean, there is no reason to consider systems evolving in more general manifolds than Euclidean space.) For undefined terminology from control theory see [13]. It is assumed that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0) = 0$. *Controls* or *inputs* are measurable locally essentially bounded functions $u : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$. The set of all such functions is denoted by $L^m_{\infty,e}$, and one denotes $\|u\|_\infty = (\text{ess})\sup\{|u(t)|, t \geqslant 0\} \leqslant \infty$; when this is finite, one obtains the usual space L^m_∞, endowed with the (essential) supremum norm. (Everywhere, $|\cdot|$ denotes Euclidean norm in the appropriate space of vectors, and $\|\cdot\|$ denotes induced norm for matrices, while $\|\cdot\|_\infty$ is used for supremum norm.) For each $x_0 \in \mathbb{R}^n$ and each $u \in L^m_\infty, x(t, x_0, u)$ denotes the trajectory of the system (1) with initial state $x(0) = x_0$ and input u. This is *a priori* defined only on some maximal interval $[0, T_{x0,u})$, with $T_{x0}, u \leqslant +\infty$. If the initial state and input are clear from the context, one writes just $x(\cdot)$ for the ensuing trajectory. The system is *(forward-) complete* if $T_{x0,u} = +\infty$ for all x_0 and u.

The questions to be studied relate to the 'stability', understood in an appropriate sense, of the input to state mapping $(x_0, u(\cdot)) \mapsto x(\cdot)$ (or, in the last section, when an output is also given, of the input to output mapping $\mapsto y(\cdot)$). To appreciate the type of problem that one may encounter, consider the following issue. Suppose that in the absence of inputs the trivial solution $x \equiv 0$ of the differential equation

$$\dot{x} = f_0(x) = f(x, 0) \tag{2}$$

is globally asymptotically stable (for simplicity, in such a situation, we will simply say that (1), or equivalently the zero-input restricted system (2), is GAS). Then one would like to know if, for solutions of (1) associated to *nonzero* controls, it holds that

$$u(\cdot) \underset{t\to\infty}{\longrightarrow} 0 \Rightarrow x(\cdot) \underset{t\to\infty}{\longrightarrow} 0$$

(the 'converging input converging state' property) or that

$$u(\cdot) \text{ bounded} \Rightarrow x(\cdot) \text{ bounded}$$

(the 'bounded input bonded state' property). Of course, for linear systems $\dot{x} = Ax + Bu$ these implications are always true. Not only that, but one has explicit estimates

$$| x(t) | \leqslant \beta(t) | x_0 | + \gamma \|u_t\|_\infty$$

where

$$\beta(t) = \left\| e^{tA} \right\| \to 0 \text{ and } \gamma = \|B\| \int_0^\infty \left\| e^{sA} \right\| ds$$

for any Hurwitz matrix A, where u_t is the restriction of u to $[0, t]$, though of as a function in L_∞^m which is zero for $s > t$. From these estimates both properties can be easily deduced.

These implications fail in general for nonlinear systems, however, as has been often pointed out in the literature (see, for instance, [1]). As a trivial illustration, take the system

$$\dot{x} = -x + (x^2 + 1)u \tag{3}$$

and the control $u(t) = (2t + 2)^{-1/2}$. With $x_0 = \sqrt{2}$ there results the unbounded trajectory $x(t) = (2t + 2)^{1/2}$. This is in spite of the fact that the system is GAS. Thus, the converging input converging state property does not hold. Even worse, the bounded input $u \equiv 1$ results in a finite-time explosion. This example is not artificial, as it arises from the simplest case of feedback linearisation design. Indeed, given the system

$$\dot{x} = x + (x^2 + 1)u,$$

the obvious stabilising control law (obtained by first cancelling the nonlinearity and then assigning dynamics $\dot{x} = -x$) is

$$u : \frac{-2x}{x^2 + 1} + v$$

where v is the new external input. In terms of this new control (which might be required in order to meet additional design objectives, or may represent the effect of an input disturbance), the closed-loop system is as in (3), and thus is ill-behaved. Observe, however, that if instead of the obvious law just given one used:

$$u := \frac{-2x}{x^2 + 1} - x + v$$

then the closed-loop system becomes instead

$$\dot{x} = -2x - x^3 + (x^2 + 1)u$$

This is still stable when $u \equiv 0$, but in addition it tolerates perturbations far better, since the term $-x^3$ dominates $u(x^2 + 1)$ for bounded u and large x. The behaviour with respect to such u is characterised qualitatively by the notion of 'ISS' system, to be discussed below. More generally, it is possible to show that up to *feedback equivalence*, GAS always implies (and is hence equivalent) to the ISS property to be defined. This is one of many motivations for the study of the ISS notion, and will be reviewed after the precise definitions have been given.

Besides being mathematically natural and providing the appropriate framework in which to state the above-mentioned feedback equivalence result, there are several other reasons for studying the ISS property, some of which are briefly mentioned in this paper. See, for instance, the applications to observer design and new small gain theorems in [12,14,15,16]; the construction of coprime stable factorisations was the main motivation in the original paper [2] which introduced the ISS concept, and the stabilisation of cascade systems using these ideas was briefly discussed in [17].

2. The Property ISS

Next, four natural definitions of input to state stability are proposed and separately justified. Later, they turn out to be equivalent. The objective is to express the fact that states remain bounded for bounded controls, with an ultimate bound which is a function of the input magnitude, and in particular that states decay when inputs do.

2.1. From GAS to ISS – A First Pass

The simplest way to introduce the notion of ISS system is a generalisation of GAS, global asymptotic stability of the trivial solution $x \equiv 0$ for (2). The GAS property amounts to the requirements that the system be complete and the following two properties hold:

1. *(Stability):* the map $x_0 \mapsto x(\cdot)$ is continuous at 0, when seen as a map from $\mathbb{R}^n$ into $C^0([0, +\infty), \mathbb{R}^n)$; and
2. *(Attractivity):* $\lim_{t \to +\infty} | x(t, x_0) | = 0$.

Note that, under the assumption that 1. holds, the convergence in the second part is automatically uniform with respect to initial states x_0 in any given compact. By analogy, one defines the system (1) to be *input to state stable* (ISS) if the system is complete and the following properties, which now involve nonzero inputs, hold:

1. the map $(x_0, u) \mapsto x(\cdot)$ is continuous at $(0, 0)$ (seen as a map from $\mathbb{R}^n \times L_\infty^m$ to $C^0([0, +\infty), \mathbb{R}^n)$), and
2. there exists a 'nonlinear asymptotic gain' $\gamma \in \mathcal{K}$ so that

$$\overline{\lim_{t\to+\infty}} \mid x(t,x_0,u) \mid \ \leqslant \gamma(\|u\|_\infty) \tag{4}$$

uniformly on x_0 in any compact and all u.

