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REMARKS ON THE TIME-OPTIMAL CONTROL OF TWO-LINK MANIPULATORS

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ABSTRACT. Various preliminary results are given related to the problem of characterizing singular and time-optimal trajectories for (planar, rigid) two-link robotic manipulators with torque constraints. Most of the results are consequences of the nice Lie-algebraic structure of the corresponding control system.

1. Introduction.

A manipulator with two rotational links (see figure 1) can be modeled as in [PA], chapter 6. We let w be the column vector (u,v)' (prime indicates transpose), where u(t) and v(t) are the torques applied at each joint, and $\theta=(\theta_1,\theta_2)'$, where $\theta_1(t)$ and $\theta_2(t)$ are the respective angles at time t. An equation of the following type results:

$$w = M(\dot{\theta})\ddot{\theta} + N(\theta,\dot{\theta}) + Q(\theta) \ . \tag{1.1}$$

(The precise equations are given by [PA], equations 6.16 and 6.20.) Here $M(\cdot)$ is a 2 by 2 matrix of inertial terms, $N(\cdot)$ includes all contributions from Coriolis and centrifugal forces, and $Q(\cdot)$ characterizes all gravity loadings. The matrix $M(\cdot)$ is symmetric positive definite for all values of its arguments, and the vector $N(\cdot)$ is homogeneous quadratic in θ; some of the properties to follow will be consequences of these general facts about M and N, but most will depend on the actual form of the matrices. The torques w(t) are bounded; for simplicity we assume constraints of the type $|u|\leq K$, $|v|\leq L$, for some positive real numbers K,L, but more arbitrary intervals for u and v would give rise to similar results. We shall model such a system in state-space form, using the components of $(\theta(t),\theta(t))$ as the state at time t (details given below).

The *time-optimal control* problem is that of designing controls $w(\cdot)$ that achieve the transfer of a given state (θ,θ) to another given such state, in minimal time. A general theoretical approach to such problems is provided by the Maximum Principle. This approach can be used as a basis for numerical methods of solution, but even for the present low (4) dimensional system, the resulting nonlinear 2-point boundary value equations are very difficult to solve without some better knowledge of the singular structure of the problem. It is to that last topic that the present note addresses itself. We present preliminary results which provide "closed forms" for some optimal controls, and we study the existence of singular trajectories. The complexity of the "closed forms" obtained seems to indicate that they should be of use mainly in guiding the design of sophisticated numerical methods.

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The literature in (numerical) optimal control of manipulators is rather extensive, and we shall not give complete references in this short introduction. The reader may wish to consult the papers [RA], [SD], and [SH], as well as the references there and other papers in the conference volume in which they appear. As far as we are aware, a systematic study of singularities as the one taken here has not been attempted in previous work. We intend to direct further research to the understanding of what implications our results have for the algorithms given in the literature. For instance, our results may help in the "prunning" of possibilities in dynamic programing numerical methods.

The study of the singular structure of the problem is of great interest in itself, for the following reason. One of the main techiques used in practical robotic control consists in dividing the design effort into two stages: (1) find an open-loop control which achieves the desired state transfer, and (2) linearize along the resulting trajectory, and use a linear controller to regulate deviations from this trajectory. The essential point is that this last step will typically depend on controllability of the obtained linearization (as a time-varying linear system), and *a trajectory is singular precisely when this linearization is uncontrollable.* Thus, our characterizations of singular trajectories should help in determining if a trajectory suggested by step (1) is suitable for step (2).

The organization of this note is as follows. The first section illustrates some of the techniques through the comparatively trivial (and classical) case of single-link manipulators (nonlinear pendulum). Notations and definitions are given after that. The main observations are all consequences of a number of Lie algebraic facts given in the next section. After this, the existence of singular aextremals and singular optimal controls is investigated.

Briefly, it is easy to establish that all singular controls must take values in the boundary of the constraint set, i.e. that for each t either $u(t)=\pm K$ or $v(t)=\pm L$. It is a much more difficult problem, not completely resolved here, to determine those singular trajectories for which one of the controls is saturated but not the other. We show that if v is saturated then u must satisfy certain constraints; through every point of the state space, with the possible exception of those with $\theta_2=0,\pm\pi/2,\pi$, there is a one-parameter family of singular extremals of this kind. In the case where u is saturated, the situation is reversed: for all points in an open dense subset of the state space there are *no* such trajectories. Regarding optimality, in a certain sense about half of the singular trajectories with v saturated are not time-optimal, as concluded from the application of high order optimality conditions. The same arguments strongly suggest that the other "half" of these trajectories are (at least) locally optimal, but this depends on some apparently open questions on the high order theory.

In general, the results with respect to the control u are what one may expect for "generic" 4-dimensional systems, except that some of the arguments are especially simple due to the particular Lie algebraic structure of the robotics problem. For the control v, on the other hand, the system behaves in ways that one would *not* expect for a 4 dimensional system.

2. The single-link case.

First consider, as a trivial illustration of the technique, the case of a *single*-link manipulator, i.e. a pendulum with torque control. This model results when the mass and length of the second link approach zero in the general two-link problem. The defining equation is

$$\ddot{\theta} + c \sin\theta = d\, u \,, \tag{2.1}$$

where u satisfies the constraints $|u| \leq K$, for some nonzero constants c,d,K. (Suitable regularity conditions are assumed on u(·), typically that u(·) is measurable.) The objective is to control (2.1) in minimal time from any one given state $x = (\theta,\dot{\theta})'$ to any other given state. We write the above equation in usual control-system form: $\dot{x}(t) = f(x(t)) + u(t)g(x(t))$, where $f(x) := (\dot{\theta},-c\sin\theta)'$ and g(x) is the constant vector field (0,d)'. The state space can be taken to be either $\mathbf{S}^1 \times \Re$ or $\Re^2$, the latter for local questions or by lifting the system. By general existence results, for each pair of states, if there is a control driving one to the other, then there is one doing so in minimal time.

