# Input to State Stability: Basic Concepts and Results

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## 1 Introduction

The analysis and design of nonlinear feedback systems has recently undergone an exceptionally rich period of progress and maturation, fueled, to a great extent, by (1) the discovery of certain basic conceptual notions, and (2) the identification of classes of systems for which systematic decomposition approaches can result in effective and easily computable control laws. These two aspects are complementary, since the latter approaches are, typically, based upon the inductive verification of the validity of the former system properties under compositions (in the terminology used in [62], the "activation" of theoretical concepts leads to "constructive" control).

This expository presentation addresses the first of these aspects, and in particular the precise formulation of questions of robustness with respect to disturbances, formulated in the paradigm of *input to state stability*. We provide an intuitive and informal presentation of the main concepts. More precise statements, especially about older results, are given in the cited papers, as well as in several previous surveys such as [103, 105] (of which the present paper represents an update), but we provide a little more detail about relatively recent work. Regarding applications and extensions of the basic framework, we give some pointers to the literature, but we do not focus on feedback design and specific engineering problems; for the latter we refer the reader to textbooks such as [27, 43, 44, 58, 60, 66, 96].

## 2 ISS as a Notion of Stability of Nonlinear I/O Systems

Our subject is the study of *stability-type questions for input/output ("i/o") systems*. We later define more precisely what we mean by "system," but, in an intuitive sense, we have in mind the situation represented in Fig. 1, where the "system" may well represent a component ("module" or "subsystem") of a more complex, larger, system. In typical applications of control theory,


**Fig. 1.** I/O system

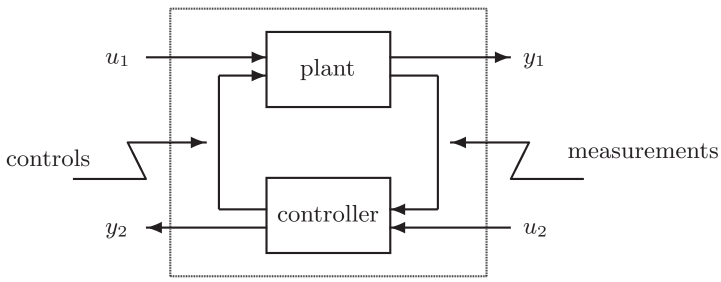


**Fig. 2.** Plant and controller

our "system" may in turn represent a plant/controller combination (Fig. 2), where the input $u = (u_1, u_2)$ incorporates actuator and measurement noises respectively, as well as disturbances or tracking signals, and where $y = (y_1, y_2)$ might consist respectively of some measure of performance (distance to a set of desired states, tracking error, etc.) and quantities directly available to a controller.

The goals of our work include:

- Helping develop a "toolkit" of concepts for studying systems via decompositions
- The quantification of system response to external signals
- The unification of state-space and input/output stability theories

### 2.1 Desirable Properties

We wish to formalize the idea of "stability" of the mapping $u(\cdot) \mapsto y(\cdot)$. Intuitively, we look for a concept that encompasses the properties that inputs that are bounded, "eventually small," "integrally small," or convergent, produce outputs with the respective property:

$$u \quad \left\{\begin{matrix} \text{bounded} \\ \text{(ev)small} \\ \text{(integ)small} \\ \to 0 \end{matrix}\right\} \quad \stackrel{?}{\Rightarrow} \quad y \quad \left\{\begin{matrix} \text{bounded} \\ \text{(ev)small} \\ \text{(integ)small} \\ \to 0 \end{matrix}\right\}$$

and, in addition, we will also want to account appropriately for initial states and transients. A special case is that in which the output $y$ of the system is just the internal state. The key notion in our study will be one regarding such stability from inputs to states; only later do we consider more general outputs. In terms of states, thus, the properties that we would like to encompass in a good stability notion include the the *convergent-input convergent-state* (CICS) and the *bounded-input bounded-state* (BIBS) properties.

We should remark that, for simplicity of exposition, we concentrate here solely on stability notions relative to globally attractive steady states. However, the general theory allows consideration of more arbitrary attractors (so that norms get replaced by, for example, distances to certain compact sets), and one may also consider local versions, as well as robust and/or adaptive concepts associated to the ones that we will define.

### 2.2 Merging Two Different Views of Stability

Broadly speaking, there are two main competing approaches to system stability: the *state-space approach* usually associated with the name of Lyapunov, and the *operator* approach, of which George Zames was one of the main proponents and developers and which was the subject of major contributions by Sandberg, Willems, Safonov, and others. Our objective is in a sense (Fig. 3) that of merging these "Lyapunov" and "Zames" views of stability. The operator approach studies the i/o mapping

$$\begin{array}{ccc} (x^0, u(\cdot)) & \mapsto & y(\cdot) \\ \mathbb{R}^n \times [\mathcal{L}_q(0,+\infty)]^m & \to & [\mathcal{L}_q(0,+\infty)]^p \end{array}$$

(with, for instance, $q = 2$ or $q = \infty$, and assuming the operator to be well-defined and bounded) and has several advantages, such as allowing the use

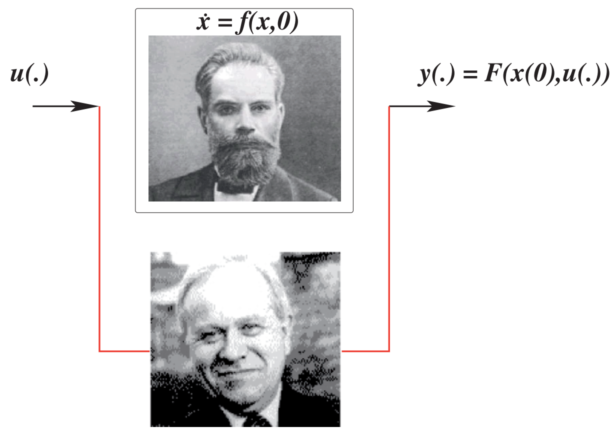


**Fig. 3.** Lyapunov state-space and Zames-like external stability

of Hilbert or Banach space techniques, and elegantly generalizing many properties of linear systems, especially in the context of robustness analysis, to certain nonlinear situations. The state-space approach, in contrast, is geared to the study of systems without inputs, but is better suited to the study of nonlinear dynamics, and it allows the use of geometric and topological ideas. The ISS conceptual framework is consistent with, and combines several features of, both approaches.

### 2.3 Technical Assumptions

In order to keep the discussion as informal and simple as possible, we make the assumption from now on that we are dealing with systems with inputs and outputs, in the usual sense of control theory [104]:

$$\dot{x}(t) \;=\; f(x(t),u(t))\,, \quad y(t) = h(x(t))$$

(usually omitting arguments $t$ from now on) with states $x(t)$ taking values in Euclidean space $\mathbb{R}^n$, inputs (also called "controls" or "disturbances" depending on the context) being measurable locally essentially bounded maps $u(\cdot) : [0,\infty) \to \mathbb{R}^m$, and output values $y(t)$ taking values in $\mathbb{R}^p$, for some positive integers $n, m, p$. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz with $f(0,0) = 0$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous with $h(0) = 0$. Many of these assumptions can be weakened considerably, and the cited references should be consulted for more details. We write $x(t, x^0, u)$ to denote the solution, defined on some maximal interval $[0, t_{\max}(x^0, u))$, for each initial state $x^0$ and input $u$. In particular, for systems with no inputs

$$\dot{x}(t) \;=\; f(x(t))\,,$$

we write just $x(t, x^0)$. The *zero-system* associated to $\dot{x} = f(x,u)$ is by definition the system with no inputs $\dot{x} = f(x,0)$. We use $|x|$ to denote Euclidean norm and $\|u\|$, or $\|u\|_\infty$ for emphasis, the (essential) supremum norm (possibly $+\infty$, if $u$ is not bounded) of a function, typically an input or an output. When only the restriction of a signal to an interval $I$ is relevant, we write $\|u_I\|_\infty$ (or just$\|u_I\|$), for instance $\|u_{[0,T]}\|_\infty$ when $I = [0,T]$, to denote the sup norm of that restriction.

### 2.4 Comparison Function Formalism

A *class* $\mathcal{K}_\infty$ *function* is a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is continuous, strictly increasing, unbounded, and satisfies $\alpha(0) = 0$ (Fig. 4), and a *class* $\mathcal{KL}$ *function* is a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t$ and $\beta(r,t) \searrow 0$ as $t \to \infty$.

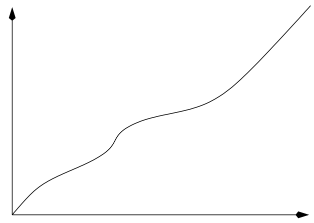

**Fig. 4.** $\mathcal{K}_\infty$-function

## 2.5 Global Asymptotic Stability

For a system with no inputs $\dot{x} = f(x)$, there is a well-known notion of global asymptotic stability (for short from now on, *GAS*, or "*0-GAS*" when referring to the zero-system $\dot{x} = f(x, 0)$ associated to a given system with inputs $\dot{x} = f(x, u)$) due to Lyapunov, and usually defined in "$\varepsilon$-$\delta$" terms. It is an easy exercise to show that this standard definition is in fact equivalent to the following statement:

$$(\exists\, \beta \in \mathcal{KL}) \quad |x(t, x^0)| \;\leq\; \beta\,(|x^0|, t) \quad \forall\, x^0\,,\ \forall\, t \geq 0\,.$$

Observe that, since $\beta$ decreases on $t$, we have, in particular:

$$|x(t, x^0)| \;\leq\; \beta\,(|x^0|, 0) \quad \forall\, x^0\,,\ \forall\, t \geq 0\,,$$

which provides the Lyapunov-stability or "small overshoot" part of the GAS definition (because $\beta\,(|x^0|, 0)$ is small whenever $|x^0|$ is small, by continuity of $\beta(\cdot, 0)$ and $\beta(0, 0) = 0$), while the fact that $\beta \to 0$ as $t \to \infty$ gives:

$$|x(t, x^0)| \;\leq\; \beta\,(|x^0|, t) \underset{t \to \infty}{\longrightarrow} 0 \quad \forall\, x^0\,,$$

which is the attractivity (convergence to steady state) part of the GAS definition.

We also remark a property proved in [102], Proposition 7, namely that for each $\beta \in \mathcal{KL}$ there exist two class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2$ such that:

$$\beta(r, t) \;\leq\; \alpha_2\left(\alpha_1(r) e^{-t}\right) \quad \forall\, s, t \geq 0\,,$$

which means that the GAS estimate can be also written in the form:

$$|x(t, x^0)| \;\leq\; \alpha_2\left(\alpha_1(|x^0|) e^{-t}\right)$$

and thus suggests a close analogy between GAS and an exponential stability estimate $|x(t, x^0)| \leq c\,|x^0|\, e^{-at}$.

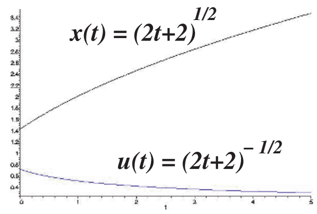


**Fig. 5.** Diverging state for converging input, for example

### 2.6 0-GAS Does Not Guarantee Good Behavior with Respect to Inputs

A *linear* system in control theory is one for which both $f$ and $h$ are linear mappings:

$$\dot{x} = Ax + Bu\,, \quad y = Cx$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. It is well-known that a linear system is 0-GAS (or "internally stable") if and only if the matrix $A$ is a *Hurwitz* matrix, that is to say, all the eigenvalues of $A$ have negative real parts. Such a 0-GAS linear system automatically satisfies all reasonable input/output stability properties: bounded inputs result in bounded state trajectories as well as outputs, inputs converging to zero imply solutions (and outputs) converging to zero, and so forth; see, e.g., [104]. But *the 0-GAS property is not equivalent*, in general, to input/output, or even input/state, stability of any sort. This is in general false for nonlinear systems. For a simple example, consider the following one-dimensional ($n = 1$) system, with scalar ($m = 1$) inputs:

$$\dot{x} \;=\; -x + (x^2 + 1)u\,.$$

This system is clearly 0-GAS, since it reduces to $\dot{x} = -x$ when $u \equiv 0$. On the other hand, solutions diverge even for some inputs that converge to zero. For example, take the control $u(t) = (2t+2)^{-1/2}$ and $x^0 = \sqrt{2}$, There results the unbounded trajectory $x(t) = (2t+2)^{1/2}$ (Fig. 5). This is in spite of the fact that the unforced system is GAS. Thus, the converging-input converging-state property does not hold. Even worse, the bounded input $u \equiv 1$ results in a finite-time explosion. This example is not artificial, as it arises in feedback-linearization design, as we mention below.

### 2.7 Gains for Linear Systems

For linear systems, the three most typical ways of defining input/output stability in terms of operators

$$\left\{L^2, L^\infty\right\} \to \left\{L^2, L^\infty\right\}$$

are as follows. (In each case, we mean, more precisely, to ask that there should exist positive $c$ and $\lambda$ such that the given estimates hold for all $t \geq 0$ and all solutions of $\dot{x} = Ax + Bu$ with $x(0) = x^0$ and arbitrary inputs $u(\cdot)$.)

$$\text{“}L^\infty \to L^\infty\text{”} : \qquad c\,|x(t,x^0,u)| \;\leq\; |x^0|\,e^{-\lambda t} + \sup_{s\in[0,t]} |u(s)|$$

$$\text{“}L^2 \to L^\infty\text{”} : \qquad c\,|x(t,x^0,u)| \;\leq\; |x^0|\,e^{-\lambda t} + \int_0^t |u(s)|^2\,ds$$

$$\text{“}L^2 \to L^2\text{”} : \qquad c\int_0^t |x(s,x^0,u)|^2\,ds \;\leq\; |x^0| + \int_0^t |u(s)|^2\,ds$$

(the missing case $L^\infty \to L^2$ is less interesting, being too restrictive). For linear systems, these are all equivalent in the following sense: if an estimate of one type exists, then the other two estimates exist too. The actual numerical values of the constants $c, \lambda$ appearing in the different estimates are not necessarily the same: they are associated to the various types of norms on input spaces and spaces of solutions, such as "$H_2$" and "$H_\infty$" gains, see, e.g., [23]. Here we are discussing only the question of *existence* of estimates of these types. It is easy to see that existence of the above estimates is simply equivalent to the requirement that the $A$ matrix be Hurwitz, that is to say, to 0-GAS, the asymptotic stability of the unforced system $\dot{x} = Ax$.