(The class $\mathcal{K}$ consists of all functions $\gamma : R_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ which are continuous, strictly increasing, and satisfy $\gamma(0) = 0$. The uniformity requirement means, explicitly: for each r and ϵ positive, there is a $T > 0$ so that $\mid x(t,x_0,u) \mid \ \leqslant \varepsilon + \gamma(\|u\|_\infty)$ for all u and all $\mid x_0 \mid \ \leqslant r$ and $t \geqslant T$.)

In the language of robust control, the inequality (4) is an 'ultimate boundedness' condition. Note that this is a direct generalisation of attractivity to the case $u \not\equiv 0$; the 'lim sup' is now required since the limit need not exist.

2.2. From Lyapunov to Dissipation – A Second Pass

A potentially different concept of input to state stability arises when generalising classical Lyapunov conditions to certain classes of dissipation inequalities.

A *storage* or energy function is a $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ which is continuously differentiable, proper (that is, radically unbounded) and positive definite (that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$). A (classical) *Lyapunov function* for the zero-input system (2) is a storage function for which there exists some function α of class $\mathcal{K}_\infty$ – that is, of class $\mathcal{K}$ and so that also $\alpha(s) \to +\infty$ as $s \to +\infty$ – so that

$$\nabla V(x) \cdot f_0(x) \leqslant -\alpha(\mid x \mid)$$

holds for all $x \in \mathbb{R}^n$. This means that $dV(x(t))/dt \leqslant -\alpha(\mid x(t) \mid)$ along all trajectories.

By analogy, when nonzero inputs must be taken into account, it is sensible to define an *ISS-Lyapunov* function as a storage function for which there exist two class K_∞ functions α and θ such that

$$\nabla V(x) \cdot f(x,u) \leqslant \theta(\mid u \mid) - \alpha(\mid x \mid) \tag{5}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Thus, along trajectories one now obtains the inequality $dV(x(t))/dt \leqslant \theta(\mid u(t) \mid) - \alpha(\mid x(t) \mid)$.

A *smooth* ISS-Lyapunov function is a V which satisfies these properties and is in addition infinitely differentiable. Smoothness is an extremely useful property in this context, as one may then use iterated derivatives of V along trajectories for various design as well as analysis questions, in particular in so-called 'backstepping' design techniques.

In the terminology of [3,5], (5) is a *dissipation inequality* with storage function V and *supply* function $w(u,x) = \theta(\mid u \mid) - \alpha(\mid x \mid)$. (In the context of dissipative systems one often postulates the equivalent integral form $V(x(t,x_0,u)) - V(x_0) \leqslant \int_0^t w(u(s), x(s))ds$, which must hold along all trajectories, and no differentiability is required of V. Moreover, outputs $y = h(x)$ are used instead of states in the estimates, so the present setup corresponds to the case $h(x) = x$.) The estimate (5) is a generalisation of the one used by Brockett in [18] when defining 'finite gain at the origin'; in that paper, the function θ is restricted to be quadratic, and the concepts are only defined locally, but the ideas are very similar.

2.3. Gain Margins – A Third Pass

Yet another possible approach to formalising input to state stability is motivated both by the classical concept of total stability and as a generalisation of the usual gain margin for linear systems.

In [6], a *(nonlinear) stability margin* for system (1) is defined as any function $\rho \in \mathcal{K}_\infty$ with the following property: for each admissible – possibly nonlinear and/or time-varying – feedback law k bounded by ρ, that is, so that

$$\mid k(t,x) \mid \ \leqslant \rho(\mid x \mid)$$

for all (t,x), the closed-loop system

$$\dot{x} = f(x, k(t,x)) \tag{6}$$

is GAS, uniformly on k. (More precisely, an admissible feedback law is a measurable function $k : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^m$ for which (6) is well-posed; that is, for each initial state $x(0)$ there is an absolutely continuous solution, defined at least for small times, and any two such solutions coincide on their interval of existence. Uniformity in k means that all limits in the definition of GAS are independent of the particular k, as long as the inequality $\mid k(t,x) \mid \ \leqslant \rho(\mid x \mid)$ holds.) A system is said to be *robustly stable* if there exists some such ρ.

Observe that for arbitrary nonlinear GAS systems, in general only small perturbations can be tolerated (cf. total stability results). The requirement that $\rho \in \mathcal{K}_\infty$ is thus highly nontrivial: it means that for large states relatively large perturbations should not affect stability.

2.4. Estimates – A Fourth Pass

A final proposed notion of input to state stability can be introduced by means of an estimate similar to that which holds in the linear case:

$$\mid x(t,x_0,u) \mid \ \leqslant \left\| e^{tA} \right\| \mid x_0 \mid + \left(\|B\| \int_0^\infty \left\| e^{sA} \right\| ds \right) \|u_t\|_\infty$$

It is first necessary to review an equivalent – if somewhat less widely known – definition of GAS. This is a characterisation in terms of comparison functions. Recall that a function of class $\mathcal{KL}$ is a

$$\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$$

so that $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geqslant 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each $s \geqslant 0$ (example of relevance to the linear case: $ce^{-at}s$, with $a > 0$ and a constant c). It is not difficult to prove (this is essentially in [19]; see also [2]) that the system (2) is GAS if and only if there exists a $\beta \in \mathcal{KL}$ so that

$$| x(t, x_0) | \leqslant \beta(| x_0 |, t) \tag{7}$$

for all t, x_0. (Note that sufficiency is trivial, since forward completeness follows from the fact that trajectories stay bounded, the estimate $| x(t, x_0) | \leqslant \beta(| x_0 |, 0)$ provides stability, and $| x(t, x_0) | \leqslant \beta | x_0 |, t) \to 0$ shows attractivity. The converse is established by formulating and solving a differential inequality for $| x(t, x_0) |$.)

In this context, it is then natural to consider the following '$\beta + \gamma$' property: there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that, for all initial states and controls, and all $t \geqslant 0$:

$$| x(t, x_0, u) | \leqslant \beta(| x_0 |, t) + \gamma(\|u_t\|_\infty) \tag{8}$$

(One could use a 'max' instead of the sum of the two estimates, but the same concept would result. Also, it makes no difference to write $\|u\|_\infty$ instead of the norm of the restriction $\|u_t\|_\infty$.) This is a direct generalisation of both the linear estimate and the characterisation of GAS in terms of comparison functions.