We are mainly interested in deciding if such a control must be "bang-bang". Recall that u is a bang-bang control if it switches finitely many times between the values K and -K. Let (x(t),u(t)) be a time-optimal trajectory, defined on the closed time interval I. Then, there is a nontrivial solution p(·) of the adjoint equation which satisfies the condition in the maximum principle. Let J be the (closed) set of zeroes of the corresponding ("switching") function $\phi(t) := \langle p(t), g(x(t))\rangle$. This is an absolutely continuous function, because p and x are, and g is smooth. If we prove that J is a finite set, then u will be forced to take the values $u(t) = \mathrm{sign}(\phi(t))$, with switches at the set J.

Since I is compact, it will be enough to show that J is discrete, or equivalently that for each τ in J there is a neighborhood N of τ in I such that $\phi(t) \neq 0$ for all $t \in N$, $t \neq \tau$. Take any such τ. A straightforward calculation shows that the derivative of ϕ is $\dot{\phi}(t) = \langle p(t), [f,g](x(t))\rangle$, where [,] is the standard Lie bracket. But [f,g] is the constant vector field (d,0)', and since $\det(g,[f,g]) \neq 0$, it is linearly independent of g (at every point of the state space), so by nontriviality of p, $\dot{\phi}(\tau)$ must be nonzero. Thus there is a neighborhood of τ where $\dot{\phi}(t)$ is still nonzero. In this neighborhood, ϕ cannot have any other zero besides τ, since being a continuously differentiable function this would imply the existence of a zero of its derivative. Thus the set J is indeed discrete. We established the following (well-known) result:

Lemma 2.1: For the pendulum model (2.1), every control corresponding to a singular extremal, and hence every optimal control, is bang-bang.∎

Note how Lie algebraic information about the system allows us to conclude that all trajectories are bang-bang. Of course, the above remarks are well-known for the case of the pendulum; in fact, the proof for this example, in [LM], p.427, that optimal controls are bang-bang, is --with different terminology-- precisely the same that we gave.

3. Definitions and Notations.

Most functions and vector fields appearing will have a particularly nice form; we introduce some appropiate terminology to refer to this form. If $M = \{(\theta_1,\cdots,\theta_r,z_1,\cdots,z_s)\}$ is a product of circles and Euclidean spaces, a *polynomial* (resp., *rational*,) function from M to an Euclidean space will be one that is polynomial (resp., rational,) on the z_i and on trigonometric functions of the θ_i. (It is understood that such a rational function is everywhere defined on M.) An *algebraic subset* of M is one defined by an equation f=0, where $f: M \to \Re$ is polynomial. This set is *nontrivial* if different from M. A *generic* subset is the complement of a nontrivial algebraic subset. Note that finite unions and intersections of generic sets are still generic. Generic sets are open dense, and calculations involving polynomial and rational functions in this generalized sense can be carried out in principle using methods and algorithms from elimination theory. Note that the definitions are consistent with viewing a circle as a an algebraic subset of the plane. It is sometimes useful in computations to apply the substitutions $\varsigma := \tan(\theta/2)$: then $\cos\theta = (1-\varsigma^2)/(1+\varsigma^2)$ and $\sin\theta = 2\varsigma/(1+\varsigma^2)$, and we can work with true rational functions.

The model we shall use is that in (1.1), as in Paul's book, pp.159-on. The parameters appearing are the masses m_1,m_2 and lengths ℓ_1,ℓ_2 of links, and the acceleration of gravity g. It is possible to reparametrize time and controls in such a way that $m_1=1$, $\ell_1=1$, and g=1 can be assumed. This is done by letting the time scale change according to $\tau = (\ell_2/g)^{1/2}$ and controls according to $w(\tau) = (m_2\ell_2 g)^{-1}w(\tau)$. We model (1.1) in state space form

$$\dot{x}(t) = f(x(t)) + u(t)g_1(x(t)) + v(t)g_2(x(t)) \,, \tag{3.1}$$

where $x(t) = x = (\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)'$ is in $(\mathbf{S}^1)^2 \times \Re^2$ for each t, and the vector fields f, g_1, g_2 are as induced from (1.1). It will be convenient to partition these equations into the parts corresponding to $y = (\theta_1,\theta_2)$ and $z = (\dot{\theta}_1,\dot{\theta}_2)$:

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= F(y,z) + G(y)w \,. \end{aligned} \tag{3.2}$$

Remark 3.1: With this notations, the following properties hold.

a. each coordinate of F(y,z) has the form $q(y,z)\sin y_2 + c(y)$, with q homogeneous quadratic in z ;
b. $\det G(y) \neq 0$ for all y ;
c. all entries of F and G are "rational" in the above sense;
d. G is symmetric positive definite, and depends only on y_2 .∎

Remark 3.2: What the natural state space for (3.1) should be is not a trivial question to decide, and seems not to be very clear in the literature. The equations suggest $x \in M$ with M the manifold $(\mathbf{S}^1)^2 \times \Re^2$. On the other hand, there are frequently constraints on the possible achievable angles, for instance $\theta_1 \in (-\pi/2,\pi/2)$. (Open, rather than closed, intervals should be used since there are no good mathematical results for systems evolving in manifolds with boundary.) It also seems reasonable to assume that the coordinates of the vector $Q(\theta)$ in (1.1) are in the interior of the control constraint set, for each θ, since otherwise fixed link positions cannot be mantained without the assistance of passive locking mechanisms. If Q_1 and Q_2 are the coordinates of Q, this condition is, then:

$$-K < Q_1(\theta) < K \text{ and } -L < Q_2(\theta) < L \text{ for each } \theta \,. \tag{3.3}$$

(See [STK] for a discussion of this hypothesis.) For small z, this means that all slow enough trajectories are achievable, and in particular that each state with z=0 is an equilibrium state. Let $S_o = \{(y,z) \text{ such that } z=0\}$. It follows that states in S_o are mutually reachable, when condition (3.3) holds. At this stage we shall not have any need to introduce this restriction, however. It will only appear at one minor point. Another interesting complication may be due to obstacles in the workspace. Further, it does not make physical sense to look at states which cannot be reached from the set S_o, so one may restrict attention to this reachable set (which is reasonable to assume open, because of the local controllability of S_o). Since the conclusions from our results are basically local, a great deal of this discussion is at this point not very important. All results will remain the same if we restrict the state space and consider only extremals that respect the given extra constraints.