### 2.8 Nonlinear Coordinate Changes

A "geometric" view of nonlinear dynamics leads one to adopt the view that

*notions of stability should be invariant under (nonlinear) changes of variables*

– meaning that if we make a change of variables in a system which is stable in some technical sense, the system in new coordinates should again be stable in the same sense. For example, suppose that we start with the exponentially stable system $\dot{x} = -x$, but we make the change of variables $y = T(x)$ and wish to consider the equation $\dot{y} = f(y)$ satisfied by the new variable $y$. Suppose that $T(x) \approx \ln x$ for large $x$. If it were the case that the system $\dot{y} = f(y)$ is globally exponentially stable ($|y(t)| \leq ce^{-\lambda t}\,|y(0)|$ for some positive constants $c, \lambda$), then there would exist some time $t_0 > 0$ so that $|y(t_0)| \leq |y(0)|/2$ for all $y(0)$. But $\dot{y} = T'(x)\dot{x} \approx -1$ for large $y$, so $y(t_0) \approx y(0) - t_0$, contradicting $|y(t_0)| \leq |y(0)|/2$ for large enough $y(0)$. In conclusion, exponential stability is *not* a natural mathematical notion when nonlinear coordinate changes are allowed. This is why the notion of asymptotic stability is important.

Let us now discuss this fact in somewhat more abstract terms, and see how it leads us to GAS and, when adding inputs, to ISS. By a *change of coordinates* we will mean a map

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

such that the following properties hold: $T(0) = 0$ (since we want to preserve the equilibrium at $x = 0$), $T$ is continuous, and it admits an inverse map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ which is well-defined and continuous as well. (In other words, $T$ is a homeomorphism which fixes the origin. We could also add the requirement that $T$ should be differentiable, or that it be differentiable at least for $x \neq 0$, but the discussion to follow does not need this additional condition.) Now suppose that we start with a system $\dot{x} = f(x)$ that is exponentially stable:

$$|x(t, x^0)| \;\leq\; c\,|x^0|e^{-\lambda t} \quad \forall\, t \geq 0 \qquad \text{(some } \; c, \lambda > 0)$$

and we perform a change of variables:

$$x(t) \;=\; T(z(t))\,.$$

We introduce, for this transformation $T$, the following two functions:

$$\underline{\alpha}(r) := \min_{|x| \geq r} |T(x)| \quad \text{and} \quad \overline{\alpha}(r) := \max_{|x| \leq r} |T(x)| \;,$$

which are well-defined because $T$ and its inverse are both continuous, and are both functions of class $\mathcal{K}_\infty$ (easy exercise). Then,

$$\underline{\alpha}(|x|) \;\leq\; |T(x)| \;\leq\; \overline{\alpha}(|x|) \quad \forall\, x \in \mathbb{R}^n$$

and therefore, substituting $x(t, x^0) = T(z(t, z^0))$ in the exponential stability estimate:

$$\underline{\alpha}(|z(t, z^0)|) \;\leq\; c\,\overline{\alpha}(|z^0|)\,e^{-\lambda t}$$

where $z^0 = T^{-1}(x^0)$. Thus, the estimate in $z$-coordinates takes the following form:

$$|z(t, z^0)| \;\leq\; \beta\,(|z^0|, t)$$

where $\beta(r, t) = \underline{\alpha}^{-1}\left(c\overline{\alpha}\left(re^{-\lambda t}\right)\right)$ is a function of class $\mathcal{KL}$. (As remarked earlier, any possible function of class $\mathcal{KL}$ can be written in this factored form, actually.)

In summary, we re-derived the concept of global asymptotic stability simply by making coordinate changes on globally exponentially stable systems. So let us see next where these considerations take us when looking at systems with inputs and starting from the previously reviewed notions of stability for linear systems. Since there are now inputs, in addition to the state transformation $x(t) = T(z(t))$, we must now allow also transformations $u(t) = S(v(t))$, where $S$ is a change of variables in the space of input values $\mathbb{R}^m$. Arguing exactly as for the case of systems without inputs, we arrive to the following three concepts:

$$L^\infty \to L^\infty \rightsquigarrow \alpha\,(|x(t)|) \;\leq\; \beta(|x^0|\,, t) \;+\; \sup_{s \in [0,t]} \gamma(|u(s)|),$$

$$L^2 \to L^\infty \rightsquigarrow \alpha\left(|x(t)|\right) \;\leq\; \beta(|x^0|, t) \;+\; \int_0^t \gamma(|u(s)|)\, ds$$

$$L^2 \to L^2 \rightsquigarrow \int_0^t \alpha\left(|x(s)|\right)\, ds \;\leq\; \alpha_0(|x^0|) \;+\; \int_0^t \gamma(|u(s)|)\, ds\,.$$

From now on, we often write $x(t)$ instead of the more cumbersome $x(t, x^0, u)$ and we adopt the convention that, any time that an estimate like the ones above is presented, an unless otherwise stated, we mean that there should exist comparison functions ($\alpha, \alpha_0 \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$) such that the estimates hold for all inputs and initial states. We will study these three notions one at a time.

### 2.9 Input-to-State Stability

The "$L^\infty \to L^\infty$" estimate, under changes of variables, leads us to the first concept, that of *input to state stability* (ISS). That is, there should exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t)| \;\leq\; \beta(|x^0|, t) \;+\; \gamma\left(\|u\|_\infty\right) \qquad \text{(ISS)}$$

holds for all solutions. By "all solutions" we mean that this estimate is valid for all inputs $u(\cdot)$, all initial conditions $x^0$, and all $t \geq 0$. Note that we did not now include the function "$\alpha$" in the left-hand side. That is because, redefining $\beta$ and $\gamma$, one can assume that $\alpha$ is the identity: if $\alpha(r) \leq \beta(s,t)+\gamma(t)$ holds, then also $r \leq \alpha^{-1}(\beta(s,t)+\gamma(t)) \leq \alpha^{-1}(2\beta(s,t))+\alpha^{-1}(2\gamma(t))$; since $\alpha^{-1}(2\beta(\cdot,\cdot)) \in \mathcal{KL}$ and $\alpha^{-1}(2\gamma(\cdot)) \in \mathcal{K}_\infty$, an estimate of the same type, but now with no "$\alpha$," is obtained. In addition, note that the supremum $\sup_{s\in[0,t]} \gamma(|u(s)|)$ over the interval $[0,t]$ is the same as $\gamma(\|u_{[0,t]}\|_\infty) = \gamma(\sup_{s\in[0,t]}(|u(s)|))$, because the function $\gamma$ is increasing, and that we may replace this term by $\gamma(\|u\|_\infty)$, where $\|u\|_\infty = \sup_{s\in[0,\infty)} \gamma(|u(s)|)$ is the sup norm of the input, because the solution $x(t)$ depends only on values $u(s), s \leq t$ (so, we could equally well consider the input that has values $\equiv 0$ for all $s > t$).

It is important to note that a potentially weaker definition might simply have requested that this condition hold merely for all $t \in [0, t_{\max}(x^0, u))$. However, this definition turns out to be equivalent to the one that we gave. Indeed, if the estimate holds a priori only on such a maximal interval of definition, then, since the right-hand is bounded on $[0,T]$, for any $T > 0$ (recall that inputs are by definition assumed to be bounded on any bounded interval), it follows that the maximal solution of $x(t, x^0, u)$ is bounded, and therefore that $t_{\max}(x^0, u) = +\infty$ (see, e.g., Proposition C.3.6 in [104]). In other words, the ISS estimate holds for all $t \geq 0$ automatically, if it is required to hold merely for maximal solutions.

Since, in general, $\max\{a,b\} \leq a+b \leq \max\{2a, 2b\}$, one can restate the ISS condition in a slightly different manner, namely, asking for the existence of some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ (in general different from the ones in the ISS definition) such that

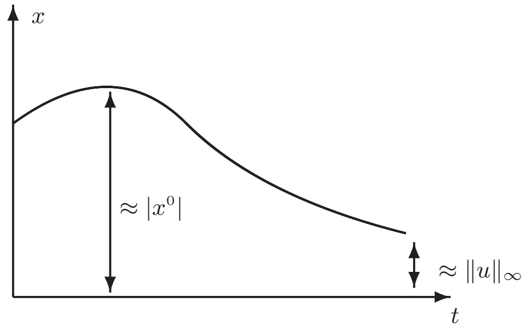


**Fig. 6.** ISS combines overshoot and asymptotic behavior

$$|x(t)| \;\leq\; \max\{\beta(|x^0|,t)\,,\,\gamma\,(\|u\|_\infty)\}$$

holds for all solutions. Such redefinitions, using "max" instead of sum, will be possible as well for each of the other concepts to be introduced later; we will use whichever form is more convenient in each context, leaving implicit the equivalence with the alternative formulation.

Intuitively, the definition of ISS requires that, for $t$ large, the size of the state must be bounded by some function of the sup norm – that is to say, the amplitude, – of inputs (because $\beta(|x^0|\,,t) \to 0$ as $t \to \infty$). On the other hand, the $\beta(|x^0|\,,0)$ term may dominate for small $t$, and this serves to quantify the magnitude of the transient (overshoot) behavior as a function of the size of the initial state $x^0$ (Fig. 6). The *ISS superposition theorem*, discussed later, shows that ISS is, in a precise mathematical sense, the conjunction of two properties, one of them dealing with asymptotic bounds on $|x^0|$ as a function of the magnitude of the input, and the other one providing a transient term obtained when one ignores inputs.

### 2.10 Linear Case, for Comparison

For internally stable linear systems $\dot{x} = Ax + Bu$, the variation of parameters formula gives immediately the following inequality:

$$|x(t)| \;\leq\; \beta(t)\,|x^0| \,+\, \gamma\|u\|_\infty\,,$$

where

$$\beta(t) = \|e^{tA}\| \;\to\; 0 \quad\text{ and }\quad \gamma = \|B\|\int_0^\infty \|e^{sA}\|ds \;<\; \infty\,.$$

This is a particular case of the ISS estimate, $|x(t)| \leq \beta(|x^0|,t) + \gamma\,(\|u\|_\infty)$, with linear comparison functions.

## 2.11 Feedback Redesign

The notion of ISS arose originally as a way to precisely formulate, and then answer, the following question. Suppose that, as in many problems in control theory, a system $\dot{x} = f(x, u)$ has been stabilized by means of a feedback law $u = k(x)$ (Fig. 7), that is to say, $k$ was chosen such that the origin of the closed-loop system $\dot{x} = f(x, k(x))$ is globally asymptotically stable. (See, e.g., [103] for a discussion of mathematical aspects of state feedback stabilization.) Typically, the design of $k$ was performed by ignoring the effect of possible *input disturbances* $d(\cdot)$ (also called actuator disturbances). These "disturbances" might represent true noise or perhaps errors in the calculation of the value $k(x)$ by a physical controller, or modeling uncertainty in the controller or the system itself. What is the effect of considering disturbances? In order to analyze the problem, we incorporate $d$ into the model, and study the new system $\dot{x} = f(x, k(x) + d)$, where $d$ is seen as an input (Fig. 8). We then ask what is the effect of $d$ on the behavior of the system.

Disturbances $d$ may well destabilize the system, and the problem may arise even when using a routine technique for control design, feedback linearization. To appreciate this issue, we take the following very simple example. We are given the system

$$\dot{x} = f(x, u) = x + (x^2 + 1)u \,.$$

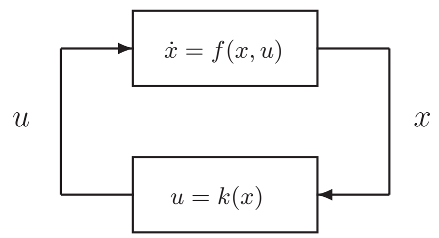


**Fig. 7.** Feedback stabilization, closed-loop system $\dot{x} = f(x, k(x))$

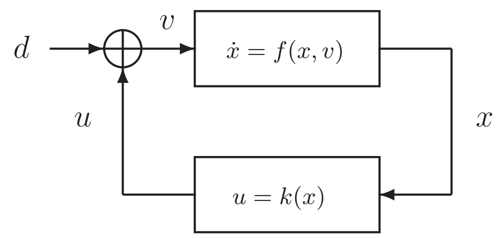


**Fig. 8.** Actuator disturbances, closed-loop system $\dot{x} = f(x, k(x) + d)$

In order to stabilize it, we first substitute $u = \frac{\tilde{u}}{x^2+1}$ (a preliminary feedback transformation), rendering the system linear with respect to the new input $\tilde{u}$: $\dot{x} = x + \tilde{u}$, and then we use $\tilde{u} = -2x$ in order to obtain the closed-loop system $\dot{x} = -x$. In other words, in terms of the original input $u$, we use the feedback law:

$$k(x) \;=\; \frac{-2x}{x^2+1}$$

so that $f(x, k(x)) = -x$. This is a GAS system. Next, let us analyze the effect of the disturbance input $d$. The system $\dot{x} = f(x, k(x) + d)$ is:

$$\dot{x} \;=\; -x + (x^2+1)\,d\,.$$

As seen before, this system has solutions which diverge to infinity even for inputs $d$ that converge to zero; moreover, the constant input $d \equiv 1$ results in solutions that explode in finite time. Thus $k(x) = \frac{-2x}{x^2+1}$ was not a good feedback law, in the sense that its performance degraded drastically once that we took into account actuator disturbances.