2.5. All are Equivalent

The following result was recently proved by Yuan Wang and the author:

Theorem 1 ([6]). For any system (1), the following properties:

1. ISS (nonlinear asymptotic gain),
2. there is an ISS-Lyapunov function (dissipativity),
3. there is a smooth ISS-Lyapunov function,
4. there is a nonlinear stability margin (robust stability), and
5. there is some $\beta + \gamma$ estimate,

are all equivalent. □

The proof is heavily based on a result obtained by Wang, Lin, and the author in [7], which states essentially that a parametric family of systems $\dot{x} = f(x, d)$, with arbitrary time-varying 'disturbances' $d(t)$ taking values on a compact set D, is uniformly globally asymptotically stable if and only if there exist a smooth storage function V and an $\alpha \in \mathcal{K}_\infty$ so that

$$\nabla V(x) \cdot f(x, d) \leqslant -\alpha(| x |)$$

for all $x \in \mathbb{R}^n$ and values $d \in D$. Note that the construction of a smooth V is not entirely trivial (this subsumes as particular cases several standard converse Lyapunov theorems).

There is yet another equivalent notion of ISS, obtained in the recent work [20]. This notion allows replacing sup norms on controls with a fading-memory L^1 estimate, as follows, and is of great use in robust control applications.

Theorem 2. A system (1) is ISS if and only if there exist a $\mathcal{KL}$-function β, and two functions $\gamma_i, \gamma_0 \in \mathcal{K}$, so that

$$| x(t, \xi, u) | \leqslant \beta | \xi |, t) + \gamma_0 \left(\int_0^t e^{s-t} \gamma_i(| u(s) |) ds \right)$$

for all $t \geqslant 0$. □

2.6. Checking the ISS Property

Of course, verifying the ISS property is in general very hard – after all, in the particular case of systems with no inputs, this amounts to checking global asymptotic stability. Nonetheless, the dissipation inequality (5) provides in principle a good tool, playing the same role as Lyapunov's direct method for asymptotic stability. Actually, even more useful is the following variant, which is the original definition of 'ISS-Lyapunov function' in [2]. Consider a storage function with the property that there exist two class $\mathcal{K}$ functions α and χ so that the implication

$$| x | \geqslant \chi(| u |) \Rightarrow \nabla V(x) \cdot f(x, u) \leqslant -\alpha(| x |) \tag{9}$$

holds for each state $x \in \mathbb{R}^n$ the control value $u \in \mathbb{R}^m$. It is shown in [6] that the existence of such a V provides yet another necessary and sufficient characterisation of the ISS property. (Other variants are also equivalent: for instance, asking that α be of class $\mathcal{K}_\infty$.)

As an illustration, consider the following system, which will appear again later in the context of an example regarding the stabilisation of the angular momentum of a rigid body. The state space is $\mathbb{R}$, the control value space is $\mathbb{R}^2$, and dynamics are given by:

$$\dot{x} = -x^3 + x^2u_1 - xu_2 + u_1u_2 \tag{10}$$

This system is GAS when $u \equiv 0$, and for large states the term $-x^3$ dominates, so it can be expected to be ISS. Indeed, using the storage function $V(x) = x^2/2$ there results

$$\nabla V(x) \cdot f(x,u) \leqslant -\left(\frac{2}{9}\right)x^4$$

provided that $3 \mid u_1 \mid \leqslant \mid x \mid$ and $3 \mid u_2 \mid \leqslant x^2$. A sufficient condition for this to hold is that $\mid u \mid \leqslant \nu(\mid x \mid)$, where $\nu(r) := \min\{r/3, r^2/3\}$. Thus V is an ISS-Lyapunov function as above, with $\alpha(r) = (2/9)r^4$ and $\chi = \nu^{-1}$.

Another example is as follows. Let SAT : $\mathbb{R} \to \mathbb{R}$ be the standard saturation function: $\mathrm{SAT}[r] = r$ if $\mid r \mid \leqslant 1$, and $\mathrm{SAT}[r] = \mathrm{sign}(r)$ otherwise. Consider the following one-dimensional one-input system:

$$\dot{x} = -\mathrm{SAT}[x+u]. \tag{11}$$

This is an ISS system, as will be proved next by showing that

$$V(x) := \frac{\mid x \mid^3}{3} + \frac{x^2}{2} \tag{12}$$

is an ISS-Lyapunov function. Observe that V is once differentiable, as required. This is a very particular case of a more general result dealing with linear systems with saturated controls, treated in [9]; more will be said later about the general case (which employs a straightforward generalisation of this V).

To prove that V satisfies a dissipation inequality, first note that, since $\mid r - \mathrm{SAT}[r] \mid \leqslant r\mathrm{SAT}[r]$ for all r,

$$\begin{aligned} \mid x - \mathrm{SAT}[x+u] \mid &\leqslant \mid x + u - \mathrm{SAT}[x+u] \mid + \\ \mid u \mid &\leqslant (x+u)\mathrm{SAT}[x+u] + \mid u \mid \end{aligned} \tag{13}$$

for all values and $u \in \mathbb{R}$. It follows that

$$\begin{aligned} -x\mathrm{SAT}[x+u] &= x(-x) + x(x - \mathrm{SAT}[x+u]) \\ &\leqslant -x^2 + \mid x \mid (x+u)\mathrm{SAT}[x+u] \\ &\quad + \mid x \mid\mid u \mid \end{aligned}$$

for all x, u. On the other hand, using that $\mathrm{SAT}[r] \leqslant 1$ for all r,

$$\begin{aligned} -\mid x \mid x\mathrm{SAT}[x+u] &= \mid x \mid [-(x+u)\mathrm{SAT}[x+u] \\ &\quad + u\mathrm{SAT}[x+u]] \\ &\leqslant -\mid x \mid (x+u)\mathrm{SAT}\,[x+u] \\ &\quad + \mid x \mid\mid u \mid \end{aligned}$$

Adding the two inequalities, it holds that

$$-(1 + \mid x \mid)x\mathrm{SAT}[x+u] \leqslant -x^2 + 2 \mid x \mid\mid u \mid \tag{14}$$

so that indeed

$$\nabla V(x) \cdot f(x,u) \leqslant -\frac{x^2}{2} + 2u^2$$

as desired (note that $\nabla V(x) = x(1 + \mid x \mid)$).

2.7. Relations Among Estimates, Zero-State Responses, and Linear Gains

There are many relationships among the various estimates which appear in the alternative characterisations of ISS. Two of them are as follows.