We shall take $(\mathbf{S}^1)^2 \times \Re^2$ as the state space, but abuse terminology and notations by using local representations of objects associated to the $\mathbf{S}^1$ components, thinking of elements of $\mathbf{S}^1$ as numbers mod 2π. Thus vector fields on the state space are identified with vector functions $h = (h_1,h_2,h_3,h_4)'$ on $\Re^4$ which are periodic in the first two components.

Correspondingly, the joint differential equation for $x(t)$ and the adjoint covector $p(t)$ that is concluded in the Maximum Principle should be understood as a Hamiltonian system in an appropiate cotangent bundle, but because of the locality of the arguments, we shall write $p(t)$ as an element of $\Re^4$ again.■

An important notational convention is the following. We use the shorthand notation:

$$X_1X_2\cdots X_k := [X_1,[X_2,[\cdots,[X_{k-1},X_k]\cdots]]] \tag{3.4}$$

for iterated Lie compositions of vector fields. Note that $X^kY = \mathrm{ad}_X^kY$ with this notation. By an *iterated bracket* we shall mean an expression as above for which each X_i is one of f, g_1, or g_2, the vector fields appearing in the model (3.1). If α is a function $\Re^4\to\Re$, and q a vector field we denote by $L_q(\alpha) := \mathrm{grad}(\alpha).q$ the action of the vector field q on α (Lie derivation).

3.1. Singular extremals

Unless otherwise stated, I always denotes a nontrivial closed interval, and statements like "$\gamma\neq0$ a.e." mean precisely "$\gamma(t)\neq0$ a.e. for $t\in I$". An *extremal* (on the interval I) is given by functions (x,p,w) on I satisfying the simultaneous equations (3.1) and

$$\dot p(t) = D[f(x(t))+u(t)g_1(x(t))+v(t)g_2(x(t))]p(t) \tag{3.5}$$

(where $D[\cdots]$ denotes transpose of Jacobian matrix), and such that $p\neq0$ everywhere, w is suitably regular, $x(t)$ is not constant on I, and almost everywhere: $u<p(t),g_1(x(t))> + v<p(t),g_2(x(t))> = \max\{\upsilon<p(t),g_1(x(t))> + \nu<p(t),g_2(x(t))>\}$, where the maximum is over all (υ,ν) in U. We often write (x,p,u,v) to refer explicitly to the coordinates of w; on the other hand, if w is irrelevant to the discussion, we write just (x,p). Note that w is not required to satisfy any magnitude constraints in this definition: w is any (suitably regular) function $I\to\Re^2$. If w takes values in U, we talk of an *admissible* extremal.

The Maximum Principle states that every time-optimal trajectory (x,w) is part of an admissible extremal (x,p,w), along which

$$<p,f> + u<p,g_1> + v<p,g_2> = \text{constant} \geq 0 \quad . \tag{3.6}$$

A control w is *u-bang-bang* if $u=\pm K$ with finitely many switchings, and similarly for *v-bang-bang*; w is bang-bang if both components u and v are. We interpret these statements in the sense of equivalence modulo a set of measure zero. Thus a statement concluding that a control must be bang-bang must really be interpreted as equality to a bang-bang control almost everywhere. If an extremal is such that the *u-switching function* $\phi_u(t) := <p(t),g_1(x(t))>$ has only finitely many zeroes, then $u(t)$ must be equal almost everywhere to the function $K.\mathrm{sign}(\phi(t)) = \pm K$, which switches among K and -K at the zeroes of ϕ, and so u is bang-bang. Similarly for v and its associated switching function ϕ_v. The extremal (x,p,u,v) is *u-singular* if ϕ_u is zero a.e. on I. Similarly for *v-singular* extremals. A *singular extremal* is one that is u- or v-singular. A *nonsingular* extremal will be one which is not singular *on any subinterval.*

Remark 3.3: Singular admissible extremals are such that along the corresponding $x(t)$ and $u(t)$ the variational system ("linearization along the trajectory") is not controllable. (For a good discussion of all this, as well as an example of analogous problems in satellite control, see [BO].) If a trajectory is nonsingular but any of the controls takes values in the boundary of the control constraint set, the corresponding linearization is controllable, but physically one may not apply to the system controls which are small perturbations of the saturated controls. Thus it is of interest, independently of time-optimality questions, to study u-singular and v-singular extremals, corresponding to the study of the controllability of the linearizations with respect to either the control u or the control v.■

4. Useful algebraic facts.

4.1. Some Lie identities.

A large number of facts about Lie brackets of the vector fields f, g_1, g_2 appearing in equation (3.1) will be essential in the material to follow. We omit proofs. A few of these facts can be proved easily from the properties listed in remark (3.1); most were obtained from a very heavy amount of symbolic computation using MACSYMA running on a TOPS20 system.

Lemma 4.1: $\{g_1,g_2,fg_1,fg_2\}$ are linearly independent everywhere.

This is easy to see from the form of the equations (3.2). It is closely related to the fact that the kind of manipulator considered here is linearizable under feedback.

Lemma 4.2: Any iterated bracket with two more g_i's than f's is identically zero. In particular, $[g_1,g_2]=0$.

Lemma 4.3: There exist functions α_1, α_2, α_3 of the variable y_2, "rational" in the above defined sense, such that

- $g_1fg_1 = \alpha_1g_2$,
- $g_2fg_2 = \alpha_2g_2$, and
- $g_1fg_2 = g_2fg_1 = -g_1g_2f = -g_2g_1f = \alpha_3g_2$.

The function α_1 is of the form $c.\sin(2y_2)$, where c is a positive constant. The functions α_2 and α_3 have zeroes in particular at $y_2=0,\pi$ (and possibly other zeroes, depending on the manipulator parameters), but neither is identically zero.■

From lemma (4.1) it follows that g_1 and g_2 are independent, and from lemma (4.3) that g_1fg_1 is a nonzero multiple of g_2 almost everywhere, so:

Corollary 4.4: g_1fg_1 and g_1 are linearly independent if and only if $y_2 \neq 0,\pm\pi/2,\pi$.