The key observation for what follows is that, if we add a correction term "$-x$" to the above formula for $k(x)$, so that we now have:

$$\tilde{k}(x) \;=\; \frac{-2x}{x^2+1}\; \boldsymbol{-x}$$

then the system $\dot{x} = f(x, \tilde{k}(x) + d)$ with disturbance $d$ as input becomes, instead:

$$\dot{x} \;=\; -\,2x\; \boldsymbol{-\, x^3} \;+\; (x^2+1)\,d$$

and this system is much better behaved: it is still GAS when there are no disturbances (it reduces to $\dot{x} = -2x - x^3$) but, in addition, it is ISS (easy to verify directly, or appealing to some of the characterizations mentioned later). Intuitively, for large $x$, the term $-x^3$ serves to dominate the term $(x^2+1)d$, for all bounded disturbances $d(\cdot)$, and this prevents the state from getting too large.

### 2.12 A Feedback Redesign Theorem for Actuator Disturbances

This example is an instance of a general result, which says that, whenever there is some feedback law that stabilizes a system, there is also a (possibly different) feedback so that the system with external input $d$ (Fig. 9) is ISS.

**Theorem 2.1.** [99] *Consider a system affine in controls*

$$\dot{x} \;=\; f(x,u) \;=\; g_0(x) + \sum_{i=1}^{m} u_i g_i(x) \qquad (g_0(0) = 0)$$

*and suppose that there is some differentiable feedback law $u = k(x)$ so that*

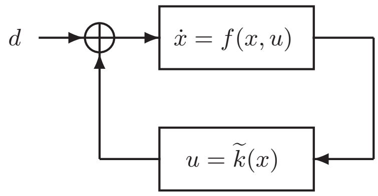


**Fig. 9.** Different feedback ISS-stabilizes

$$\dot{x} \;=\; f(x, k(x))$$

*has $x = 0$ as a GAS equilibrium. Then, there is a feedback law $u = \widetilde{k}(x)$ such that*

$$\dot{x} \;=\; f(x, \widetilde{k}(x) + d)$$

*is ISS with input $d(\cdot)$.*

The proof is very easy, once that the appropriate technical machinery has been introduced: one starts by considering a smooth Lyapunov function $V$ for global asymptotic stability of the origin in the system $\dot{x} = f(x, k(x))$ (such a $V$ always exists, by classical converse theorems); then $\hat{k}(x) := -(L_G V(x))^T = -(\nabla V(x) G(x))^T$, where $G$ is the matrix function whose columns are the $g_i$, $i = 1, \ldots, m$ and $T$ indicates transpose, provides the necessary correction term to add to $k$. This term has the same degree of smoothness as the vector fields making up the original system. Somewhat less than differentiability of the original $k$ is enough for this argument: continuity is enough. However, if no continuous feedback stabilizer exists, then no smooth $V$ can be found. (Continuous stabilization of nonlinear systems is basically equivalent to the existence of what are called smooth control-Lyapunov functions, see, e.g., [103].) In that case, if only discontinuous stabilizers are available, the result can still be generalized, see [79], but the situation is harder to analyze, since even the notion of "solution" of the closed-loop system $\dot{x} = f(x, k(x))$ has to be carefully defined.

There is also a redefinition procedure for systems that are not affine on inputs, but the result as stated above is false in that generality, and is much less interesting; see [101] for a discussion.

The above feedback redesign theorem is merely the beginning of the story. See for instance the book [60], and the references given later, for many further developments on the subjects of recursive feedback design, the "backstepping" approach, and other far-reaching extensions.

## 3 Equivalences for ISS

Mathematical concepts are useful when they are "natural" in the sense that they can be equivalently stated in many different forms. As it turns out, ISS can be shown to be equivalent to several other notions, including asymptotic gain, existence of robustness margins, dissipativity, and an energy-like stability estimate. We review these next.

### 3.1 Nonlinear Superposition Principle

Clearly, if a system is ISS, then the system with no inputs $\dot{x} = f(x, 0)$ is GAS: the term $\|u\|_\infty$ vanishes, leaving precisely the GAS property. In particular, then, the system $\dot{x} = f(x, u)$ is *0-stable*, meaning that the origin of the system without inputs $\dot{x} = f(x, 0)$ is stable in the sense of Lyapunov: for each $\varepsilon > 0$, there is some $\delta > 0$ such that $|x^0| < \delta$ implies $|x(t, x^0)| < \varepsilon$. (In comparison-function language, one can restate 0-stability as: there is some $\gamma \in \mathcal{K}$ such that $|x(t, x^0)| \le \gamma(|x^0|)$ holds for all small $x^0$.)

On the other hand, since $\beta(|x^0|, t) \to 0$ as $t \to \infty$, for $t$ large one has that the first term in the ISS estimate $|x(t)| \le \max\{\beta(|x^0|, t), \gamma(\|u\|_\infty)\}$ vanishes. Thus an ISS system satisfies the following *asymptotic gain property ("AG"):* there is some $\gamma \in \mathcal{K}_\infty$ so that:

$$\overline{\lim_{t\to+\infty}} |x(t, x^0, u)| \ \le\ \gamma(\|u\|_\infty) \quad \forall\, x^0,\, u(\cdot) \qquad \text{(AG)}$$

(see Fig. 10). In words, for all large enough $t$, the trajectory exists, and it gets arbitrarily close to a sphere whose radius is proportional, in a possibly nonlinear way quantified by the function $\gamma$, to the amplitude of the input. In the language of robust control, the estimate (AG) would be called an "ultimate boundedness" condition; it is a generalization of attractivity (all trajectories converge to zero, for a system $\dot{x} = f(x)$ with no inputs) to the case of systems with inputs; the "lim sup" is required since the limit of $x(t)$ as $t \to \infty$ may well not exist. From now on (and analogously when defining other properties), we will just say "the system is AG" instead of the more cumbersome "satisfies the AG property."

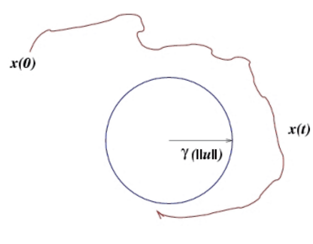


**Fig. 10.** Asymptotic gain property

Observe that, since only large values of $t$ matter in the limsup, one can equally well consider merely tails of the input $u$ when computing its sup norm. In other words, one may replace $\gamma(\|u\|_\infty)$ by $\gamma(\overline{\lim}_{t\to+\infty}|u(t)|)$, or (since $\gamma$ is increasing), $\overline{\lim}_{t\to+\infty}\gamma(|u(t)|)$.

The surprising fact is that these two necessary conditions are also sufficient. We call this the *ISS superposition theorem*:

**Theorem 3.1.** [110] *A system is ISS if and only if it is 0-stable and AG.*

This result is nontrivial. The basic difficulty is in establishing uniform convergence estimates for the states, i.e., in constructing the $\beta$ function in the ISS estimate, independently of the particular input. As in optimal control theory, one would like to appeal to compactness arguments (using weak topologies on inputs), but there is no convexity to allow this. The proof hinges upon a lemma given in [110], which may be interpreted [41] as a relaxation theorem for differential inclusions, relating global asymptotic stability of an inclusion $\dot{x} \in F(x)$ to global asymptotic stability of its convexification.

A minor variation of the above superposition theorem is as follows. Let us consider the *limit property* (LIM):

$$\inf_{t\geq 0}|x(t,x^0,u)| \leq \gamma(\|u\|_\infty) \quad \forall\, x^0,\, u(\cdot) \tag{LIM}$$

(for some $\gamma \in \mathcal{K}_\infty$).

**Theorem 3.2.** [110] *A system is ISS if and only if it is 0-stable and LIM.*

### 3.2 Robust Stability

Let us call a system *robustly stable* if it admits a *margin of stability* $\rho$, by which we mean some smooth function $\rho \in \mathcal{K}_\infty$ which is such that the system

$$\dot{x} = g(x,d) := f(x, d\rho(|x|))$$

is GAS uniformly in this sense: for some $\beta \in \mathcal{KL}$,

$$|x(t,x^0,d)| \leq \beta(|x^0|\,,t)$$

for all possible $d(\cdot) : [0,\infty) \to [-1,1]^m$. An alternative way to interpret this concept (cf. [109]) is as uniform global asymptotic stability of the origin with respect to all possible time-varying feedback laws $\Delta$ bounded by $\rho$: $|\Delta(t,x)| \leq \rho(|x|)$. In other words, the system

$$\dot{x} \;=\; f(x,\Delta(t,x))$$

(Fig. 11) is stable uniformly over all such perturbations $\Delta$. In contrast to the ISS definition, which deals with all possible "open-loop" inputs, the present notion of robust stability asks about all possible closed-loop interconnections. One may think of $\Delta$ as representing uncertainty in the dynamics of the original system, for example.

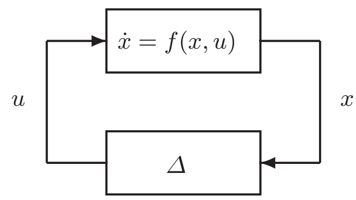


**Fig. 11.** Margin of robustness

**Theorem 3.3.** [109] *A system is ISS if and only if it is robustly stable.*

Intuitively, the ISS estimate $|x(t)| \leq \max\{\beta(|x^0|, t), \gamma(\|u\|_\infty)\}$ tells us that the $\beta$ term dominates as long as $|u(t)| \ll |x(t)|$ for all $t$, but $|u(t)| \ll |x(t)|$ amounts to $u(t) = d(t).\rho(|x(t)|)$ with an appropriate function $\rho$. This is an instance of a "small gain" argument, about which we will say more later.

One analog for linear systems is as follows: if $A$ is a Hurwitz matrix, then $A + Q$ is also Hurwitz, for all small enough perturbations $Q$; note that when $Q$ is a nonsingular matrix, $|Qx|$ is a $\mathcal{K}_\infty$ function of $|x|$.

### 3.3 Dissipation

Another characterization of ISS is as a dissipation notion stated in terms of a Lyapunov-like function.

We will say that a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a *storage function* if it is positive definite, that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, and proper, that is, $V(x) \to \infty$ as $|x| \to \infty$. This last property is equivalent to the requirement that the sets $V^{-1}([0, A])$ should be compact subsets of $\mathbb{R}^n$, for each $A > 0$, and in the engineering literature it is usual to call such functions *radially unbounded*. It is an easy exercise to show that $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if and only if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \quad \forall\, x \in \mathbb{R}^n$$

(the lower bound amounts to properness and $V(x) > 0$ for $x \neq 0$, while the upper bound guarantees $V(0) = 0$). We also use this notation: $\dot{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the function:

$$\dot{V}(x, u) := \nabla V(x).f(x, u)$$

which provides, when evaluated at $(x(t), u(t))$, the derivative $dV/dt$ along solutions of $\dot{x} = f(x, u)$.

An *ISS-Lyapunov function* for $\dot{x} = f(x, u)$ is by definition a smooth storage function $V$ for which there exist functions $\gamma, \alpha \in \mathcal{K}_\infty$ so that

$$\dot{V}(x, u) \leq -\alpha(|x|) + \gamma(|u|) \quad \forall\, x, u\,. \tag{L-ISS}$$

In other words, an ISS-Lyapunov function is a smooth (and proper and positive definite) solution of a *partial differential inequality* of this form, for appropriate $\alpha, \gamma$. Integrating, an equivalent statement is that, along all trajectories of the system, there holds the following dissipation inequality:

$$V(x(t_2)) - V(x(t_1)) \;\leq\; \int_{t_1}^{t_2} w(u(s), x(s))\, ds$$

where, using the terminology of [126], the "supply" function is $w(u,x) = \gamma(|u|) - \alpha(|x|)$. Note that, for systems with no inputs, an ISS-Lyapunov function is precisely the same as a Lyapunov function in the usual sense. Massera's Theorem says that GAS is equivalent to the existence of smooth Lyapunov functions; the following theorem provides a generalization to ISS:

**Theorem 3.4.** [109] *A system is ISS if and only if it admits a smooth ISS-Lyapunov function.*

Since $-\alpha(|x|) \leq -\alpha(\overline{\alpha}^{-1}(V(x)))$, the ISS-Lyapunov condition can be restated as

$$\dot{V}(x,u) \;\leq\; -\tilde{\alpha}(V(x)) \,+\, \gamma(|u|) \quad \forall\, x, u$$

for some $\tilde{\alpha} \in \mathcal{K}_\infty$. In fact, one may strengthen this a bit [93]: for any ISS system, there is a always a smooth ISS-Lyapunov function satisfying the "exponential" estimate $\dot{V}(x,u) \leq -V(x) + \gamma(|u|)$.

The sufficiency of the ISS-Lyapunov condition is easy to show, and was already in the original paper [99]. A sketch of proof is as follows, assuming for simplicity a dissipation estimate in the form $\dot{V}(x,u) \leq -\alpha(V(x)) + \gamma(|u|)$. Given any $x$ and $u$, either $\alpha(V(x)) \leq 2\gamma(|u|)$ or $\dot{V} \leq -\alpha(V)/2$. From here, one deduces by a comparison theorem that, along all solutions,

$$V(x(t)) \;\leq\; \max\left\{\beta(V(x^0), t)\,,\, \alpha^{-1}(2\gamma(\|u\|_\infty))\right\}\,,$$

where we have defined the $\mathcal{KL}$ function $\beta(s,t)$ as the solution $y(t)$ of the initial value problem

$$\dot{y} = -\frac{1}{2}\alpha(y) + \gamma(u)\,, \quad y(0) = s\,.$$

Finally, an ISS estimate is obtained from $V(x^0) \leq \overline{\alpha}(x^0)$.

The proof of the converse part of the theorem is much harder. It is based upon first showing that ISS implies robust stability in the sense already discussed, and then obtaining a converse Lyapunov theorem for robust stability for the system $\dot{x} = f(x, d\rho(|x|)) = g(x,d)$, which is asymptotically stable uniformly on all Lebesgue-measurable functions $d(\cdot) : \mathbb{R}_{\geq 0} \to B(0,1)$. This last theorem was given in [73], and is basically a theorem on Lyapunov functions for differential inclusions. A classical result of Massera [84] for differential equations becomes a special case.