Assume that V is a storage function satisfying the estimates in Eq. (5):

$$\nabla V(x) \cdot f(x,u) \leqslant \alpha_4(\mid u \mid) - \alpha_3(\mid x \mid) \tag{15}$$

for some $\mathcal{K}_\infty$ functions α_3 and α_4. Since V is proper, continuous, and positive definite, there are as well two other class $\mathcal{K}_\infty$ functions α_1 and α_2 such that

$$\alpha_1(\mid x \mid) \leqslant V(x) \leqslant \alpha_2(\mid x \mid) \tag{16}$$

for all $x \in \mathbb{R}^n$. It then holds that one may pick an asymptotic gain γ in Eq. (4) of the form:

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ_3^{-1} \circ \alpha_4. \tag{17}$$

Moreover, if instead of (15) there holds a slightly stronger estimate of the form

$$\nabla V(x) \cdot f(x,u) \leqslant \alpha_4(\mid u \mid) - \alpha_3(\mid x \mid) - \alpha(\mid x \mid)$$

where α is any class $\mathcal{K}$ function, then the γ function in the '$\beta + \gamma$' property (8) can also be picked as in Eq. (17). These conclusions are implicit in the proofs given in [2] and [6].

For trajectories starting at the particular initial state $x_0 = 0$, for any input function u, and assuming only that V satisfies (15)–(16), it holds that $\mid x(t,0,u) \mid \leqslant \gamma(\|u\|_\infty)$ for all $t \geqslant 0$, not merely asymptotically, for the same γ as in (17), that is,

$$\|x(\cdot,0,u)\|_\infty \leqslant \gamma(\|u\|_\infty).$$

Thus the zero-state response has a 'nonlinear gain' bounded by this γ.

A particular case of interest is when both (α_1, α_2) and (α_3, α_4) are *convex estimate pairs* in the following sense: a pair of class $\mathcal{K}$ functions (α, β) is a convex estimate pair if α and β are convex functions and there is some real number $k \geqslant 1$ such that $\beta(r) \leqslant k\alpha(r)$ for all $r \geqslant 0$. Note that for any convex function α in $\mathcal{K}$ and any $k \geqslant 1$ it holds that $\alpha^{-1}(k\alpha(r)) \leqslant kr$ for all non-negative r, from which it follows that $\alpha^{-1}(\beta(r)) \leqslant kr$ if k is as in this definition. One concludes that if each of (α_1, α_2) and

(α_3, α_4) is a convex estimate pair, then the gain γ can be taken to be bounded by a linear function. In other words, the input to state operator, starting from $x_0 = 0$, is bounded as an operator with respect to sup norms:

$$\|x(\cdot, 0, u)\|_\infty \leqslant g\|u\|_\infty$$

This is the standard situation in linear systems theory, where V is quadratic (and hence admits estimates in terms of α_1 and α_2 of the form $c_i r^2$, where c_1 and c_2 are respectively the smallest and largest singular values of the associated form) and the supply function can likewise be taken of the form $c_4 \mid u \mid^2 - c_3 \mid x \mid^2$. So finiteness of linear gain, that is, operator boundedness, follows from convexity of the estimation functions. Somewhat surprisingly, for certain linear systems subject to actuator saturation, convex (but not quadratic) estimates are also possible, and this again leads to finite linear gains. For example, this applies to the function V in Eq. (12), as an ISS-Lyapunov function for system (11): there one may pick $\alpha = \alpha_1 = \alpha_2 = r^3/3 + r^2/2$, which is convex since $\alpha''(r) = 2r + 1 > 0$, while α_3 and α_4 can be taken quadratic (cf. Eq. (14)).

As an additional remark, note that, just from the fact that V is non-negative and $V(0) = 0$, and integrating the dissipation inequality (5), for $x_0 = 0$ there results the inequality $\int_0^{+\infty} \alpha(\mid x(t, 0, u) \mid)dt \leqslant \int_0^{+\infty} \theta(\mid u(t) \mid)dt$. In this manner, it is routine to use dissipation inequalities for proving operator boundedness in various pth norms (in particular, when $\alpha(r) = c_1 r^2$ and $\theta(r) = c_2 r^2$ one is estimating 'H^∞' norms). But in the current context, more general nonlinearities than powers are being considered.

It is also interesting to note that, if V and α are so that the estimate (9) is satisfied, then there is some θ so that the dissipation estimate (5) also holds, with these same V and α.

2.8. Set ISS

A useful variation of the notion of ISS system is obtained when one studies stability with respect to a closed subset K of the state space $\mathbb{R}^n$, not necessarily $K = \{0\}$. It is possible to generalise the various definitions given; for instance, the definition (8) becomes

$$\mid x(t, x_0, u) \mid_K \leqslant \beta(\mid x_0 \mid_K, t) + \gamma(\|u_t\|_\infty)$$

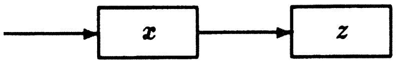
where $\mid x \mid_K$ denotes the distance from x to the set K. (When $u \equiv 0$, this equation implies in particular that the set K must be invariant for the unforced system.) This notion of set input to state stability was introduced and studied in [8,21]. The equivalence of various alternative definitions can be given in much the same way as the equivalence for the particular case $K = \{0\}$ (at least for compact K), since the general results in [7] are already formulated for set stability; see [22] for details. Set-ISS is of interest in various contexts, among them the design of robust control laws (the sets in question correspond to equilibria for different parameter values) and those cases in which only 'practical' stabilisation, that is to say, stabilisation to a neighbourhood of the origin, is all that is possible.

3. Interconnections

It is by now well known, and easy to prove, that the cascade of two ISS systems is again ISS (in particular, a cascade of an ISS and a GAS system is GAS). It is interesting to observe that this statement can be understood very intuitively in terms of the dissipation formalism, and it provides further evidence of the naturality of the ISS notion. In addition, proceeding in this manner, one obtains a Lyapunov function (with strictly negative derivative along trajectories) for the cascade.

Theorem 3. Consider the system in cascade form:

$$\dot z = f(z, x)$$
$$\dot x = g(x, u)$$



where $f(0, 0) = g(0, 0) = 0$, the second eq. is ISS, and the first eq. is ISS when x is seen as an input. Then the composite system is ISS. □

The proof can be based on the following argument. First, one shows that it is possible to obtain storage functions V_1 and V_2 so that V_1 satisfies a dissipation estimate

$$\nabla V_1(z) \cdot f(z, x) \leqslant \theta(\mid x \mid) - \alpha(\mid z \mid)$$

for the first subsystem, while V_2 is a storage function for the x-subsystem so that

$$\nabla V_2(x) \cdot g(x, u) \leqslant \tilde\theta(\mid u \mid) - 2\theta(\mid x \mid)$$

(see [23] for details). Then $V_1(z) + V_2(x)$ is a storage function for the composite system, which satisfies the dissipation inequality with derivative bounded by $\tilde\theta(\mid u \mid) - -\theta(\mid x \mid) - \alpha(\mid z \mid)$.

A beautiful common generalisation of both the cascade result and the usual Small-Gain Theorem was recently obtained by Jiang, Teel and Praly.