A large number of expressions can be derived from the above. For instance, from the form of g_1 and g_2 it follows that $L_g(\alpha_i) = 0$ for $g = g_1$ or g_2 and the α_i in lemma (4.3). Thus we can conclude that, for instance,

$$\begin{aligned} g_1ffg_1 &= fg_1fg_1 = L_f(\alpha_1)g_2 + \alpha_1fg_2 \text{ , and} \\ g_2ffg_2 &= fg_2fg_2 = L_f(\alpha_2)g_2 + \alpha_2fg_2 \text{ .} \end{aligned} \tag{4.1}$$

Let e_4 be the constant vector field $(0,0,0,1)'$.

Lemma 4.5: There is are a positive constant γ and a "rational" function α of y_2 such that $g_2 = \alpha g_1 + \gamma e_4$.

The function α is in fact $\alpha = -[1+(\ell_1/\ell_2)\cos(y_2)]$, so it is nonzero everywhere if the parameters of the system are such that $\ell_1>\ell_2$. In that case, g_1 can be expressed in terms of g_2 and e_4.

The values of certain vector fields and determinants at various special points will also be useful. We have the following calculations:

$$g_1ffg_2 = 0 \quad \text{at all x of form } (y_1,0,0,0) \text{ .} \tag{4.2}$$

$$\det(g_2,fg_2,ffg_2,fffg_2) = c.\cos(y_1)^2 \text{ at } (y_1,0,0,0) \text{ .} \tag{4.3}$$

$$\det(f,g_2,fg_2,ffg_2) = c'.z_1.\cos(y_1) \text{ at } (y_1,z_1,0,0) \text{ .} \tag{4.4}$$

$$\det(g_1,g_2,fg_1,ffg_1) = \beta(y_2).z_1.\sin(2y_2) \text{ .} \tag{4.5}$$

In the above, both c and c' are nonzero constants, and β is a function satisfying $\beta(y_2)\neq0$ for all y_2.

4.2. Certain Lie Algebras

If L is a Lie algebra of vector fields, by the ***rank*** of L ***at the point*** x we mean the dimension of the subspace $\{X(x),X\in L\}$ of the tangent space at x. The smallest Lie algebra of vector fields containing given vector fields $X,Y,\cdots$ is denoted by $\{X,Y,\cdots\}_{LA}$. It is well known that in principle important controllability properties of the system (3.1) can be deduced from the structure of the Lie algebra $\{f,g_1,g_2\}_{LA}$ and of the ideal of $\{f,g_1,g_2\}_{LA}$ generated by $\{g_1,g_2\}$. From lemma (4.1), it is clear that the latter has rank 4 (full rank) at each point. This implies that certain degenerate behaviors can not occur in the time optimal problem; we omit details since we shall be able to conclude much more from other results.

We now want to study the algebras L^1, L^1_o, L^2, and L^2_o, which are obtained as follows for any given nonzero numbers k,ℓ.

$$L^1 := \{f-kg_1,g_2\}_{LA} \,, \tag{4.6}$$
$$L^1_o := \{(f+kg_1)^i g_2,\ i\geq 0\}_{LA} \,,$$
$$L^2 := \{f+\ell g_2,g_1\}_{LA} \,, \text{ and}$$
$$L^2_o := \{(f+\ell g_2)^i g_1,\ i\geq 0\}_{LA} \,.$$

For this purpose, it will be useful to consider those states (y,z) that are equilibrium points and satisfy $y_2=0$ (and, necessarily, $z_1=z_2=0$). Setting derivatives equal to zero in equation (1.1) results in the condition $Q(\theta)=w$, which gives rise to the equations

$$\sin(y_1) = c_1 u = c_2 v \,, \tag{4.7}$$

where the c_i are positive constants that depend on the masses and lengths of the links and are easily computed. Assume that u is set to a fixed constant value $k = K$ or $-K$. Then (3.1) becomes a single-control system. There is a corresponding control u_o, obtained from solving (4.7) for v, such that $f(x_o) + kg_1(x_o) + u_o g_2(x_o) = 0$, where $x_o = (0,y_1,0,0)$, and y_1 is also obtained from (4.7). (There may not be any such y_1, depending on c_1; in fact, the existence of such a point y_1 would contradict condition (3.3) if that condition is imposed on the system parameters.)

We now study the linearization of (3.1), seen as a single-input system, about (u_o,x_o). Let X be the vector field $f-kg_1$, and let $Y=g_2$. Then this linearization is a controllable linear system precisely when

$$\det\,(Y,XY,XXY,XXXY) \neq 0 \text{ at } x_o \,. \tag{4.8}$$

Now note that $XY = [f,g_2] + k[g_1,g_2] = fg_2$, $XXY = ffg_2 + \alpha_3 Y$, and $XXXY = fffg_2 - kg_1ffg_2 + v$, where v is in the span of Y and XY. Because of fact (4.2), we conclude that the determinant in (4.8) equals that in equation (4.3), so the linearization is controllable iff $y_1 \neq \pm\pi/2$. Recall that x_o was choosen so that $\sin(y_1) = c_2 k$. Let $k_o := 1/c_2$. When applying all this to the choices $k = \pm K$, we conclude:

Lemma 4.6: The above linearized systems at the equilibrium points with $z_1=z_2=y_2=0$ are controllable iff the system parameters satisfy $K \neq \pm k_o$.

Since the parameters are all physically quantities, we may expect that indeed these systems are all controllable.

Lemma 4.7: With the possible exception of the particular cases when $k = \pm k_o$, every state ξ can be steered to a state of the form $x = (y_1,z_1,0,0)$, where $z_1\neq 0$ and $\cos(y_1)\neq 0$, using controls of the form $u\equiv k$, v arbitrary real-valued piecewise analytic function. Same result using $u=-k$.