### 3.4 Using "Energy" Estimates Instead of Amplitudes

In linear control theory, $H_\infty$ theory studies $L^2 \to L^2$ induced norms. We already saw that, under coordinate changes, we are led to the following type of estimate:

$$\int_0^t \alpha\left(|x(s)|\right) ds \;\leq\; \alpha_0(|x^0|) \,+\, \int_0^t \gamma(|u(s)|)\, ds$$

along all solutions, and for some $\alpha, \alpha_0, \gamma \in \mathcal{K}_\infty$. More precisely, let us say, just for the purposes of the next theorem, that a system *satisfies an integral–integral estimate* if for every initial state $x^0$ and input $u$, the solution $x(t, x^0, u)$ is defined for all $t > 0$ and an estimate as above holds. (In contrast to ISS, we now have to explicitly demand that $t_{\max} = \infty$.)

**Theorem 3.5.** [102] *A system is ISS if and only if it satisfies an integral–integral estimate.*

This theorem is quite easy, in view of previous results. A sketch of proof is as follows. If the system is ISS, then there is an ISS-Lyapunov function satisfying $\dot{V}(x, u) \leq -V(x) + \gamma(|u|)$, so, integrating along any solution we obtain

$$\int_0^t V(x(s))\, ds \,\leq\, \int_0^t V(x(s))\, ds \,+\, V(x(t)) \,\leq\, V(x(0)) \,+\, \int_0^t \gamma(|u(s)|)\, ds$$

and thus an integral–integral estimate holds. Conversely, if such an estimate holds, one can prove that $\dot{x} = f(x, 0)$ is stable and that an asymptotic gain exists. We show here just the "limit property" $\inf_{t \geq 0} |x(t)| \leq \theta(\|u\|_\infty)$. Indeed, let $\theta := \alpha^{-1} \circ \gamma$. Pick any $x^0$ and $u$, and suppose that $\inf_{t\geq 0} |x(t)| > (\alpha^{-1} \circ \gamma)(\|u\|)$, so that there is some $\varepsilon > 0$ so that $\alpha(x(t)) \geq \varepsilon + \gamma(|u(t)|)$ for all $t \geq 0$. Then, $\int_0^t \alpha(x(s))ds \geq \varepsilon t + \int_0^t \gamma(|u(s)|)ds$, which implies $\alpha_0(|x^0|) > \varepsilon t$ for all $t$, a contradiction. Therefore, the LIM property holds with this choice of $\theta$.

## 4 Cascade Interconnections

One of the main features of the ISS property is that it behaves well under composition: a cascade (Fig. 12) of ISS systems is again ISS, see [99]. In this section, we will sketch how the cascade result can also be seen as a consequence of the dissipation characterization of ISS, and how this suggests a more general feedback result. We will not provide any details of the rich theory of ISS small-gain theorems, and their use in nonlinear feedback design, for which the references should be consulted, but we will present a very simple example to illustrate the ideas. We consider a cascade as follows:

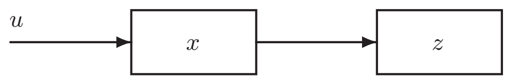


**Fig. 12.** Cascade

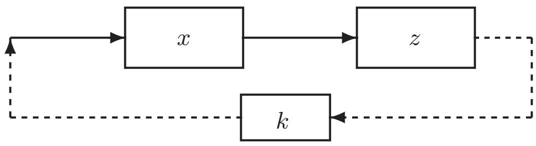


**Fig. 13.** Adding a feedback to the cascade

$$\dot{z} = f(z, x),$$
$$\dot{x} = g(x, u),$$

where each of the two subsystems is assumed to be ISS. Each system admits an ISS-Lyapunov function $V_i$. But, moreover, it is always possible (see [106]) to redefine the $V_i$'s so that the comparison functions for both are matched in the following way:

$$\dot{V}_1(z, x) \leq \theta(|x|) - \alpha(|z|),$$
$$\dot{V}_2(x, u) \leq \tilde{\theta}(|u|) - 2\theta(|x|)\,.$$

Now it is obvious why the full system is ISS: we simply use $V := V_1 + V_2$ as an ISS-Lyapunov function for the cascade:

$$\dot{V}((x, z), u) \;\leq\; \tilde{\theta}(|u|) - \theta(|x|) - \alpha(|z|)\,.$$

Of course, in the special case in which the $x$-subsystem has no inputs, we have also proved that the cascade of a GAS and an ISS system is GAS.

More generally, one may allow a "small gain" feedback as well (Fig. 13). That is, we allow inputs $u = k(z)$ as long as they are small enough:

$$|k(z)| \;\leq\; \tilde{\theta}^{-1}((1 - \varepsilon)\alpha(|z|))\,.$$

The claim is that the closed-loop system

$$\dot{z} = f(z, x)$$
$$\dot{x} = g(x, k(x))$$

is GAS. This follows because the same $V$ is a Lyapunov function for the closed-loop system; for $(x, z) \neq 0$:

$$\tilde{\theta}(|u|) \leq (1 - \varepsilon)\alpha(|z|) \;\rightsquigarrow\; \dot{V}(x, z) \leq -\theta(|x|) - \varepsilon\alpha(|z|) \;< 0\,. \qquad \checkmark$$

A much more interesting version of this result, resulting in a composite system with inputs being itself ISS, is the *ISS small-gain theorem* due to Jiang, Teel, and Praly [53].

### 4.1 An Example of Stabilization Using the ISS Cascade Approach

We consider a model of a rigid body in 3-space (Fig. 14), controlled by two torques acting along principal axes. This is a simple model of a satellite controlled by an opposing jet pair. If we denote by $\omega = (\omega_1, \omega_2, \omega_3)$ the angular velocity of a body-attached frame with respect to inertial coordinates, and let $I = \mathrm{diag}(I_1, I_2, I_3)$ be the principal moments of inertia, the equations are:

$$I\dot{\omega} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} I\omega + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} u$$

Ignoring kinematics, we just look at angular momenta, and we look for a feedback law to globally stabilize this system to $\omega = 0$. Under feedback and coordinate transformations, one can bring this into the following form of a system in $\mathbb{R}^3$ with controls in $\mathbb{R}^2$:

$$\begin{aligned} \dot{x}_1 &= x_2 x_3, \\ \dot{x}_2 &= u_1, \\ \dot{x}_3 &= u_2\,. \end{aligned}$$

(We assume that $I_2 \neq I_3$, and use these transformations: $(I_2 - I_3)x_1 = I_1\omega_1$, $x_2 = \omega_2$, $x_3 = \omega_3$, $I_2\widetilde{u}_1 = (I_3 - I_1)\omega_1\omega_2 + u_1$, $I_3\widetilde{u}_2 = (I_1 - I_2)\omega_1\omega_3 + u_2$.) Our claim is that the following feedback law globally stabilizes the system:

$$\begin{aligned} u_1 &= -x_1 - x_2 - x_2 x_3 \\ u_2 &= -x_3 + x_1^2 + 2x_1 x_2 x_3\,. \end{aligned}$$

Indeed, as done in [18] for the corresponding local problem, we make the following transformations: $z_2 := x_1 + x_2$, $z_3 := x_3 - x_1^2$, so the system becomes:

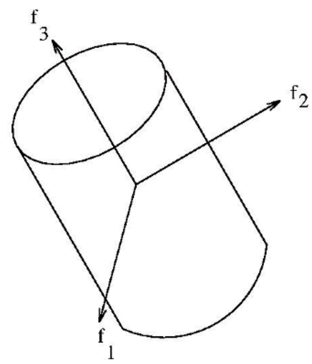


**Fig. 14.** Rigid body

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + \alpha(x_1, z_2, z_3) \quad (\deg_{x_1} \alpha \leq 2),\\ \dot{z}_2 &= -z_2,\\ \dot{z}_3 &= -z_3\,.\end{aligned}$$

Now, the $x_1$-subsystem is easily seen to be ISS, and the $z_1, z_2$ subsystem is clearly GAS, so the cascade is GAS. Moreover, a similar construction produces a feedback robust with respect to input disturbances.

## 5 Integral Input-to-State Stability

We have seen that several different properties, including "integral to integral" stability, dissipation, robust stability margins, and asymptotic gain properties, all turned out to be exactly equivalent to input to state stability. Thus, it would appear to be difficult to find a general and interesting concept of nonlinear stability that is truly distinct from ISS. One such concept, however, does arise when considering a mixed notion which combines the "energy" of the input with the amplitude of the state. It is obtained from the "$L^2 \to L^\infty$" gain estimate, under coordinate changes, and it provides a genuinely new concept [102].

A system is said to be *integral-input to state stable* (iISS) provided that there exist $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the estimate

$$\alpha\left(|x(t)|\right) \;\leq\; \beta(|x^0|\,, t) \;+\; \int_0^t \gamma(|u(s)|)\, ds \qquad \text{(iISS)}$$

holds along all solutions. Just as with ISS, we could state this property merely for all times $t \in t_{\max}(x^0, u)$, but, since the right-hand side is bounded on each interval $[0, t]$ (because, recall, inputs are by definition assumed to be bounded on each finite interval), it is automatically true that $t_{\max}(x^0, u) = +\infty$ if such an estimate holds along maximal solutions. So forward-completeness can be assumed with no loss of generality.

### 5.1 Other Mixed Notions

We argued that changes of variables transformed linear "finite $L^2$ gain" estimates into an "integral to integral" property, which we then found to be equivalent to the ISS property. On the other hand, finite operator gain from $L^p$ to $L^q$, with $p \neq q$ both finite, lead one naturally to the following type of "weak integral to integral" mixed estimate:

$$\int_0^t \underline{\alpha}(|x(s)|)\, ds \;\leq\; \kappa(|x^0|) \;+\; \alpha\left(\int_0^t \gamma(|u(s)|)\, ds\right)$$

for appropriate $\mathcal{K}_\infty$ functions (note the additional "$\alpha$"). See [12] for more discussion on how this estimate is reached, as well as the following result:

**Theorem 5.1.** [12] *A system satisfies a weak integral to integral estimate if and only if it is iISS.*

Another interesting variant is found when considering mixed *integral/ supremum* estimates:

$$\underline{\alpha}(|x(t)| \;\leq\; \beta(|x^0|,t) \,+\, \int_0^t \gamma_1(|u(s)|)\,ds \,+\, \gamma_2(\|u\|_\infty)$$

for suitable $\beta \in \mathcal{KL}$ and $\underline{\alpha}, \gamma_i \in \mathcal{K}_\infty$. One then has:

**Theorem 5.2.** [12] *A system satisfies a mixed estimate if and only if it is iISS.*

### 5.2 Dissipation Characterization of iISS

There is an amazingly elegant characterization of iISS, as follows. Recall that by a storage function we mean a continuous $V : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and proper. Following [11], we will say that a smooth storage function $V$ is an *iISS-Lyapunov function* for the system $\dot{x} = f(x,u)$ if there are a $\gamma \in \mathcal{K}_\infty$ and an $\alpha : [0,+\infty) \to [0,+\infty)$ which is merely *positive definite* (that is, $\alpha(0) = 0$ and $\alpha(r) > 0$ for $r > 0$) such that the inequality:

$$\dot{V}(x,u) \;\leq\; -\alpha(|x|) \,+\, \gamma(|u|) \qquad \text{(L-iISS)}$$

holds for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. By contrast, recall that an ISS-Lyapunov function is required to satisfy an estimate of the same form but where $\alpha$ is required to be of class $\mathcal{K}_\infty$; since every $\mathcal{K}_\infty$ function is positive definite, an ISS-Lyapunov function is also an iISS-Lyapunov function.

**Theorem 5.3.** [11] *A system is iISS if and only if it admits a smooth iISS-Lyapunov function.*

Since an ISS-Lyapunov function is also an iISS one, ISS implies iISS. However, iISS is a strictly weaker property than ISS, because $\alpha$ may be bounded in the iISS-Lyapunov estimate, which means that $V$ may increase, and the state become unbounded, even under bounded inputs, so long as $\gamma(|u(t)|)$ is larger than the range of $\alpha$. This is also clear from the iISS definition, since a constant input with $|u(t)| = r$ results in a term in the right-hand side that grows like $rt$. As a concrete example using a nontrivial $V$, consider the system

$$\dot{x} \;=\; -\tan^{-1}x \,+\, u,$$

which is not ISS, since $u(t) \equiv \pi/2$ results in unbounded trajectories. This system is nonetheless iISS: if we pick $V(x) = x\tan^{-1}x$, then

$$\dot{V} \;\leq\; -(\tan^{-1}|x|)^2 \,+\, 2\,|u|$$

so $V$ is an iISS-Lyapunov function. An interesting general class of examples is given by *bilinear* systems

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i A_i\right) x + Bu$$

for which the matrix $A$ is Hurwitz. Such systems are always iISS (see [102]), but they are not in general ISS. For instance, in the case when $B = 0$, boundedness of trajectories for all constant inputs already implies that $A + \sum_{i=1}^{m} u_i A_i$ must have all eigenvalues with nonpositive real part, for all $u \in \mathbb{R}^m$, which is a condition involving the matrices $A_i$ (for example, $\dot{x} = -x + ux$ is iISS but it is not ISS).

The notion of iISS is useful in situations where an appropriate notion of detectability can be verified using LaSalle-type arguments. There follow two examples of theorems along these lines.

**Theorem 5.4.** [11] *A system is iISS if and only if it is 0-GAS and there is a smooth storage function $V$ such that, for some $\sigma \in \mathcal{K}_\infty$:*

$$\dot{V}(x,u) \leq \sigma(|u|)$$

*for all $(x,u)$.*

The sufficiency part of this result follows from the observation that the 0-GAS property by itself already implies the existence of a smooth and positive definite, but not necessarily proper, function $V_0$ such that $\dot{V}_0 \leq \gamma_0(|u|) - \alpha_0(|x|)$ for all $(x,u)$, for some $\gamma_0 \in \mathcal{K}_\infty$ and positive definite $\alpha_0$ (if $V_0$ were proper, then it would be an iISS-Lyapunov function). Now one uses $V_0 + V$ as an iISS-Lyapunov function ($V$ provides properness).