Theorem 4 [7]. Consider a system in composite feedback form (cf. Fig. 1):

$$\dot z = g(z,x,v)$$
$$\dot x = f(x,z,u)$$

where u, v are the inputs to the composite system. Assume:

- Each of $\dot x = f(x,z,u)$ and $\dot z = g(z,x,v)$ is an ISS system, when (z,u) and (x, v) are considered as inputs respectively; let γ_1 and γ_2 denote the gains for the x and z subsystems, in the sense of the estimate of type (8).
- The following small-gain condition holds: there is a $\rho \in \mathcal{K}_\infty$ so that $(\gamma_1 + \rho) \circ (\gamma_2 + \rho) \leqslant r$ for all $r \geqslant 0$.

Then, the composite system is ISS. □

(Note that, in the special case in which the $\gamma_i(r) = g_i r$, the small gain condition is satisfied iff $g_1 g_2 < 1$, thus generalising the usual case.) It is important to note that the result in [12] is far more general; for instance, it deals with partially observed systems and with 'practical stability' notions. Also, the small gain condition can be stated just in terms of the gains with respect to the z and x variables. Related to these results is previous work on small-gain conditions, also relying on comparison functions, in [24,25].

A different cascade form, with an input feeding into both subsystems, is of interest in the context of stabilisation of saturated linear systems (using an approach originally due to Teel, cf. [26], and developed – in far more detail than we can do justice to here – in the recent paper [11]) and in other applications. This provides yet another illustration of the use of ISS ideas. The structure is (cf. Fig. 2):

$$\dot z = f(z,x,u)$$
$$\dot x = g(x,u).$$

First assume that a (locally Lipschitz) feedback law k can be found which makes the system $\dot z = f(z, xk(z))$ GAS *uniformly on* x, that is, $f(0,x,k(0)) = 0$ for all x and an estimate as in (7), $|z(t)| \leqslant \beta\,|z(0)|, t)$ holds, which is independent of $x(t)$. Suppose also that the x subsystem is ISS. Then, the feedback law $u = k(z)$ gives closed-loop equations $\dot z = f(z,x,k(z))$, $\dot x = g(x,k(z))$; because x is essentially irrelevant in the first equation, these equations behave just as a cascade of a GAS system (the z-system) and an ISS one, so the GAS property results as before. (More precisely, this is because it is still possible to find a Lyapunov function which depends only on z for the z-subsystem, due to the assumed uniformity property; see [7].) The interesting fact is that the same global conclusions hold under more local assumptions on the z-subsystem. Assume:

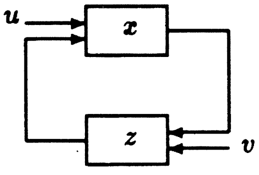


Fig. 1. Composite feedback form.

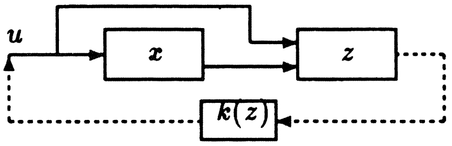


Fig. 2. Special cascade configuration.

- The z-subsystem is stabilisable with small feedback, uniformly on x small, meaning that for each $0 < \varepsilon \leqslant \varepsilon_0$ there is a (locally Lipschitz) feedback law k_ε with $|k_\varepsilon(z)| \leqslant \varepsilon$ for all z so that $\dot z = f(z,x,k_\varepsilon(z))$ is GAS uniformly on $|x| \leqslant \varepsilon_0$; further, under the feedback law $u = k_\varepsilon(z)$ the composite system is forward complete (solutions exist for all $t > 0$).
- The x subsystem is ISS.

(Later we discuss an interesting class of examples where these properties are verified.) Then, the claim is that, for any small enough $\varepsilon > 0$, the composite system under the feedback law $u = k_\varepsilon(z)$ is GAS. Stability is clear: for small x and z, trajectories coincide with those that would result if uniformity would hold globally on x (cf. the previous case). We are left to show that every solution $(x(\cdot), z(\cdot))$ satisfies $x(t) \to 0$ and $z(t) \to 0$ as $t \to +\infty$.

To establish this fact, pick any ε as follows. Let γ be a 'nonlinear asymptotic gain' as in Eq. (4), so that $\overline{\lim}_{t\to+\infty} |x(t,x_0,u)| \leqslant \gamma(\|u\|_\infty)$ for all inputs and initial conditions. Now take any $0 < \varepsilon < \varepsilon_0$ so that $\gamma(\varepsilon) < \varepsilon_0$. Pick any k_ε so that $|k_\varepsilon(z)| \leqslant \varepsilon$ for all z. Consider any solution $(x(\cdot), z(\cdot))$. Seeing $v(t) = k_\varepsilon(z(t))$ as an input to the x-subsystem, with $\|v\|_\infty \leqslant \varepsilon$, the choice of γ means that for some T, $t \geqslant T$ implies $|x(t)| < \varepsilon_0$. It follows that $z(t) \to 0$. Now the second equation is an ISS system with an input $v(t) \to 0$, so also $x(t) \to 0$, as required.

4. An Example

As a simple illustration of the use of the ISS concept, we may consider the oft-studied problem of globally stabilising to zero the angular momentum of a rigid body which is controlled by means of two external torques applied along principal axes, and suggest an alternative way of achieving this objective using ISS ideas. (This may represent a model of a satellite under the action of a pair of opposing jets.) The components of the state variable $\omega = (\omega_1, \omega_2, \omega_3)$ denote the angular velocity coordinates with respect to a body-fixed reference frame with origin at the centre of gravity and consisting of the principal axes. Letting the positive numbers I_1, I_2, I_3 denote the respective principal moments of inertia (positive numbers), this is a system on $\mathbb{R}^3$, with controls in $\mathbb{R}^2$ and equations:

$$I\dot{\omega} = S(\omega)I\omega + Bu, \tag{18}$$

where I is the diagonal matrix with entries I_1, I_2, I_3 and where B is a matrix in $\mathbb{R}^{3\times 2}$ whose columns describe the axes on which the control torques apply. Since it is being assumed that the two torques act along two principal axes, without loss of generality the columns of B are $(0,1,0)'$ and $(0,0,1)'$ respectively. The matrix $S(\omega)$ is the rotation matrix

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

Dividing by the I_j's, and applying the obvious feedback and coordinate transformations, there results a system on $\mathbb{R}^3$ of the form:

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2 \end{aligned}$$

where u_1 and u_2 are the controls.