Proof: Note that the control v is *not* required to satisfy any magnitude constraints in this lemma. We prove the result for $u\equiv k$, the other case being analogous. The "computed torque" method is useful in establishing the desired controllability result. Introduce a new control ν equal to the angular acceleration of the second joint. The control v can be obtained as a combination of ν and the constant control u (in other words, we can partially linearize the dynamics using state feedback). More precisely, for any x, and any ν and $u=k=\pm K$, we may solve $f[4](x) + k.g_1[4](x) - v.g_2[4](x) = \nu$ for v, because $g_2[4](x)$ is always nonzero. (Here [4] indicates 4th coordinate, i.e. component with respect to z_2.) Thus, given any state ξ and any pair (y_2,z_2), it is always possible to find a control v(t) steering ξ into some state with the desired (y_2,z_2). In particular, we may steer ξ into a state with $y_2=z_2=0$. If the resulting state, say x, has third coordinate $z_1\neq 0$, we may apply now, for small time, the control v for which $\nu\equiv 0$. (And, of course, $u\equiv k$.) This will result in a state that still has $y_2=z_2=0$, such that also $z_1\neq 0$ (for small enough time), and, since $y_1=z_1$, such that $\cos(y_1)\neq 0$. Thus the controllability result is established unless $z_1=0$. Finally, assume that we ended up in a state x with $z_1=0$. There are two possibilities: either x is an equilibrium state (for the single-input system with $u\equiv k$, and with $y_2=0$,) or it is not. If not, then apply the control v such that $\nu\equiv 0$; since (y_1,z_1) does not remain constant, z_1 must become nonzero at some time, and we are done. And if x is an equilibrium state, lemma (4.6) says that for some value u_o the linearized system about this x and $u=k$, $v=u_o$ is controllable with respect to v. Thus the system is locally controllable about x, and hence we can reach any point in a neighborhood of x, and in particular points with $y_2=z_2=0$ and $z_1\neq 0$.■

Corollary 4.8: With the possible exception of the particular cases $k = \pm k_o$, the algebra L^1_o has rank 4 at each point.

Proof: If x is reachable from ξ then L^1_o has the same rank at x and ξ (see for instance [SJ]). The conclusion then follows from lemma (4.7) and property (4.4).■.

Lemma 4.9: For any ℓ, the algebra L^2_o has rank 4 at each point.

Proof: Let $X := f+\ell g_2$, $Y := g_1$. Consider the set of vector fields $\{Y,XY,YXY,XYXY\}$. By lemma (4.3) and formula (4.1), this set spans the same space as $\{g_1,fg_1,g_2,fg_2\}$ whenever x is such that $y_2 \neq 0,\pm\pi/2,\pi$. Thus the rank of L^2_o is 4 except possibly at points with $y_2 = 0,\pm\pi/2,\pi$. Now we argue as in the case of L^1_o. From the explicit system equations it follows that $g_1[3]$ is always nonzero, and hence, analogously to the previous case, we may always control an arbitrary state into one with $y_1=z_1=0$ (while mantaining $v\equiv k$). If a state with $z_2\neq 0$ is reached, then any control for small enough time will result in a final state with y_2 different from the above exceptional points. We are left with the cases where $y_1=z_1=z_2=0$ and y_2 is one of the above. From the equations (or physical intuition) it follows that when y_2 is either 0, $\pi/2$, or $-\pi/2$, the fourth component $g_1[4]$ of g_1 is positive. Thus applying a control of the type $u\equiv\beta$ constant<<0 (and $v\equiv k$) results, for small time, in $z_2<0$, and we reach a point with $y_2\neq 0$. Finally, if $y_2=\pi$ then one verifies that $g_2[4]>0$ there, and (equilibrium point for $u=v=0$) $f[4]$ vanishes there. Thus with $u\equiv 0$ one obtains for small time a motion for which z_2 has the same sign as k, and hence is also nonzero in case that ℓ is nonzero. We shall not need the case $\ell=0$, but it is worth including it for completness. When $x1=z_1=y_1=0$ and $y_2=\pi$, $\mathrm{sign}(g_1[4]) = \mathrm{sign}(\ell_1-\ell_2)$. Apply the control $u\equiv b>>0$ and $v\equiv 0$. If $\ell_1\neq\ell_2$, this results as before in a well-defined sign for z_2 and hence in a good y_2. If $\ell_1=\ell_2$, we argue by contradiction that y_2 becomes nonzero. Otherwise, z_2 and y_2 would remain zero along the trajectory, and this gives rise to an equation for z_1 of the type (small term)$-\beta$.(positive term bounded away from zero). Thus z_1, x1 become positive, and the last equation becomes $z_2 = c.\sin(x1)$, c a nonzero constant, which contradicts the fact that $z_2\equiv 0$.■

5. Singular extremals.

In this section and the next we apply the results of the previous section to the study of singular extremals. Recall that an extremal is u-singular if $\phi_u(t) = \langle p(t),g_1(t)\rangle$ vanishes along it, and similarly for v-singularity. The methods to be used all rely in the following classical fact, used already in the introductory section on the single-link case (c.f. lemma (2.1)). Assume that (x,p,w) is an extremal and that $\phi_u(t) = \phi_u(t') = 0$ for some $t,t'\in I$ (for instance, if the extremal is u-singular on the interval I). Consider the derivative of the (absolutely continuous) function ϕ_u. This derivative admits a very simple expression, namely, $\phi_u(t) = \langle p(t),[f,g_1](x(t)\rangle + v(t)\langle p,[g_2,g_1](x(t))\rangle$, which in our application, because of the property in lemma (4.2) just equals $\langle p,fg_1\rangle$. (From now on we omit the argument x(t) when clear from the context, and we use the notation introduced earlier for iterated Lie brackets.) It follows that ϕ_u is again absolutely continuous, and in particular that there is a t'' in the interval $(t,t')\subseteq I$ such that $\phi_u(t'') = 0$.

If ϕ_u had three zeroes, this induces two zeroes in ϕ_u, and we can try to repeat the process as long as we can differentiate. Typically, at some point either u or v will appear in a derivative, and this may serve to obtain an expression for the singular control. A similar argument applies to v-singular controls. A very trivial but interesting consequence of this method is the following. Let Z_u (respectively, Z_v,) be the closed subset of I consisting of the limit points of the set of zeroes of ϕ_u (resp., ϕ_v). Then:

Lemma 5.1: $Z_u \cap Z_v$ is empty.