**Theorem 5.5.** [11] *A system is iISS if and only if there exists an output function $y = h(x)$ (continuous and with $h(0) = 0$) which provides zero-detectability ($u \equiv 0$ and $y \equiv 0 \Rightarrow x(t) \to 0$) and dissipativity in the following sense: there exists a storage function $V$ and $\sigma \in \mathcal{K}_\infty$, $\alpha$ positive definite, so that:*

$$\dot{V}(x,u) \leq \sigma(|u|) - \alpha(h(x))$$

*holds for all $(x,u)$.*

The paper [12] contains several additional characterizations of iISS.

## 5.3 Superposition Principles for iISS

We now discuss asymptotic gain characterizations for iISS.

We will say that a system is *bounded energy weakly converging state* (BEWCS) if there exists some $\sigma \in \mathcal{K}_\infty$ so that the following implication holds:

$$\int_0^{+\infty} \sigma(|u(s)|)\,ds < +\infty \quad \Rightarrow \quad \liminf_{t\to+\infty} |x(t,x^0,u)| = 0 \tag{BEWCS}$$

(more precisely: if the integral is finite, then $t_{\max}(x^0,u) = +\infty$ and the liminf is zero), and that it is *bounded energy frequently bounded state* (BEFBS) if there exists some $\sigma \in \mathcal{K}_\infty$ so that the following implication holds:

$$\int_0^{+\infty} \sigma(|u(s)|)\,ds < +\infty \quad \Rightarrow \quad \liminf_{t\to+\infty} |x(t,x^0,u)| < +\infty \tag{BEFBS}$$

(again, meaning that $t_{\max}(x^0,u) = +\infty$ and the liminf is finite).

**Theorem 5.6.** [6] *The following three properties are equivalent for any given system $\dot{x} = f(x,u)$:*

- *The system is iISS*
- *The system is BEWCS and 0-stable*
- *The system is BEFBS and 0-GAS*

These characterizations can be obtained as consequences of characterizations of input/output to state stability (IOSS), cf. Sect. 8.4. The key observation is that a system is iISS with input gain (the function appearing in the integral) $\sigma$ if and only if the following auxiliary system is IOSS with respect to the "error" output $y = e$:

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{e} &= \sigma(|u|)\,. \end{aligned}$$

The proof of this equivalence is trivial, so we include it here. If the system is iISS, then:

$$\begin{aligned} \alpha(|x(t,x^0,u)|) &\le \beta(|x^0|\,,t) + \int_0^t \sigma(|u(s)|)\,ds \;=\; \beta(|x^0|\,,t) + e(t) - e(0) \\ &\le \beta(|x^0|\,,t) + 2\|y_{[0,t]}\|_\infty \end{aligned}$$

and, conversely, if it is IOSS, then for any $x^0$ and picking $e(0) = 0$, we have:

$$\begin{aligned} |x(t,x^0,u)| &\le \beta(|(x^0,0)|\,,t) + \gamma(\|u\|_\infty) + \|y_{[0,t]}\|_\infty \\ &\le \beta(|x^0|\,,t) + \gamma(\|u\|_\infty) + \int_0^t \sigma(|u|)\,ds\,. \end{aligned}$$

### 5.4 Cascades Involving iISS Systems

We have seen that cascades of ISS systems are ISS, and, in particular, any system of the form:

$$\begin{aligned} \dot{x} &= f(x,z) \\ \dot{z} &= g(z) \end{aligned}$$

for which the $x$-subsystem is ISS when $z$ is viewed as an input and the $g$-subsystem is GAS, is necessarily GAS. This is one of the most useful properties of the ISS notion, as it allows proving stability of complex systems by a decomposition approach. The iISS property on the first subsystem, in contrast, is not strong enough to guarantee that the cascade is GAS. As an illustration, consider the following system:

$$\dot{x} = -\mathrm{sat}(x) + xz,$$
$$\dot{z} = -z^3\,,$$

where $\mathrm{sat}(x) := \mathrm{sgn}(x)\min\{1, |x|\}$. It is easy to see that the $x$-subsystem with input $z$ is iISS, and the $z$-subsystem is clearly GAS. On the other hand [13], if we pick $z(0) = 1$ and any $x(0) \geq 3$, then $x(t) \geq e^{(\sqrt{1+2t}-1)}$, so $x(t) \to \infty$ as $t \to \infty$; so the complete system is not GAS. However, under additional conditions, it is possible to obtain a cascade result for a system of the above form. One such result is as follows.

**Theorem 5.7.** [13] *Suppose that the $x$-subsystem is iISS and affine in $z$, and that the $z$-subsystem is GAS and locally exponentially stable. Then, the cascade is GAS.*

Note that the counterexample shown above is so that the $x$-subsystem is indeed affine in $z$, but the *exponential* stability property fails. This theorem is a consequence of a more general result, which is a bit technical to state. We first need to introduce two concepts. The first one qualifies the speed of convergence in the GAS property, and serves to relax exponential stability: we say that the system $\dot{z} = g(z)$ is GAS($\alpha$), for a given $\alpha \in \mathcal{K}_\infty$, if there exists a class-$\mathcal{K}_\infty$ function $\theta(\cdot)$ and a positive constant $k > 0$ so that

$$|z(t)| \;\leq\; \alpha\left(e^{-kt}\theta(|z^0|)\right)$$

holds for all $z^0$. (Recall that GAS is always equivalent to the existence of *some* $\alpha$ and $\theta$ like this.) The second concept is used to characterize the function $\gamma$ appearing in the integral in the right-hand side of the iISS estimate, which we call the "iISS gain" of the system: given any $\alpha \in \mathcal{K}_\infty$, we say that the function $\gamma$ is "class-$\mathcal{H}_\alpha$" if it is of class $\mathcal{K}$ and it also satisfies:

$$\int_0^1 \frac{(\gamma(\alpha(s))}{s}\,ds \;<\; \infty\,.$$

The main result says that if the same $\alpha$ can be used in both definitions, then the cascade is GAS:

**Theorem 5.8.** [13] *Suppose that the $x$-subsystem is iISS with a class-$\mathcal{H}_\alpha$ iISS gain, and that the $z$-subsystem is GAS($\alpha$). Then, the cascade is GAS.*

See [13] for various corollaries of this general fact, which are based upon checking that the hypotheses are always satisfied, for example for the abovementioned case of $x$- subsystem affine in $z$ and exponentially stable $z$- subsystem.

### 5.5 An iISS Example

As an example of the application of iISS ideas, we consider as in [11] a robotic device studied by Angeli in [3]. This is a manipulator with one rotational and one linear actuator (Fig. 15). A simple model is obtained considering the arm as a segment with mass $M$ and length $L$, and the hand as a material point with mass $m$. The equations for such a system are four-dimensional, using as state variables angular position and velocity $\theta, \dot{\theta}$ and linear extension and velocity $r, \dot{r}$, and they follow from the second order equation

$$\begin{aligned}(mr^2 + ML^2/3)\,\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= \tau \\ m\ddot{r} - mr\dot{\theta}^2 &= F\,,\end{aligned}$$

where the controls are the torque $\tau$ and linear force $F$. We write the state as $(q, \dot{q})$, with $q = (\theta, r)$, We wish to study the standard tracking feedback controller with equations

$$\tau = -k_1\dot{\theta} - k_2(\theta - \theta_d)\,,\ F = -k_3\dot{r} - k_4(r - r_d)$$

where $q_d, r_d$ are the desired trajectories. It is well-known that, for *constant* tracking signals $q_d$, $r_d$, one obtains convergence: $\dot{q} \to 0$ and $q \to q_d$ as $t \to \infty$. In the spirit of the ISS approach, however, it is natural to ask what is the sensitivity of the design to *additive measurement noise*, or equivalently, since these errors are potentially arbitrary functions, what is the effect of *time-varying* tracking signals. One could ask if the system is ISS, and indeed the paper [83] proposed the reformulation of tracking problems using ISS as a way to characterize performance.

It turns out that, for this example, even bounded signals may destabilize the system, by a sort of "nonlinear resonance" effect, so the system cannot be ISS (not even bounded-input bounded-state) with respect to $q_d$ and $r_d$. Fig. 16 plots a numerical example of a de-stabilizing input; the corresponding $r(t)$-component is in Fig. 17. To be precise, the figures show the "$r$" component of the state of a certain solution which corresponds to the shown input; see [11]

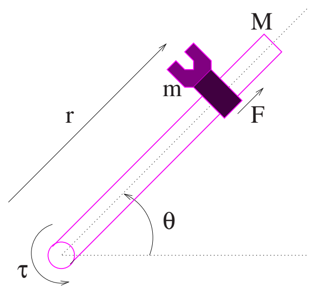


**Fig. 15.** A linear/rotational actuated arm

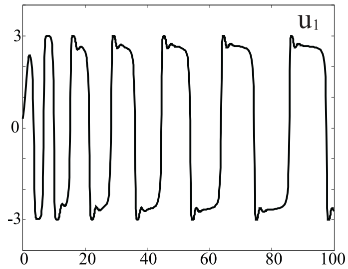


**Fig. 16.** Destabilizing input

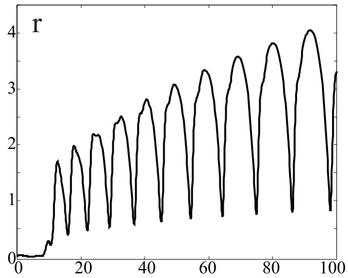


**Fig. 17.** Corresponding $r(\cdot)$

for details on how this input and trajectory were calculated. Thus, the question arises of how to qualitatively formulate the fact that some other inputs are not destabilizing. We now show that iISS provides one answer to this question,

In summary, we wish to show that the closed loop system

$$\begin{aligned}(mr^2 + ML^2/3)\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= u_1 - k_1\dot{\theta} - k_2\theta, \\ m\ddot{r} - mr\dot{\theta}^2 &= u_2 - k_3\dot{r} - k_4 r,\end{aligned}$$

with states $(q, \dot{q})$, $q = (\theta, r)$, and $u = (k_2\theta_d, k_4 r_d)$ is iISS.

In order to do so, we consider the mechanical energy of the system:

$$V(q, z) \ := \ \frac{1}{2}\dot{q}^T H(q)\dot{q} + \frac{1}{2}q^T K q$$

and note [11] the following passivity-type estimate:

$$\frac{d}{dt}V(q(t), \dot{q}(t)) \ \leq \ -c_1|\dot{q}(t)|^2 + c_2\,|u(t)|^2$$

for sufficiently small $c_1 > 0$ and large $c_2 > 0$. Taking $\dot{q}$ as an output, the system is zero-detectable and dissipative, since $u \equiv 0$ and $\dot{q} \equiv 0$ imply $q \equiv 0$, and hence, appealing to the given dissipation characterizations, we know that it is indeed iISS.

## 6 Input to State Stability with Respect to Input Derivatives

The ISS property imposes a very strong requirement, in that stable behavior must hold with respect to totally arbitrary inputs. Often, on the other hand, only stability with respect to specific classes of signals is expected. An example is in regulation problems, where disturbance rejection is usually formulated in terms of signals generated by a given finite-dimensional exosystem. Another example is that of parameter drift in adaptive control systems, where bounds on rates of change of parameters (which we may see as inputs) are imposed. This question motivated the work in [10] on ISS notions in which one asks, roughly, that $x(t)$ should be small provided that $u$ and its derivatives of some fixed order be small, but not necessarily when just $u$ is small. The precise definition is as follows.

For any given nonnegative integer $k$, we say that the system $\dot{x} = f(x, u)$ is *differentiably k-ISS* (D$^k$ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}_\infty$, $i = 0, \ldots, k$, such that the estimate:

$$|x(t, x^0, u)| \; \leq \; \beta(|x^0|, t) + \sum_{i=0}^{k} \gamma_i \left( \|u^{(i)}\|_\infty \right) \qquad \text{(D}^k\text{ISS)}$$

holds for all $x^0$, all inputs $u \in W^{k,\infty}$, and all $t \in t_{\max}(x^0, u)$. (By $W^{k,\infty}$ we are denoting the Sobolev space of functions $u$: $[0, \infty) \to \mathbb{R}^m$ for which the $(k-1)$st derivative $u^{(k-1)}$ exists and is locally Lipschitz, which means in particular that $u^{(k)}$ exists almost everywhere and is locally essentially bounded.) As with the ISS property, forward completeness is automatic, so one can simply say "for all $t$" in the definition. Notice that $D^0$ISS is the same as plain ISS, and that, for every $k$, D$^k$ISS implies D$^{k+1}$ISS.

### 6.1 Cascades Involving the $D^k$ISS Property

Consider any cascade as follows:

$$\dot{x} = f(x, z)$$
$$\dot{z} = g(z, u)$$

where we assume that $g$ is smooth. The following result generalizes the fact that cascading ISS and GAS subsystems gives a GAS system.

**Theorem 6.1.** [10] *If each subsystem is D$^k$ISS, then the cascade is D$^k$ISS. In particular the cascade of a D$^k$ISS and a GAS system is GAS.*

Actually, somewhat less is enough: the $x$-subsystem need only be D$^{k+1}$ISS, and we may allow the input to appear in this subsystem.