To globally stabilise this system, and following the ideas of [18] for the corresponding local problem, one performs first a change of coordinates into new coordinates (x, z_1, z_2), where $x = x_1$ and

$$x_2 = -x_1 + z_1, x_3 = x_1^2 + z_2.$$

The system is now viewed as a cascade of two subsystems. One of these is described by the x variable, with z_1 and z_2 now thought of as inputs, and the second one is the z_1, z_2 subsystem. The first subsystem is precisely the one in example (10), and it is therefore ISS. Since a cascade of an ISS and a GAS system is again GAS, it is only necessary to stabilise the z_1, z_2 subsystem. In other words, looking at the system in the new coordinates:

$$\begin{aligned} \dot{x} &= -x^3 + x^2 z_1 - x z_2 + z_1 z_2 \\ \dot{z}_1 &= u_1 + (-x + z_1)(x^2 + z_2) \\ \dot{z}_2 &= u_2 - 2x_1(-x + z_1)^2(x^2 + z_2), \end{aligned}$$

any feedback that stabilises the last two equations will also make the composite system GAS. One may therefore use

$$u_1 = -x_1 - x_2 - x_2 x_3, u_2 = -x_3 + x_1^2 + 2x_1 x_2 x_3,$$

which renders the last two equations $\dot{z}_1 = -z_1$ and $\dot{z}_2 = -z_2$. As a remark, note that a conceptually different approach to the same problem can be based upon zero dynamics techniques ([27,28]). In that context, one uses Lie derivatives of a Lyapunov function for the x-subsystem in building a global feedback law (see the discussion in [13], Section 4.8). For the present rigid body stabilisation problem, the feedback stabilising law obtained using that approach would be as follows ([27]):

$$u_1 = -x_1 - x_2 - x_2 x_3 - 2x_1 x_3, u_2 = -x_3 + 3x_1^2 + 2x_1 x_2 x_3.$$

5. Linear Systems with Actuator Saturation

For linear systems subject to actuator saturation, more precise results regarding stabilisation can be obtained. The objective is to study control problems for plants P that can be described as in Fig. 3, where W indicates a linear transfer matrix. For simplicity, we consider here just the state-observation case, that is, systems of the type

$$\dot{x} = Ax + B\text{SAT}[u]. \tag{19}$$

By an L_p-stable system one means that the zero-initial state response induces a bounded operator $L_p \to L_p$. The following result was recently obtained by W. Liu,

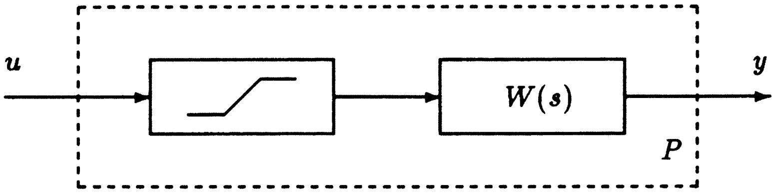


Fig. 3. Saturated-input linear system.

Y. Chitour and the author (see also [29] for related results on input-to-state dependence for such systems):

Theorem 5 [9]. Assume that the pair (A, B) is controllable and that A is neutrally stable (i.e., there is some symmetric positive definite Q so that $A^TQ + QA \leqslant 0$). Then, there exists a matrix F so that the system

$$\dot{x} = Ax + B\mathrm{SAT}[Fx + u]$$

is L_p-stable for each $1 \leqslant p \leqslant \infty$. □

The fact that GAS can be achieved for such systems is a well-known and classical application of dissipation ideas, and a quadratic Lyapunov function suffices; obtaining the ISS property, and in particular operator stability, is far harder. Not surprisingly, the proof involves establishing a dissipation inequality involving a suitable storage function. What is perhaps surprising is that the storage function that is used is only of class C^1, in general not smooth: V is of the form $x'Px + |x|^3$, for some positive definite P. One establishes by means of such a V that the system is ISS. Since the used V admits convex estimates (in the sense discussed in Section 2.7), stronger operator stability conclusions can be obtained. The second example given in Section 2.6 (system (11) and storage function (12)) illustrates the detailed calculations in a very simple case.

The hypotheses in Theorem 5 can be relaxed considerably. For instance, controllability can be weakened, and the result is also valid if, instead of SAT, one uses a more general bounded saturation function σ which satisfies: (1) near the origin, σ is in a sector $[\kappa_1, \kappa_2] : 0 < k_1 \leqslant \frac{\sigma(r)}{r} \leqslant \kappa_2$ for all $0 < |r| \leqslant 1$; and (2) sign $(r)\sigma(r) > \kappa > 0$ if $|r| > 1$.

A different line of work concerns linear systems subject to control saturation in the case in which the matrix A is not stable, but still has no eigenvalues with positive real part. This is the case, for instance, if A has a Jordan block of size at least two corresponding to an eigenvalue at the origin (the multiple integrator). In that case, L_p stabilisation is not possible, but, since the system is open-loop null-controllable (assuming, as in Theorem 5, that the pair (A, B) is controllable, or at least stabilisable as a linear pair), it is realistic to search for a globally stabilising feedback.

A first result showing that a smooth, globally stabilising feedback always exists was given in work by Sussmann and the author [30]. A remarkable design in terms of combinations of saturations was supplied by Teel ([26]), for the particular case of single-input multiple integrators, and a general construction based on Teel's ideas was completed recently in work of Sussmann, Yang and the author ([10]). For simplicity, call a function $\mathbb{R}^n \to \mathbb{R}^m$ each of whose coordinates has the form

$$\varphi_1 x + \alpha_1 \mathrm{SAT}[\varphi_2 x + \alpha_2 \mathrm{SAT}[\ldots \mathrm{SAT}$$
$$[\varphi_{s-1} x + \alpha_{s-1} \mathrm{SAT}[\varphi_s x]] \ldots]]$$

for some s and some real numbers α_i and linear functionals φ_i a *cascade of saturations*, and one for which coordinates have the form

$$\alpha_1 \mathrm{SAT}[\varphi_1 x] + \alpha_2 \mathrm{SAT}[\varphi_2 x] + \ldots + \alpha_s \mathrm{SAT}[\varphi_s x]$$

a *superposition of saturations*. (In the terminology of artificial neural networks, this last form is a 'single hidden layer net'.) There are two results, one for each of these controller forms:

Theorem 6 [10]. Consider the system (19), where the pair (A, B) is stabilisable and A has no eigenvalues with positive real part. Then there exist a cascade of saturations k and a super-position of saturations ℓ so that $\dot{x} = Ax + B\ \mathrm{SAT}[k(x)]$ and $\dot{x} = Ax + B\mathrm{SAT}[\ell(x)]$ are both GAS. □

(The coefficients α_i in the second case can be chosen arbitrarily small, which means that the second result could also be stated as stability of $\dot{x} = Ax + B\ell(x)$ since the saturation is then irrelevant.)