Proof: Assume that t is in both sets. Thus there are sequences $\{t_n\}$ and $\{s_n\}$ converging to t such that $\phi_u(t_n) = \phi_v(s_n) = 0$ for all n. By the above arguments, there are also sequences $\{t'_n\}$ and $\{s'_n\}$ converging to t such that $\phi_u(t'_n) = \phi_v(s'_n) = 0$. By continuity, all of ϕ_u, ϕ_v, ϕ_u, and ϕ_v vanish at t. Thus, at t, $\langle p,g_1\rangle = \langle p,fg_1\rangle = \langle p,g_2\rangle = \langle p,fg_2\rangle = 0$. Since g_1, g_2, fg_1, fg_2 are linearly independent (c.f. lemma (4.1)), it follows that p(t)=0, contradicting nontriviality of the adjoint vector.■

Remark 5.2: Note that the same argument actually proves a bit more: if $t\in Z_u$ and $\phi_v(t) = 0$, then $\phi_v(t)\neq 0$. Thus ϕ_v does change sign at t, if it vanishes there at all and t is in the interior of I.■

Corollary 5.3: If the extremal (x,p,u,v) is u-singular [respectively, v-singular,] on the interval I, then it is v- [respectively, u-] bang-bang on I. In particular, there are no extremals that are simultaneously u- and v-singular.

Proof: We prove the result for u-singular extremals; the other case is analogous. By assumption, ϕ_u vanishes identically on the interval I. Thus $Z_u = I$, so by lemma (5.1) Z_v is empty. Thus the set of zeroes of ϕ_v in the compact interval I must be finite, as desired.■

It follows that every singular (and hence also each optimal) control takes values in the boundary of the control constraint set, and one is led to study extremals which are, either u-singular and v-bang-bang, or viceversa. Moreover, one can be much more precise about the points where v may switch values:

Lemma 5.4: If (x,p,w) is u-singular and $t\in I$ is such that $\phi_v(t)=0$, then $z_1(t)=0$ or $y_2(t)=0,\pm\pi/2,\pi$.

Proof: Arguing as before, we know that ϕ_u is absolutely continuous; its derivative is

$$\phi_u = \langle p,ffg_1\rangle + u\langle p,g_1fg_1\rangle + v\langle p,g_2fg_2\rangle \quad (5.1)$$

Since ϕ_u vanishes identically, this expression is zero almost everywhere. By lemma (4.3), the last two terms are multiples of ϕ_v, so they vanish at t. It follows that $\langle p,g_1\rangle = \langle p,g_2\rangle = \langle p,fg_1\rangle = \langle p,ffg_1\rangle = 0$. The result is then a consequence of formula (4.5).■

Further results can be obtained by taking one more derivative of ϕ_u and by using the fact (see remark (5.2) that ϕ_v must change sign.

We wish to show that certain kinds of degenerate behavior are ruled out in the two-link manipulator problem. The first is illustrated by the superficially analogous case of the linear system consisting of a parallel connection of two second order integrators, i.e. the system with equations $\ddot{y}_1 = u$, $\ddot{y}_2 = v$. The control constraints are $-1\leq u\leq 1$ and $-1\leq v\leq 1$. (We thank Elmer Gilbert for suggesting a comparison with this example.) Introducing states (y_1,z_1,y_2,z_2) as before, assume that v is set to a constant value, say $v\equiv 1$. Consider the problem of minimal time transfer from (0,0,0,0) to (0,0,1/2,1). Then, *any* measurable control u(t) satisfying the magnitude constraints and for which $\int_0^1\int_0^t u(\tau)d\tau dt = 0$ and $\int_0^1 u(t)dt = 0$ hold, is such that (u,v) is optimal. In other words, any control that transfers the first state to the second is optimal, as long as $v\equiv 1$.

Remark 5.5: What goes wrong in the above example is that the algebra L_o^2 is of rank less than full; in fact it has rank 2 at every point. In general, consider the single-control system with $x = X+uY$ (as one obtains when $v\equiv 1$). If the ideal L_o of $\{X,Y\}_{LA}$ generated by Y is of maximal rank r less than the dimension of the state space at the point ξ, then it is possible to change coordinates locally around ξ in such a way that the equations take the partitioned form $q = h(q,\tau,u)$ $\tau = 1$. (For a reference, see for instance [IS], p.41.) Then, for small T and for q' near q, *any* control u(t) taking q to q' in time T in the first equation will be such that it transfers x=(q,0) to x'=(q',T) in minimal time. As a converse of this fact, if L_o has full rank along a singular extremal, then the corresponding control is *uniquely* determined as a "feedback" function of (x,p) (in particular, u is piecewise smooth and optimal controls are "singular of finite order"). This is because the successive derivatives of the switching function result in expressions which contain combinations of the controls with coefficients of the type $\langle p,W\rangle$, with all possible W in L_o appearing eventually. These coefficients cannot all vanish bacause of the rank assumption on the Lie ideal and the nontriviality of p. This allows solving for u; some details will be given in the material to follow. In any case, corollary (4.8) and lemma (4.9) guarantee that this kind of behavior *does not* occur in our manipulator model (with the two possible exceptions $\pm k_o$, in the case of L_o^1).■

5.1. u-singular extremals.

If (x,p,w) is u-singular on I, we know by the above considerations that there is partition of I into finitely many intervals in each of which v is (a.e.) either L or -L. So we shall restrict attention to such subintervals. It is not difficult to prove that: If (x,p,w) is singular, it cannot hold that $\sin(2y_2)\equiv 0$. Motivated by this, we shall restrict attention to u-singular extremals contained in a generic subset of the state space, namely,

$$S_u := \{x=(y_1,y_2,z_1,z_2)' \text{ s.t. } z_1\neq 0 \text{ and } y_2\neq 0,\pm\pi/2,\pi\}\ .$$

The constraint on z_1 is most probably unnecessary, but it simplifies the treatment. Because of lemma (5.4), u-singular extremals included in S_u will have $v\equiv$constant. Note that in any case there is a gap in that a trajectory might intersect the set where $\sin(2y_2)=0$ at a pathological set of nonzero measure, so the classification of u-singular trajectories will not be complete even after the results to follow.