It is not difficult to see that a system is D$^1$ISS if and only if the following system:

$$\dot{x} = f(x, z)$$
$$\dot{z} = -z + u$$

is ISS, and recursively one can obtain a similar characterization of $\mathrm{D}^k$ISS. More generally, a system is $\mathrm{D}^k$ISS if and only if it is $D^{k-1}$ISS when cascaded with any ISS "smoothly invertible filter" as defined in [10]. Also very useful is a close relationship with the IOSS concept studied in Sect. 8.2. Consider the following auxiliary system with input $u$ and output $y$:

$$\begin{aligned} \dot{x} &= f(x, u_0) \\ \dot{u}_0 &= u_1 \\ \vdots \quad & \quad \vdots \\ \dot{u}_{k-1} &= u_k \\ y &= [u_0, u_1, \ldots, u_{k-1}]. \end{aligned}$$

**Theorem 6.2.** [10] *The auxiliary system is IOSS if and only if the original system is* $D^k$*ISS.*

The paper [10] also discusses some relations between the notion of $\mathrm{D}^1$ISS and ISS, for systems of the special form $\dot{x} = f(x + u)$, which are of interest when studying observation uncertainty.

### 6.2 Dissipation Characterization of $\mathbf{D}^k$ISS

**Theorem 6.3.** [10] *A system is* $D^1$*ISS if and only if there exists a smooth function* $V(x, u)$ *such that, for some* $\alpha, \delta_0, \delta_1, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$*,*

$$\alpha_1(|x| + |u|) \;\leq\; V(x, u) \;\leq\; \alpha_2(|x| + |u|)$$

*and*

$$D_x V(x, u)\, f(x, u) + D_u V(x, u)\dot{u} \;\leq\; -\alpha(|x|) + \delta_0(|u|) + \delta_1(|\dot{u}|)$$

*for all* $(x, u, \dot{u}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$*.*

Notice that "$\dot{u}$" is just a dummy variable in the above expression. Analogous characterizations hold for $D^k$ISS.

### 6.3 Superposition Principle for $\mathbf{D}^k$ISS

We will say that a forward-complete system satisfies the *k-asymptotic gain (k-AG) property* if there are some $\gamma_0, \gamma_1, \ldots, \gamma_k \in \mathcal{K}$ so that, for all $u \in W^{k,\infty}$, and all $x^0$, the estimate

$$\overline{\lim_{t\to\infty}}\, |x(t, \xi, u)| \;\leq\; \gamma_0(\|u\|_\infty) + \gamma_1(\|\dot{u}\|_\infty) + \ldots + \gamma_k(\|u^{(k)}\|_\infty)$$

holds.

**Theorem 6.4.** [10] *A system is* $D^k$*ISS if and only if it is 0-stable and k-AG.*

### 6.4 A Counter-Example Showing that $D^1$ISS $\neq$ ISS

Consider the following system:

$$\dot{x} \;=\; \|x\|^2 \, U(\theta)' \, \Phi \, U(\theta) \, x,$$

where $x \in \mathbb{R}^2$, and $u = \theta(\cdot)$ is the input,

$$U(\theta) = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix},$$

and where $\Phi$ is any $2\times 2$ Hurwitz matrix such that $\Phi' + \Phi$ has a strictly positive real eigenvalue. It is shown in [10] that this system is not forward complete, and in particular it is not ISS, but that it is $D^1$ISS. This latter fact is shown by proving, through the construction of an explicit ISS-Lyapunov function, that the cascaded system

$$\dot{x} \;=\; \|x\|^2 \, U(\theta)' \, \Phi \, U(\theta) \, x, \qquad \dot{\theta} = -\theta + u$$

is ISS.

It is still an open question if $D^2$ISS is strictly weaker than $D^1$ISS, and more generally $D^{k+1}$ISS than $D^k$ISS for each $k$.

## 7 Input-to-Output Stability

Until now, we only discussed stability of states with respect to inputs. For systems with outputs $\dot{x} = f(x, u)$, $y = h(x)$, if we simply replace states by outputs in the left-hand side of the estimate defining ISS, we then arrive to the notion of *input-to-output stability* (IOS): there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|y(t)| \;\leq\; \beta(|x^0|, t) \;+\; \gamma\,(\|u\|_\infty) \qquad \text{(IOS)}$$

holds for all solutions, where $y(t) = h(x(t, x^0, u))$. By "all solutions" we mean that this estimate is valid for all inputs $u(\cdot)$, all initial conditions $x^0$, and all $t \geq 0$, and we are imposing as a requirement that the system be forward complete, i.e., $t_{\max}(x^0, u) = \infty$ for all initial states $x^0$ and inputs $u$. As earlier, $x(t)$, and hence $y(t) = h(x(t))$, depend only on past inputs ("causality"), so we could have used just as well simply the supremum of $|u(s)|$ for $s \leq t$ in the estimate.

We will say that a system is *bounded-input bounded-state stable (BIBS)* if, for some $\sigma \in \mathcal{K}_\infty$, the following estimate:

$$|x(t)| \;\leq\; \max\left\{\sigma(|x^0|),\; \sigma(\|u\|_\infty)\right\}$$

holds along all solutions. (Note that forward completeness is a consequence of this inequality, even if it is only required on maximal intervals, since the state is upper bounded by the right-hand side expression.)

We define an *IOS-Lyapunov function* as any smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that, for some $\alpha_i \in \mathcal{K}_\infty$:

$$\alpha_1(|h(x)|) \ \leq \ V(x) \ \leq \ \alpha_2(|x|) \qquad \forall\, x \in \mathbb{R}^n,\, u \in \mathbb{R}^m$$

and, for all $x, u$:

$$V(x) > \alpha_3(|u|) \ \Rightarrow \ \nabla V(x)\, f(x,u) < 0\,.$$

**Theorem 7.1.** [113] *A BIBS system is IOS if and only if it admits an IOS-Lyapunov function.*

A concept related to IOS is as follows. We call a system *robustly output stable* (ROS) if it is BIBS and there is some smooth $\lambda \in \mathcal{K}_\infty$ such that

$$\dot{x} = g(x,d) := f(x, d\lambda(|y|))\,, \quad y = h(x)$$

is globally output-asymptotically stable uniformly with respect to all $d(\cdot) : [0,\infty) \to [-1,1]^m$: for some $\beta \in \mathcal{KL}$,

$$|y(t,x^0,d)| \leq \beta(|x^0|\,,t)$$

for all solutions. Then, IOS implies ROS, but the converse does not hold in general [112]. We have the following dissipation characterization of ROS:

**Theorem 7.2.** [113] *A system is ROS if and only if it is BIBS and there are $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, and $\alpha_3 \in \mathcal{KL}$, and a smooth function $V : \mathbb{R}^n \to \mathbb{R}$, so that*

$$\alpha_1(|h(x)|) \leq V(x) \leq \alpha_2(|x|)$$

*and*

$$|h(x)| \geq \chi(|u|) \ \Rightarrow \ \dot{V} \leq -\alpha_3(V(x), |x|)$$

*for all $(x,u)$.*

The area of *partial stability* studies stability of a subset of variables in a system $\dot{x} = f(x)$. Letting $y = h(x)$ select the variables of interest, one may view partial stability as a special case of output stability, for systems with no inputs. Note that, for systems with no inputs, the partial differential inequality for IOS reduces to $\nabla V(x)\, f(x) < 0$ for all nonzero $x$, and that for ROS to $\dot{V} \leq -\alpha_3(V(x), |x|)$. In this way, the results in [113] provide a far-reaching generalization of, and converse theorems to, sufficient conditions [125] for partial stability.

There is also a superposition principle for IOS. We will say that a forward-complete system satisfies the *output asymptotic gain* (OAG) property if

$$\overline{\lim_{t\to\infty}}\, |y(t)| \ \leq \ \gamma(\|u\|_\infty) \qquad \text{(OAG)}$$

for some $\gamma \in \mathcal{K}_\infty$ and all solutions. One would like to have a characterization of IOS in terms of OAG, which is an analog of the AG gain property in the state case, and a stability property. Let us define a system to be *output-Lagrange stable (OL)* if it satisfies an estimate, for some $\sigma \in \mathcal{K}_\infty$:

$$|y(t)| \;\leq\; \sigma(|y(0)|) \,+\, \sigma(\|u\|_\infty)$$

along all solutions. Under this assumption, we recover a separation principle:

**Theorem 7.3.** [6] *An OL system is OAG if and only if it is IOS.*

Observe that the OL property asks that the output be uniformly bounded in terms of the amplitudes of the input and of the initial output (not of the initial state), which makes this property a very strong constraint. If we weaken the assumption to an estimate of the type

$$|y(t)| \;\leq\; \sigma(|x^0|) \,+\, \sigma(\|u\|_\infty)$$

then IOS implies the conjunction of OL and this property, but the converse fails, as shown by the following counter-example, a system with no inputs:

$$\dot{x}_1 \;=\; -x_2\,|x_2|\;, \quad \dot{x}_2 \;=\; x_1\,|x_2|\;, \qquad y = x_2\,.$$

The set of equilibria is $\{x_2 = 0\}$, and trajectories are half circles traveled counterclockwise. We have that $|y(t)| \leq |x(0)|$ for all solutions, and $y(t) \to 0$ as $t \to \infty$, so both properties hold. However, there is no possible IOS estimate $|y(t)| \leq \beta(|x^0|\,,t)$, since, in particular, for a state of the form $x(0) = (1,\varepsilon)$, the time it takes for $y(\cdot)$ to enter an $\varepsilon$-neighborhood of 0 goes to $\infty$ as $\varepsilon \to 0$; see [6] for more discussion.

## 8 Detectability and Observability Notions

Recall (see [104] for precise definitions) that an *observer* for a given system with inputs and outputs $\dot{x} = f(x,u)$, $y = h(x)$ is another system which, using only information provided by past input and output signals, provides an asymptotic (i.e., valid as $t \to \infty$) estimate $\hat{x}(t)$ of the state $x(t)$ of the system of interest (Fig. 18). One may think of the observer as a physical system or as an algorithm implemented by a digital computer. The problem of state estimation is one of the most important and central topics in control theory, and it arises

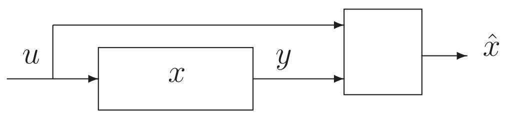


**Fig. 18.** Observer provides estimate $\hat{x}$ of state $x$; $\hat{x}(t) - x(t) \to 0$ as $t \to \infty$

in signal processing applications (Kalman filters) as well as when solving the problem of stabilization based on partial information. It is well understood for linear systems, but, a huge amount of research notwithstanding, the theory of observers is not as developed in general.

We will not say much about the general problem of building observers, which is closely related to "incremental" ISS-like notions, a subject not yet studied enough, but will focus on an associated but easier question. When the ultimate goal is that of stabilization to an equilibrium, let us say $x = 0$ in Euclidean space, sometimes a weaker type of estimate suffices: it may be enough to obtain a *norm-estimator* which provides merely an *upper bound* on the norm $|x(t)|$ of the state $x(t)$; see [50, 57, 93]. Before defining norm-estimators, and studying their existence, we need to introduce an appropriate notion of detectability.

## 8.1 Detectability

Suppose that an observer exists, for a given system. Since $x^0 = 0$ is an equilibrium for $\dot{x} = f(x, 0)$, and also $h(0) = 0$, the solution $x(t) \equiv 0$ is consistent with $u \equiv 0$ and $y \equiv 0$. Thus, the estimation property $\hat{x}(t) - x(t) \to 0$ implies that $\hat{x}(t) \to 0$. Now consider *any* state $x^0$ for which $u \equiv 0$ and $y \equiv 0$, that is, so that $h(x(t, x^0, 0)) \equiv 0$. The observer output, which can only depend on $u$ and $y$, must be the same $\hat{x}$ as when $x^0 = 0$, so $\hat{x}(t) \to 0$; then, using once again the definition of observer $\hat{x}(t) - x(t, x^0, 0) \to 0$, we conclude that $x(t, x^0, 0) \to 0$. In summary, a *necessary* condition for the existence of an observer is that the "subsystem" of $\dot{x} = f(x, u)$, $y = h(x)$ consisting of those states for which $u \equiv 0$ produces the output $y \equiv 0$ must have $x = 0$ as a GAS state (Fig. 19); one says in that case that the system is *zero-detectable*. (For *linear* systems, zero-detectability is equivalent to detectability or "asymptotic observability" [104]: two trajectories which produce the same output must approach each other. But this equivalence need not hold for nonlinear systems.) In a nonlinear context, zero-detectability is not "well-posed" enough: to get a well-behaved notion, one should add explicit requirements to ask that small inputs and outputs imply that internal states are small too (Fig. 20), and that inputs and outputs converging to zero as $t \to \infty$ implies that states do, too (Fig. 21), These properties are needed so that "small" errors in measurements of inputs and outputs processed by the observer give rise to small errors. Furthermore, one should impose asymptotic bounds on states as a function

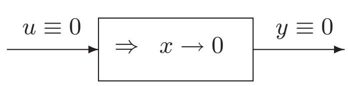


**Fig. 19.** Zero-detectability

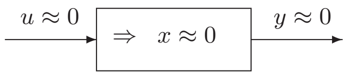


**Fig. 20.** Small inputs and outputs imply small states

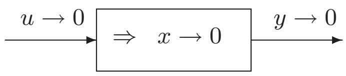


**Fig. 21.** Converging inputs and outputs imply convergent states

of input/output bounds, and it is desirable to quantify "overshoot" (transient behavior). This leads us to the following notion.

### 8.2 Dualizing ISS to OSS and IOSS

A system is *input/output to state stable* (IOSS) if, for some $\beta \in \mathcal{KL}$ and $\gamma_u, \gamma_y \in \mathcal{K}_\infty$,

$$x(t) \;\leq\; \beta(|x^0|, t) + \gamma_1\left(\|u_{[0,t]}\|_\infty\right) + \gamma_2\left(\|y_{[0,t]}\|_\infty\right) \qquad \text{(IOSS)}$$

for all initial states and inputs, and all $t \in [0, T_{\xi,u})$. Just as ISS is stronger than 0-GAS, IOSS is stronger than zero-detectability. A special case is when one has no inputs, *output to state stability:*

$$|x(t, x^0)| \;\leq\; \beta(|x^0|, t) + \gamma\left(\|y|_{[0,t]}\|_\infty\right)$$

and this is formally "dual" to ISS, simply replacing inputs $u$ by outputs in the ISS definition. This duality is only superficial, however, as there seems to be no useful way to obtain theorems for OSS by dualizing ISS results. (Note that the outputs $y$ depend on the state, not vice versa.)