For cascades of saturations, this design proceeds in very rough outline as follows (the super-position case is similar). A preliminary step is to bring the original system (19) to the following composite form:

$$\dot{z} = A_1 z + B_1(-Fx + \mathrm{SAT}[u])$$
$$\dot{x} = A_2 x + B_2 \mathrm{SAT}[u],$$

where F is a matrix which has the property that the system $\dot{x} = A_2 x + B_2 \mathrm{SAT}[Fx + u]$ is ISS. (An example of such F is provided by the case $\dot{x} = -\mathrm{SAT}[x + u]$, shown earlier to be ISS, and more generally the case treated in Theorem 5.) Further, it is assumed that for each $\varepsilon > 0$ sufficiently small there is a (locally Lipschitz) feedback law k_ε with $|k_\varepsilon(z)| \leqslant \varepsilon$ for all z and so that $\dot{z} = A_1 z + B_1 k_\varepsilon(z)$ is GAS. Now the feedback law

$$u = Fx + k_\varepsilon(z)$$

is so that for small x and ε the z-eq. is GAS independently of x (in fact, the x variable is completely cancelled out), and hence the discussion given in connection with Fig. 2 applies. Thus the composite system is stabilised, assuming only that the z-subsystem can be stabilised with small feedback.

Moreover, $Fx + k_\varepsilon(z)$ has a cascade form provided that k_ε be a saturation of a cascade. These assumptions can be in turn obtained inductively, by decomposing the z-equation recursively into lower dimensional subsystems. (More precisely, instead of SAT, one may use a scaled version with smaller lower bounds, $\mathrm{SAT}_\delta[r] = \delta\mathrm{SAT}[r/\delta]$, and the proof is the same. This provides the small feedback needed in the inductive step.) See [10] for details as well as for a far more general result, which allows many other saturation functions σ instead of SAT.

6. Feedback Equivalence

As mentioned earlier, with the concept of ISS, it is possible to prove a general result on feedback equivalence. Consider two systems

$$\dot{x} = f(x,u) \quad \text{and} \quad \dot{x} = g(x,u)$$

with the same state and input value spaces (same n, m). These systems are *feedback equivalent* if there exist a smooth $k : \mathbb{R}^n \to \mathbb{R}^m$, and an m matrix Γ consisting of smooth functions having $\det \Gamma(x) \neq 0$ for all x, such that

$$g(x,u) = f(x, k(x) + \Gamma(x)u)$$

for all x and u (see Fig. 4). The systems are *strongly* feedback equivalent if this holds with $\Gamma = I$ (see Fig. 5).

Strong equivalence is the most interesting concept when studying actuator perturbations, while feedback equivalence is a natural concept in feedback linearisation and other design techniques. More general notions of feedback equivalence are also possible, not requiring transformations to be affine in u, but the main result will hold already for these types of transformations.

The system (1) is *stabilisable* if there exists a smooth function k (with $k(0) = 0$) so that

$$\dot{x} = f(x, k(x))$$

is GAS. Equivalently, the system is strongly feedback equivalent to a GAS system. It is ISS-*stabilisable* if it is feedback equivalent to an ISS system.

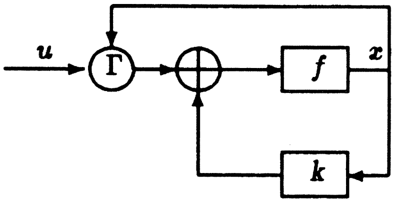


Fig. 4. Feedback equivalence.

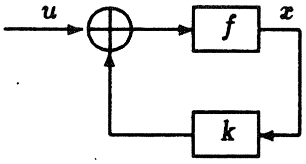


Fig. 5. Strong feedback equivalence.

Theorem 7 ([2,31]). The following properties are equivalent, for any system:

- The system is stabilisable.
- The system is ISS-stabilisable.

For systems affine in controls u (that is, $f(x,u)$ is affine in u) the above are also equivalent to strong feedback equivalence to ISS.

7. Input/Output Sability (IOS)

Until this point, only input to state ability has been discussed. It is possible to give an analogous definition for input/output operators. This will be done next, and a result will be stated which shows that this property is equivalent to internal stability under suitable reachability and observability conditions, just as with linear systems (cf., for instance, Section 6.3 in [13]).

An *i/o operator* is a causal map $F : L^m_{\infty,e} \to L^p_{\infty,e}$. (More generally, partially defined operators can be studied as well, but since only the stable case will be considered, and since stability implies that F is everywhere defined, there is no need to do so here; see [2] for more details.)

The i/o operator F is *input/output stable* (IOS) if there exist two functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that

$$| F(u)(t) | \leqslant \beta(\|u_t\|_\infty, t - T) + \gamma\left(\left\|u^t\right\|_\infty\right)$$

for all pairs $0 \leqslant T \leqslant t$ (a.e.) and all $u \in L^m_{\infty,e}$. Here u_t denotes, as earlier, the restriction of the input u to $[0,t]$ and u^t denotes its restriction to $[t, +\infty)$, in both cases seen as elements of $L^m_{\infty,e}$ having zero value outside of the considered range. This notion is well-behaved in various senses; for instance, it is closed under composition (serial connection), and $u \to 0$ implies $F(u) \to 0$.

Consider now a control system $\dot{x} = f(x,u)$ with outputs

$$y = h(x) \tag{20}$$

where $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous and satisfies $h(0) = 0$. With initial state $x_0 = 0$, this induces an operator

$$F(u)(t) := h(x(t, 0, u))$$

(*a priori* only partially defined). The system (1)–(20) is called IOS if this operator is.

The system with outputs (1)–(20) is *well-posed observable* ('strongly' observable in [2]) provided that the following property holds: there exist two functions α_1, α_2 of class $\mathcal{K}$ such that, for each triple of state, control, and output functions on $t \geqslant 0$

$$(x(\cdot), u(\cdot), y(\cdot))$$

satisfying the equations, the norms of these functions necessarily satisfy

$$\|x\|_\infty \leqslant \alpha_1(\|u\|_\infty) + \alpha_2(\|y\|_\infty). \tag{21}$$

Analogously, one has a notion of a *well-posed reachable* system (1). This is a system for which there is a function α_3 of class $\mathcal{K}$ with the following property: for each $x_0 \in \mathbb{R}^n$ there exists a time $T > 0$ and a control u so that

$$\|u\|_\infty \leqslant \alpha_3(|\, x_0 \,|)$$

and so that $x(T, 0, u) = x_0$.