We introduce a couple of functions on the state space, "rational" in the sense defined earlier. Let $p = (p_1,p_2,p_3,p_4)'$ be any vector, and consider the problem of solving the equations

$$\langle p,g_1\rangle = 0 \quad\text{and}\quad \langle p,fg_1\rangle = 0 \quad (5.2)$$

at any given x. Since $g_1[3]$ (third coordinate of g_1) is never zero, we may solve in the first equation for p_3 in terms of p_4 (p_1 and p_2 do not appear due to the special block form of g_1). Similarly, the second equation can be used to express p_1 in terms of p_2,p_3,p_4, and hence just p_2 and p_4 (note that $fg_1[1] = -g_1[3]$). Thus there are "rational" functions c(x), d(x), and e(x) such that whenever the equations (5.2) hold then

$$p_3 = c(x)p_4,\ p_1 = d(x)p_2 + e(x)p_4, \quad (5.3)$$

and in particular $p = a(x)p_2 + b(x)p_4$, where the (co-) vectors a(x) and b(x) can be given explicitely as follows in terms of the coordinates of g_1 and fg_1:

$$a(x) := (-g_1[4]/g_1[3],1,0,0) \ ,$$

$$b(x) := (fg_1[4]/g_1[3]-fg_1[3]g_1[4]/(g_1[3])^2,0,-g_1[4]/g_1[3],1) \ .$$

On the set S_u we define the scalar functions: $r(x) := -\langle a,ffg_1\rangle/\alpha_1\gamma$ and $s(x) := -\langle b,ffg_1\rangle/\alpha_1\gamma - \ell\alpha_3/\alpha_1$, for any given fixed real number ℓ. Note that α_1 is nonzero in S_u. Further, from $a[3] = a[4] = 0$ it follows that $\langle a,g_1\rangle \equiv \langle a,g_2\rangle \equiv 0$, and from $(fg_1[1],fg_1[2]) = -(g_1[3],g_1[4])$ that $\langle a,fg_1\rangle \equiv 0$. From (4.5) we conclude that (on its domain S_u)

$$r(x) \neq 0 \quad \text{at all points} \ . \tag{5.4}$$

Consider now the adjoint equation for the covector p, assuming a given extremal. From the partitioned form (3.2) it follows that p_2 depends only on p_3 and p_4, and that p_4 equals $-p_2$ plus a linear combination of p_3 and p_4. When the extremal is u-singular, p satisfies (5.2), and we may substitute for p_3 and p_4 using equations (5.3). Then p_2 and p_4 satisfy the equations

$$p_2 = \chi(x,u,v)p_4 \tag{5.5}$$

$$p_4 = -p_2 - \psi(x)p_4 \ ,$$

where χ and ψ are functions whose expression can be easily obtained, and are "rational" in x and affine in u and v. We shall consider finally the following Riccati equation:

$$q = q^2 + \psi(x)q + \chi(x,u,v) \ , \tag{5.6}$$

to be thought of, for each given trajectory x(t) and controls u(t),v(t), as a time dependent scalar first order equation.

Theorem 5.1: Assume that (x,p,w) is a u-singular extremal with $x(t)\in S_u$ for all $t\in I$. Then there is a solution q(t) of the Riccati equation (5.6) on I such that

$$u(t) = r(x(t))q(t) + s(x(t)) \quad \text{for all } t\in I \ . \tag{5.7}$$

(Where the ℓ appearing in the definition of s(x) is the value $\pm$L of v on I.) Conversely, for each $x_o\in S_u$, each $\ell=\pm$L, and each real q_o, there is a u-singular extremal (x,p,w), and a solution of equation (5.6), both defined on an interval I which contains 0 in its interior, such that $x(0)=x_o$, $q(0)=q_o$, and equation (5.7) holds. Moreover, there is for each x_o in S_u a nonempty open interval $Q(x_o)\subseteq\Re$ with the following property: If $q\in Q$ then the singular extremal so constructed, for ℓ = L or -L, is an *admissible* u-singular v-bang-bang extremal.

Proof: Given such a u-singular extremal, consider the derivative ϕ_u given as in (5.1). This must vanish identically. Furthermore, $g_1fg_1 = \alpha_1g_2$, so by lemma (4.5) and the vanishing of $\phi_u = \langle p,g_1\rangle$ it follows that $\langle p,g_1fg_1\rangle = \alpha_1\gamma p_4$. Since v does not change sign, p_4 is never zero. Hence we can solve for u in (5.1), with $v\equiv\ell$; this results in

$$u = r(x)(p_2/p_4) + s(x) \ . \tag{5.8}$$

Calculating the derivative of p_2/p_4 using the adjoint equation and the substitutions (5.3) shows that $q := p_2/p_4$ satisfies the differential equation (5.6).

Conversely, let x_o, q_o, and ℓ be given. Solve the composite system (3.1) and (3.5) using $v\equiv\ell$ and u given by (5.8), and initial conditions $x(0):= x_o$, $p_4(0):= \mathrm{sign}(\ell)$, $p_2(0):= q_o.\mathrm{sign}(\ell)$, and $p_1(0)$, $p_3(0)$ given by equations (5.3) in terms of this data. (Solutions exist for small enough intervals I, by the existence theorem applied in the submanifold of the cotangent bundle consisting of all (x,p) with $p_4\neq0$, $\langle p,g_2(x)\rangle\neq0$ and $\sin(y_2)\neq0$.) We claim that the resulting extremal is u-singular. Consider ϕ_u. By the choice of $p_1(0)$ and $p_3(0)$ it follows that $\phi_u(0) = \phi_u(0) = 0$. Furthermore, ϕ_u is identically zero because of the definition of u(t). Thus ϕ_u vanishes identically, as desired. The statement about equation (5.6) is now clear from the previous paragraph. Finally, let

$$Q(x) := \left(-\frac{K}{|r(x)|} - \frac{s(x)}{r(x)}, \frac{K}{|r(x)|} - \frac{s(x)}{r(x)}\right). \tag{5.9}$$

By (5.4), this is well-defined. Note that $q\in Q(x)$ precisely when the u obtained from (5.7) satisfies $|u|<K$. When choosing an initial condition in $Q(x_o)$, we restrict the interval I if necessary so that $|u|$ remains less than L. This completes the proof.∎

To summarize, through each point of the generic set S_u there passes a one-parameter family of u-singular, v-bang-bang admissible extremals.