### 8.3 Lyapunov-Like Characterization of IOSS

To formulate a dissipation characterization, we define an *IOSS-Lyapunov function* as a smooth storage function so that

$$\nabla V(x)\, f(x,u) \leq -\alpha_1(|x|) + \alpha_2(|u|) + \alpha_3(|y|)$$

for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$. The main result is:

**Theorem 8.1.** [65] *A system is IOSS if and only if it admits an IOSS-Lyapunov function.*

### 8.4 Superposition Principles for IOSS

Just as for ISS and IOS, there are asymptotic gain characterizations of input/output to state stability.

We say that a system satisfies the *IO-asymptotic gain (IO-AG)* property if:

$$\overline{\lim_{t \nearrow t_{\max(x^0,u)}}} |x(t,x^0,u)| \leq \gamma_u(\|u\|_\infty) + \gamma_y(\|y\|_\infty) \qquad \forall\, x^0,\ u(\cdot) \tag{IO-AG}$$

(for some $\gamma_u, \gamma_y$), and the *IO-limit* (IO-LIM) property if:

$$\inf_{t \geq 0} |x(t,x^0,u)| \leq \gamma_u(\|u\|_\infty) + \gamma_y(\|y\|_\infty) \qquad \forall\, x^0,\ u(\cdot) \tag{IO-LIM}$$

(for some $\gamma_u, \gamma_y$), where sup norms and inf are taken over $[0, t_{\max}(x^0,u))$. We also define the notion of *zero-input local stability modulo outputs* (0-LS) as follows:

$$(\forall\, \varepsilon > 0)\,(\exists\, \delta_\varepsilon) \quad \max\{|x^0|, \|y_{[0,t]}\|_\infty\} \leq \delta_\varepsilon \Rightarrow |x(t,x^0,0)| \leq \varepsilon\,. \tag{0-LS}$$

This is a notion of marginal local detectability; for linear systems, it amounts to marginal stability of the unobservable eigenvalues. We have the following result.

**Theorem 8.2.** [6] *The following three properties are equivalent for any given system $\dot{x} = f(x,u)$:*

- *The system is IOSS*
- *The system is IO-AG and zero-input O-LS*
- *The system is IO-LIM and zero-input O-LS*

Several other characterizations can also be found in [6].

### 8.5 Norm-Estimators

We define a *state-norm-estimator* (or *state-norm-observer*) for a given system as another system

$$\dot{z} = g(z,u,y)\,, \qquad \text{with output} \quad k : \mathbb{R}^\ell \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$$

evolving in some Euclidean space $\mathbb{R}^\ell$, and driven by the inputs and outputs of the original system. We ask that the output $k$ should be IOS with respect to the inputs $u$ and $y$, and the true state should be asymptotically bounded in norm by some function of the norm of the estimator output, with a transient (overshoot) which depends on both initial states. Formally:

- There are $\hat{\gamma}_1, \hat{\gamma}_2 \in \mathcal{K}$ and $\hat{\beta} \in \mathcal{KL}$ so that, for each initial state $z^0 \in \mathbb{R}^\ell$, and inputs $\mathbf{u}$ and $\mathbf{y}$, and every $t$ in the interval of definition of the solution $z(\cdot, z^0, \mathbf{u}, \mathbf{y})$

$$k\left(z(t,z^0,\mathbf{u},\mathbf{y}), \mathbf{y}(t)\right) \leq \hat{\beta}(|z^0|\,, t) + \hat{\gamma}_1\left(\|\mathbf{u}|_{[0,t]}\|\right) + \hat{\gamma}_2\left(\|\mathbf{y}|_{[0,t]}\|\right)$$

- There are $\rho \in \mathcal{K}$, $\beta \in \mathcal{KL}$ so that, for all initial states $x^0$ and $z^0$ of the system and observer, and every input $\mathbf{u}$

$$|x(t,x^0,\mathbf{u})| \le \beta(|x^0| + |z^0|,t) + \rho\left(k\left(z(t,z^0,\mathbf{u},\mathbf{y}_{x^0,\mathbf{u}}),\mathbf{y}_{x^0,\mathbf{u}}(t)\right)\right)$$

  for all $t \in [0, t_{\max}(x^0,\mathbf{u}))$, where $\mathbf{y}_{x^0,\mathbf{u}}(t) = y(t,x^0,\mathbf{u})$

**Theorem 8.3.** [65] *A system admits a state-norm-estimator if and only if it is IOSS.*

### 8.6 A Remark on Observers and Incremental IOSS

As mentioned earlier, for linear systems, "zero-detectability" and detectability coincide, where the latter is the property that *every pair* of distinct states is asymptotically distinguishable. The following is an ISS-type definition of detectability: we say that a system is *incrementally (or Lipschitz) input/output-to-state stable (i-IOSS)* if there exist $\gamma_1, \gamma_2 \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that, for any two initial states $x^0$ and $z^0$, and any two inputs $u_1$ and $u_2$,

$$|x(t,x^0,u_1)-x(t,z^0,u_2)| \le \max\{\beta(|x^0-z^0|,t), \gamma_1(\|\Delta u\|), \gamma_2(\|\Delta y\|)\} \quad \text{(i-IOSS)}$$

where $\Delta u = (u_1-u_2)$, $\Delta y = (y_{x^0,u_1}-y_{z^0,u_2})_{[0,t]}$, for all $t$ in the common domain of definition. It is easy to see that i-IOSS implies IOSS, but the converse does not hold in general. The notion of incremental-IOSS was introduced in [111]. A particular case is that in which $h(x) \equiv 0$, in which case we have the following notion: a system is *incrementally ISS* (i-ISS) if there holds an estimate of the following form:

$$|x(t,x^0,u_1)-x(t,z^0,u_2)| \le \max\{\beta(|x^0-z^0|,t),\ \gamma_1(\|\Delta u\|)\} \qquad \text{(i-ISS)}$$

where $\Delta u = u_1 - u_2$, for all $t$ in the common domain of definition. Several properties of the i-ISS notion were explored in [4], including the fact that i-ISS is preserved under cascades. Specializing even more, when there are no inputs one obtains the property *incremental GAS* (i-GAS). This last property can be characterized in Lyapunov terms using the converse Lyapunov result given in [73] for stability with respect to (not necessarily compact) sets, since it coincides with stability with respect to the diagonal of the system consisting of two parallel copies of the same system. Indeed, i-GAS is equivalent to asking that the system:

$$\dot{x} = f(x)$$
$$\dot{z} = f(z)$$

be asymptotically stable with respect to $\{(x,z) \mid x = z\}$. A sufficient condition for i-ISS in dissipation terms, using a similar idea, was given in [4].

As recalled earlier, an observer is another dynamical system, which processes inputs and outputs of the original system, and produces an estimate $\widehat{x}(t)$ of the state $x(t)$: $x(t)-\widehat{x}(t) \to 0$ as $t \to \infty$, and this difference (the

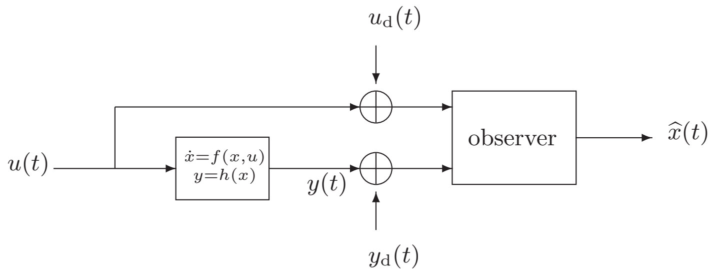


**Fig. 22.** Observer with perturbations in measurements

estimation error) should be small if it starts small (see [104], Chap. 6). As with zero-detectability, it is more natural in the ISS paradigm to ask that the estimation error $x(t)-\widehat{x}(t)$ should be small even if the measurements of inputs and outputs received by the observer are corrupted by noise. Writing $u_{\rm d}$ and $y_{\rm d}$ for the input and output measurement noise respectively, we have the situation shown pictorially in Fig. 22 (see [111] for a precise definition). Existence of an observer implies that the system is i-IOSS [111]. The converse and more interesting problem of building observers under IOSS assumptions is still a largely unsolved, although much progress has been made for systems with special structures, cf. [16, 57].

### 8.7 Variations of IOSS

The terminology IOSS was introduced in [111], and the name arises from the view of IOSS as "stability from the i/o data to the state." It combines the "strong" observability from [99] with ISS; and was called simply "detectability" in [98], where it was formulated in an input/output language and applied to controller parameterization, and it was called "strong unboundedness observability" in [53] (more precisely, this paper allowed for an additive nonnegative constant in the estimate). IOSS is related to other ISS-like formalisms for observers, see, e.g., [37, 75, 77, 92]. Both IOSS and its incremental variant are very closely related to the OSS-type detectability notions pursued in [59]; see also the emphasis on ISS guarantees for observers in [82].

The dissipation characterization amounts to a Willems'-type dissipation inequality $(d/dt)V(x(t)) \leq -\sigma_1(|x(t)|) + \sigma_2(|y(t)|) + \sigma_3(|u(t)|)$ holding along all trajectories. There have been other suggestions that one should *define* "detectability" in dissipation terms; see, e.g., [76], where detectability was defined by the requirement that there exist a differentiable storage function $V$ as here, but with the special choice $\sigma_2(r) := r^2$ (and no inputs), or as in [85], which asked for a weaker the dissipation inequality:

$$x \neq 0 \;\Rightarrow\; \frac{d}{dt}V(x(t)) < \sigma_2(|y(t)|)$$

(again, with no inputs), not requiring the "margin" of stability $-\sigma_1(|x(t)|)$. Observe also that, asking that along all trajectories there holds the estimate

$$\frac{d}{dt}V(x(t)) \;\leq\; -\sigma_1(|x(t)|) + \sigma_2(|y(t)|) + \sigma_3(|u(t)|)$$

means that $V$ satisfies a partial differential inequality (PDI):

$$\max_{u \in \mathbb{R}^m} \{\nabla V(x) \cdot f(x,u) + \sigma_1(|x|) - \sigma_2(|h(x)|) - \sigma_3(|u|)\} \;\leq\; 0$$

which is same as the Hamilton–Jacobi inequality

$$g_0(x) + \frac{1}{4}\sum_{i=1}^{m}(\nabla V(x) \cdot g_i(x))^2 + \sigma_1(|x|) - \sigma_2(|h(x)|) \;\leq\; 0$$

in the special case of quadratic input "cost" $\sigma_3(r) = r^2$ and systems $\dot{x} = f(x,u)$ affine in controls

$$\dot{x} \;=\; g_0(x) \;+\; \sum_{i=1}^{m} u_i\, g_i(x)$$

(just replace the right-hand side in the PDI by the maximum value, obtained at $u_i = (1/2)\nabla V(x) \cdot g_i(x)$). Thus the converse result amounts to providing necessary and sufficient conditions for existence of a smooth (and proper and positive definite) solution $V$ to the PDI. In this context, it is worth remarking that the mere existence of a lower semicontinuous $V$ (interpreted in an appropriate weak sense) implies the existence of a $\mathcal{C}^\infty$ solution (possibly with different comparison functions); see [64].

### 8.8 Norm-Observability

There are many notions of observability for nonlinear systems (see, e.g., [104], Chap. 6); here we briefly mention one such notion given in an ISS style, which was presented in [34]. More precisely, we define "norm-observability", which concerns the ability to determine an upper bound on norms, rather than the precise value of the state (an "incremental" version would correspond to true observability). We do so imposing a bound on the norm of the state in terms of the norms of the output and the input, and imposing an additional requirement which says, loosely speaking, that the term describing the effects of initial conditions can be chosen to decay arbitrarily fast.

A system $\dot{x} = f(x,u)$, $y = h(x)$ is *small-time initial-state norm-observable* if:

$$\forall\, \tau > 0\; \exists\, \gamma, \chi \in \mathcal{K}_\infty \;\text{ such that }\; |x^0| \leq \gamma(\|y_{[0,\tau]}\|_\infty) + \chi(\|u_{[0,\tau]}\|_\infty)\; \forall\, x^0,\, u\,,$$

it is *small-time final-state norm-observable* if:

$$\forall\,\tau > 0\,\exists\,\gamma,\chi \in \mathcal{K}_\infty \;\text{ such that }\; |x(\tau)| \leq \gamma(\|y_{[0,\tau]}\|_\infty) + \chi(\|u_{[0,\tau]}\|_\infty)\;\forall\,x^0,\,u\,,$$

and is *small-time-$\mathcal{KL}$ norm-observable* if for every $\varepsilon > 0$ and every $\nu \in \mathcal{K}$, there exist $\gamma, \chi \in \mathcal{K}_\infty$ and a $\beta \in \mathcal{KL}$ so that $\beta(r,\varepsilon) \leq \nu(r)$ for all $r \geq 0$ (i.e., $\beta$ can be chosen to decay arbitrarily fast in the second argument) such that the IOSS estimate:

$$|x(t)| \;\leq\; \beta(|x^0|\,,t) + \gamma(\|y_{[0,t]}\|_\infty) + \chi(\|u_{[0,t]}\|_\infty)\;\forall\,x^0,\,u,\,t \geq 0$$

holds along all solutions.