For linear systems, these properties are equivalent to observability and reachability from zero respectively. In general, the first one corresponds to the possibility of reconstructing the state trajectory in a regular fashion – similar notions have been studied under various names, such as 'algebraic observability' or 'topological observability' – and the second models the situation where the energy needed to control from the origin to any given state must be in some sense proportional to how far this state is from the origin. The proof of the following result is a routine argument, and is quite similar to the proofs of analogous results in the linear case as well as in the dissipation literature:

Theorem 8 [2]. If (1) is ISS, then (1)–(20) is IOS. Conversely, if (1)–(20) is *IOS,* well-posed reachable, and well-posed observable, then (1) is ISS. □

Many variants of the notion of IOS are possible, in particular in order to deal with nonzero initial states ([8], [7]), or to study notions of practical stability, in which convergence to a small neighbourhood of the origin is desired. Also of interest is the study of the IOS (or even ISS) property relative to attracting invariant sets, not necessarily the origin and not even necessarily compact; see ([7,21]), for instance.

Acknowledgements

I wish especially to thank Yuan Wang and Zhong Ping Jiang for a careful reading of this manuscript and their many suggestions for its improvement.

References

1. Varaiya PP, Liu R. 'Bounded-input bounded-output stability of nonlinear time-varying differential systems', SIAM J. Control, 4, 1966; 698–704
2. Sontag ED. 'Smooth stabilization implies coprine factorization', IEEE Transactions on Automatic Control, AC-34, 1989; 435–443
3. Willems JC. 'Mechanisms for the stability and instability in feedback systems', Proc. IEEE, 64, 1976; 24–35
4. Hill DJ, Moylan P. 'Dissipative dynamical systems: Basic input-output and state properties', J. Franklin Institute, 5, 1980; 327–357
5. Hill DJ. 'Dissipative nonlinear systems: Basic properties and stability analysis', Proc. 31st IEEE Conf. Dec. and Control, Tucson, Arizona, 1992; 3259–3264
6. Sontag ED, Wang Y. 'On characterizations of the input-to-state stability property', Systems and Control Letters, 24, 1995; 351–359. (Preliminary version in 'Notions equivalent to input-to-state stability', in Proc. IEEE Conf. Decision and Control, Orlando, Dec. 1994, IEEE Publications, 1994; 3438–3443)
7. Lin Y, Sontag ED, Wang Y. 'A smooth converse Lyapunov theorem for robust stability', SIAM J. Control and Opt., to appear. (Preliminary version in 'Recent results on Lyapunov-theoretic techniques for nonlinear stability', in Proc. Amer. Automatic Control Conference, Baltimore, June 1994; 1771–1775)
8. Sontag ED, Lin Y. 'Stabilization with respect to noncompact sets: Lyapunov characterizations and effect of bounded inputs', in Proc. Nonlinear Control Systems Design Symp., Bordeaux, June 1992, (M. Fliess, Ed.), IFAC Publications, 9–14
9. Liu W, Chitour Y, Sontag ED. 'Remarks on finite gain stabilizability of linear systems subject to input saturation', SIAM J. Control and Opt., to appear. (Preliminary version in Proc. IEEE Conf. Decision and Control, San Antonio, Dec. 1993, IEEE Publications, 1993; 1808–1813)
10. Sussmann HJ, Sontag ED, Yang Y. 'Stabilization of linear systems with bounded controls', IEEE Trans. Autom. Control, 39, 1994; 2411–2425
11. Teel AR. 'A nonlinear small-gain theorem for the analysis of control systems with saturation', IEEE Trans. Autom. Ctr., to appear
12. Jiang Z-P, Teel A, Praly L. 'Small-gain theorem for ISS systems and applications', Math of Control, Signals, and Systems, 7, 1994; 104–130
13. Sontag ED. Mathematical Control Theory, Deterministic Finite Dimensional Systems, Springer–Verlag, New York, 1990
14. Tsinias J. 'Versions of Sontag's input to state stability condition and the global stabilizability problem', SIAM Journal on Control and Optimization, 31, 1993; 928–941

15. Tsinias J. 'Sontag's "input to state stability condition" and global stabilization using state detection', Systems & Control Letters, 20, 1993; 219–226
16. Praly L, Jiang Z-P. 'Stabilization by output feedback for systems with ISS inverse dynamics', Systems and Control Letters, 21, 1993; 19–34
17. Sontag ED. "Remarks on stabilization and input-to-state stability', Proc. IEEE Conf. Decision and control, Tampa, Dec. 1989, IEEE Publications, 1989; 1376–1378
18. Brockett RW. 'Asymptotic stability and feedback stabilization', in Differential Geometric Control theory (RW Brockett, RS Millman and HJ Sussmann, eds), Birkhauser, Boston, 1983; 181–191
19. Hahn W. Stability of Motion, Springer-Verlag, New York, 1967
20. Praly L, Wang Y. 'An equivalent definition of input-to-state stability and stabilization in spite of matched unmodelled dynamics', submitted for publication
21. Lin Y. Lyapunov Function Techniques for Stabilization, PhD Thesis, Mathematics Department, Rutgers, The State University of New Jersey, New Brunswick, New Jersey, 1992
22. Sontag ED, Wang Y. 'Characterizing the input-to-state stability property for set stability', Proc. Third Nonlinear Control System Design Symposium (NOLCOS), Lake Tahoe, CA, June 1995, to appear
23. Sontag ED, Teel A. 'Changing supply functions in input/state stable systems', IEEE Transactions on Automatic Control, 1995 to appear
24. Safonov MG. Stability and Robustness of Multivariable Feedback Systems, MIT Press, Cambridge, MA, 1980.
25. Mareels IM, Hill DJ. 'Monotone stability of nonlinear feedback systems', J. Math. Sys, Estimation and Control, 2, 1992; 275–291
26. Teel AR. 'Global stabilization and restricted tracking for multiple integrators with bounded controls', Systems and Control Letters, 18, 1992; 165–171
27. Byrnes CI, Isidori A. 'New results and counterexamples in nonlinear feedback stabilization', Systems and Control Letters, 12, 1989; 437–442
28. Tsinias J. 'Sufficient Lyapunovlike conditions for stabilization', Math. of Control, Signals, and Systems, 2, 1989; 343–357
29. Chitour Y, Liu W, Sontag ED. 'On the continuity and incremental-gain properties of certain saturated linear feedback loops', Intern. J. Robust and Nonlinear Control, to appear. (Preliminary version in Proc. IEEE Conf. Decision and Control, Orlando, Florida, Dec. 1994, IEEE Publications, 1994; 127–132)
30. Sontag ED, Sussmann HJ. 'Nonlinear output feedback design for linear systems with saturating controls', Proc. IEEE Conf. Decision and Control, Honolulu, Dec. 1990, IEEE Publications, 1990; 3414–3416
31. Sontag ED. 'Further facts about input to state stabilization', IEEE Transactions on Automatic Control, AC-35, 1990; 473–476