Remark 5.6: Equivalently, since $r\neq0$ we may give an equation like (5.6) for the control u(t). This can be achieved by substituting $q = (u-s)/r$ in both sides of equation (5.6). Thus the choice of u(0) and x(0) uniquely determines a corresponding singular extremal, which can be obtained by solving a set of 5 simultaneous ode's. This type of behavior is to be expected for 4-dimensional systems.

5.2. v-singular extremals.

The situation with v-singular trajectories is in a sense the opposite of that of u-singular ones. There is here a generic set where *no* possible such trajectories exist (as opposed to the previous case, where in a generic set there are many such trajectories passing through each state in the set). Let

$$S_v := \{x \text{ s.t. } \det(g_2,fg_2,ffg_2,fffg_2+kg_1ffg_2)(x)\neq0,\ k=\pm K\}$$

Note that $\det(g_2,fg_2,ffg_2,g_1ffg_2) = 0$ at any point with $y_2=0$ (c.f. property (4.3),) and that $\det(g_2,fg_2,ffg_2,fffg_2)$ is for instance nonzero at x=0. It follows that 0 is in S_v, which is then nontrivial and hence generic since it can be defined by "rational" inequalities.

Theorem 5.2: There are no v-singular trajectories intersecting the set S_v.

Proof: If (x,p,w) is v-singular then ϕ_v and ϕ_v both vanish identically on I, just as before, and $u=k=\pm K$. Hence $0 = \phi_v = \langle p,ffg_2\rangle + k\langle p,g_1fg_2\rangle + v\langle p,g_2fg_2\rangle = \langle p,ffg_2\rangle$, the last equality because of lemma (4.3). It follows that ϕ_v is also absolutely continuous, so we may take its derivative $0 = \phi_v^{(3)}=\langle p,fffg_2+kg_1ffg_2\rangle + \langle p,g_2ffg_2\rangle = \langle p,fffg_2+kg_1ffg_2\rangle$. The last equality follows from the second equation in (4.1). Since p is nontrivial, these derivatives cannot all be zero when x(t) is in S_v, proving the result.∎

6. Optimality.

By the results in the last section, there are in particular no time-optimal v-singular extremals in the generic set S_v. We shall concentrate on the case of u-singular extremals contained in the generic set S_u. Through each point in S_u there is a one-parameter family of such extremals, and optimal ones, if any, must be among these. To help single out extremals corresponding to time-optimal trajectories from other extremals, various authors (see e.g. [KR], [HE], [MO], and references in these papers,) have found stronger constraints than those implied by the maximum principle. The simplest of these generalizes the classical *Legendre-Clebsch* condition from variational calculus. We apply these conditions to the single-control system that results when v is set identically equal to $\ell=\pm$L. The necessary condition is then that, along the singular extremal,

$$\langle p,g_1fg_1\rangle \geq 0 \ . \tag{6.1}$$

Here g_1fg_1 is in fact equal to α_1g_2, and we know that on S_u, $\langle p,g_2\rangle$ is never zero. Thus *the above inequality is strict for trajectories that remain in* S_u. Since α_1 has the same sign as

$\sin(2y_2)$, we conclude the following result.

Theorem 6.1: Let (x,p,w) be an extremal corresponding to a time-optimal trajectory contained in the generic set S_u. Thus y_2 remains always in one of the intervals $(-\pi,-\pi/2)$, $(0,\pi/2)$, $(-\pi/2,0)$, or $(\pi/2,\pi)$. If y_2 is in either of the first two of these it follows that $v = L$, and if y_2 is in one of the last two then $v = -L$.∎

There is another constraint that an optimal trajectory must satisfy, and that is the one given by equation (3.6). Consider the following set of properties to be satisfied along an extremal, the set of which we shall call the *strict generalized Legendre-Clebsch* condition, or just SGLC:

a. $<p,g_1> = 0$
b. $<p,fg_1> = 0$
c. $<p,g_1fg_1> > 0$
d. $<p,f> + \ell<p,g_2> > 0$.

It is widely believed that this condition (in the single-input case, so that in the last property the second term does not appear,) is sufficient to imply that the corresponding trajectory is time optimal, at least in a suitable local sense, however, no complete proof seems to be available except in the analogous case in dimension 3 (see [MO]). Our last remarks will deal with the possibility of finding such trajectories in the set S_u. Whether such exist will depend on the relations between the parameters defining the system. Once admissible extremals have been found as in theorem (5.1), the only remaining constraint to satisfy is the one corresponding to the strict version of property (6.1). Dealing for instance with the case in which y_2 is in one of the first two intervals in theorem (6.1), this becomes, using $q := p_2/p_4$,

$$<a,f>q + <b,f> + L\gamma > 0 . \qquad (6.2)$$

(The functions a and b are as introduced earlier.) Let $S_u{}^*$ be subset of S_u consisting of those x such that y_2 is as desired and so that some $q \in Q(x)$ satisfies (6.2). We do not yet know much about $S_u{}^*$. In any case, it can be defined using "polynomial" (in our sense) inequalities, is an open set, and

Proposition 6.1: If x is in $S_u{}^*$ then there is an admissible singular extremal passing through x and such that condition SGLC is satisfied.∎

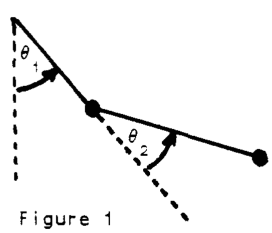

Figure 1

7. REFERENCES

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