**Theorem 8.4.** [34] *The following notions are equivalent:*

- *Small-time initial-state norm-observability*
- *Small-time final-state norm-observability*
- *Small-time-$\mathcal{KL}$ norm-observability*

To be precise, the equivalences assume *unboundedness observability* (UO), which means that for each trajectory defined on some maximal interval $t_{\max} < \infty$, the output becomes unbounded as $t \nearrow t_{\max}$, as well as a similar property for the reversed-time system. The unboundedness observability property is strictly weaker than forward completeness, which is the property that each trajectory is defined for all $t \geq 0$; see [9, 53], the latter especially for complete Lyapunov characterizations of the UO property. Similarly, one can prove equivalences among other definitions, such as asking "$\exists\,\tau$" instead of "$\forall\,\tau$," and one may obtain Lyapunov-like characterizations; the results are used in the derivation of LaSalle-like theorems for verifying stability of switched systems in [34].

## 9 The Fundamental Relationship Among ISS, IOS, and IOSS

The definitions of the basic ISS-like concepts are consistent and related in an elegant conceptual manner, as follows:

*A system is ISS if and only if it is both IOS and IOSS.*

In informal terms, we can say that:

*external stability and detectability* $\iff$ *internal stability*

as it is the case for linear systems. Intuitively, we have the three possible signals in Fig. 23. The basic idea of the proof is as follows. Suppose that external stability and detectability hold, and take an input so that $u \to 0$. Then $y \to 0$ (by external stability), and this then implies that $x \to 0$ (by detectability). Conversely, if the system is internally stable, then we prove i/o stability and detectability. Suppose that $u \to 0$. By internal stability, $x \to 0$,

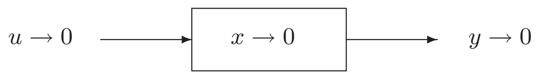


**Fig. 23.** Convergent input, state, and/or output

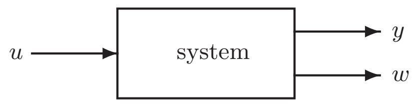


**Fig. 24.** System with error and measurement outputs $\dot{x} = f(x, u)$, $y = h(x)$, $w = g(x)$

and this gives $y(t) \to 0$ (i/o stability). Detectability is even easier: if both $u(t) \to 0$ and $y(t) \to 0$, then in particular $u \to 0$, so $x \to 0$ by internal stability. The proof that ISS is equivalent to the conjunction of IOS and IOSS must keep careful track of the estimates, but the idea is similar.

# 10 Systems with Separate Error and Measurement Outputs

We next turn to a topic which was mentioned in [105] as a suggestion for further work, but for which still only incomplete results are available. We will assume that there are *two* types of outputs (Fig. 24), which we think of, respectively as an "error" $y = h(x)$ to be kept small, as in the IOS notion, and a "measurement" $w = g(x)$ which provides information about the state, as in the IOSS notion.

Several ISS-type formulations of the central concept in regulator theory, namely the idea of using the size of $w$ in order to bound $y$, were given in [38], and are as follows.

## 10.1 Input-Measurement-to-Error Stability

We will say that a system is *input-measurement-to-error stable* (IMES) if there are $\beta \in \mathcal{KL}$, $\sigma \in \mathcal{K}$, and $\gamma \in \mathcal{K}$ such that the following estimate holds:

$$|y(t)| \;\leq\; \beta(|x^0|\,, t) \,+\, \sigma\left(\|w_{[0,t]}\|_\infty\right) \,+\, \gamma(\|u\|_\infty) \qquad \text{(IMES)}$$

for all $t \in t_{\max}(x^0, u)$, for all solutions, where we are writing $y(t) = h(x(t, x^0, u))$ and $w(t) = g(x(t, x^0, u))$. Special cases are all the previous concepts:

- When $h(x) = x$, so $y = x$, and we view $w$ as the output, we recover IOSS
- When the output $g = 0$, that is $w \equiv 0$, we recover IOS
- If both $y = x$ and $g = 0$, we recover ISS

The goal of obtaining general theorems for IMES which will specialize to all known theorems for ISS, IOS, and IOSS is so far unattained. We only know of several partial results, to be discussed next.

For simplicity from now on, we restrict to the case of systems with no inputs $\dot{x} = f(x)$, $y = h(x)$, $w = g(x)$, and we say that a system is measurement to error stable (MES) if an IMES estimate holds, i.e., for suitable $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$:

$$|y(t)| \;\leq\; \beta(|x^0|\,, t) \,+\, \gamma\left(\|w_{[0,t]}\|_\infty\right)$$

for all $t \in [0, T_{\max})$ and for all solutions.

In order to present a dissipation-like version of MES, it is convenient to introduce the following concept. We will say that a system is *relatively error-stable* (RES) if the following property holds, for some $\rho \in \mathcal{K}$ and $\beta \in \mathcal{KL}$:

$$|y(t)| > \rho(|w(t)|) \;\text{ on }\; [0,T] \;\;\Rightarrow\;\; |y(t)| \leq \beta(|x^0|\,, t) \;\text{ on }\; [0,T] \qquad \text{(RES)}$$

along all solutions and for all $T < t_{\max}(x^0, u)$. In words: while the error is much larger than the estimate provided by the measurement, the error must decrease asymptotically, with an overshoot controlled by the magnitude of the initial state. This property, together with the closely related notion of *stability in three measures* (SIT), was introduced and studied in [38]. It is easy to see that MES implies RES, but that the converse is false. In order to obtain a converse, one requires an additional concept: we say a system satisfies the *relative measurement to error boundedness* (RMEB) property if it admits an estimate of the following form, for some $\sigma_i \in \mathcal{K}$:

$$|y(t)| \leq \max\left\{\sigma_1(|h(x^0)|),\; \sigma_2\left(\|w_{[0,t]}\|_\infty\right)\right\} \qquad \text{(RMEB)}$$

along all solutions. For forward complete systems, and assuming RMEB, RES is equivalent to MES [38].

## 10.2 Review: Viscosity Subdifferentials

So far, no smooth dissipation characterization of any of these properties is available. In order to state a nonsmooth characterization, we first review a notion of weak differential. For any function $V : \mathbb{R}^n \to \mathbb{R}$ and any point $p \in \mathbb{R}^n$ in its domain, one says that a vector $\zeta$ is a *viscosity subgradient* of $V$ at $p$ if the following property holds: there is some function $\varphi : \mathbb{R}^n \to \mathbb{R}$, differentiable at zero and with $\nabla\varphi(0) = \zeta$ (that is, $\varphi(h) = \zeta \cdot h + o(h)$), such that

$$V(p + h) \;\geq\; V(p) + \varphi(h)$$

for each $h$ in a neighborhood of $0 \in \mathbb{R}^n$. In other words, a viscosity subgradient is a gradient (tangent slopes) of any supporting $\mathcal{C}^1$ function. One denotes then

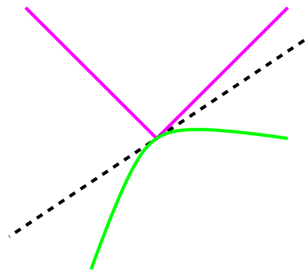

**Fig. 25.** $\partial_D V(0) = [-1, 1]$

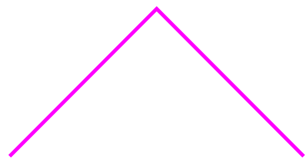

**Fig. 26.** $\partial_D V(0) = \emptyset$

$\partial_D V(p) :=$ {all viscosity subgradients of $V$ at $p$}. As an illustration, Fig. 25 shows a case where $\partial_D V(0) = [-1, 1]$, for the function $V(x) = |x|$, and Fig. 26 an example where $\partial_D V(0) = \emptyset$, for $V(x) = -|x|$. In particular, if $V$ is differentiable at $p$, then $\partial_D V(p) = \{\nabla V(p)\}$.

### 10.3 RES-Lyapunov Functions

The lower semicontinuous $V$ is an *RES-Lyapunov function* if:

- There exist $\alpha_1$ and $\alpha_2 \in \mathcal{K}_\infty$ so, on the set $C := \{p : |h(p)| > \rho(|g(p)|)\}$, it holds that
$$\alpha_1(|h(p)|) \leq V(p) \leq \alpha_2(|p|)$$
- For some continuous positive definite $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, on the set $C$ there holds the estimate
$$\zeta \cdot f(x) \;\leq\; -\alpha_3(V(p)) \qquad \forall\, \zeta \in \partial_D V(p)$$
(when $V$ is differentiable, this is just $\nabla V \cdot f(x) \leq -\alpha_3(V(p))$)

One can show (cf. [38]) that this estimate is equivalent to the existence of a locally Lipschitz, positive definite $\tilde{\alpha}_3$ such that, for all trajectories:

$$x(t) \in C \;\text{ on }\; [0, t_1] \;\Rightarrow\; V(x(t)) - V(x(0)) \leq -\int_0^t \tilde{\alpha}_3(V(x(s)))\, ds\,.$$

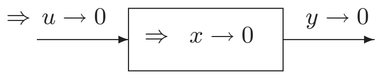


**Fig. 27.** Inverse of IOS property: small output implies input (and state) small

**Theorem 10.1.** [38] *A forward-complete system is RES if and only if it admits an RES-Lyapunov function.*

As a corollary, we have that, for RMEB systems, MES is equivalent to the existence of such a lower semicontinuous RES Lyapunov function.

# 11 Output to Input Stability and Minimum-Phase

We now mention a nonlinear "well-posed" version of Bode's minimum phase systems, which relates to the usual notion (cf. [42]) in the same manner as ISS relates to zero-GAS. We need to say, roughly, what it means for the "inverse system" to be ISS (Fig. 27). The paper [71] defines a smooth system as *output to input stable* (OIS) if there exists an integer $N > 0$, and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, so that, for every initial state $x^0$ and every $(N-1)$-times continuously differentiable input $u$, the inequality:

$$|u(t)| + |x(t)| \;\leq\; \beta(|x^0|, t) \;+\; \gamma\left(\|y^N_{[0,t]}\|_\infty\right)$$

holds for all $t \in t_{\max}(x^0, u)$, where "$y^N$" lists $y$ as well as its first $N$ derivatives (and we use supremum norm, as usual). See [71] for relationships to OSS, an interpretation in terms of an ISS property imposed on the "zero dynamics" of the system, and connections to relative degree, as well as an application to adaptive control.

# 12 Response to Constant and Periodic Inputs

Systems $\dot{x} = f(x, u)$ that are ISS have certain noteworthy properties when subject to constant or, more generally periodic, inputs, which we now discuss. Let $V$ be an ISS-Lyapunov function which satisfies the inequality $\dot{V}(x, u) \leq -V(x) + \gamma(|u|)$ for all $x, u$, for some $\gamma \in \mathcal{K}_\infty$.

To start with, suppose that $\bar{u}$ is any fixed bounded input, and let $a := \gamma(\|\bar{u}\|_\infty)$, pick any initial state $x^0$, and consider the solution $x(t) = x(t, x^0, \bar{u})$ for this input. Letting $v(t) := V(x(t))$, we have that $\dot{v}(t) + v(t) \leq a$ so, using $e^t$ as an integrating factor, we have that $v(t) \leq a + e^{-t}(v(0) - a)$ for all $t \geq 0$. In particular, if $v(0) \leq a$ it will follow that $v(t) \leq a$ for all $t \geq 0$, that is to say, the sublevel set $K := \{x \mid V(x) \leq a\}$ is a forward-invariant set for this input: if $x^0 \in K$ then $x(t) = x(t, x^0, \bar{u}) \in K$ for all $t \geq 0$. Therefore

$M_T : x^0 \mapsto x(T, x^0, \bar{u})$ is a continuous mapping from $K$ into $K$, for each fixed $T > 0$, and thus, provided that $K$ has a fixed-point property (every continuous map $M : K \to K$ has some fixed point), we conclude that for each $T > 0$ there exists some state $x^0$ such that $x(T, x^0, \bar{u}) = x^0$. The set $K$ indeed has the fixed-point property, as does any sublevel set of a Lyapunov function. To see this, we note that $V$ is a Lyapunov function for the zero-input system $\dot{x} = f(x, 0)$, and thus, if $B$ is any ball which includes $K$ in its interior, then the map $Q : B \to K$ which sends any $\xi \in B$ into $x(t_\xi, \xi)$, where $t_\xi$ is the first time such that $x(t, \xi) \in K$, is continuous (because the vector field is transversal to the boundary of $K$ since $\nabla V(x).f(x, 0) < 0$), and is the identity on $K$ (that is, $Q$ is a topological retraction). A fixed point of the composition $M \circ Q : B \to B$ is a fixed point of $M$.

Now suppose that $\bar{u}$ is periodic of period $T$, $\bar{u}(t + T) = \bar{u}(t)$ for all $t \geq 0$, and pick any $x^0$ which is a fixed point for $M_T$. Then the solution $x(t, x^0, \bar{u})$ is periodic of period $T$ as well. In other words, *for each periodic input, there is a solution of the same period*. In particular, if $\bar{u}$ is constant, we may pick for each $h > 0$ a state $x_h$ so that $x(h, x_h, \bar{u}) = x_h$, and therefore, picking a convergent subsequence $x_h \to \bar{x}$ gives that $0 = (1/h)(x(h, x_h, \bar{u}) - x_h) \to f(\bar{x}, \bar{u})$, so $f(\bar{x}, \bar{u}) = 0$. Thus we also have the conclusion that *for each constant input, there is a steady state.*

## 13 A Remark Concerning ISS and $H_\infty$ Gains

We derived the "integral to integral" version of ISS when starting from $H_\infty$-gains, that is, $L^2$-induced operator norms. In an abstract manner, one can reverse the argument, as this result shows:

**Theorem 13.1.** [33] *Assume $n \neq 4, 5$. If the system $\dot{x} = f(x, u)$ is ISS, then, under a coordinate change, for all solutions one has:*

$$\int_0^t |x(s)|^2 \, ds \;\leq\; |x^0|^2 + \int_0^t |u(s)|^2 \, ds\,.$$

(A particular case of this is that global exponential stability is equivalent to global asymptotic stability, under such nonsmooth coordinate changes. This would seem to contradict Center Manifold theory, but recall that our "coordinate changes" are not necessarily smooth at the origin, so dimensions of stable and unstable manifolds need not be preserved.) It is still an open question if the theorem generalizes to $n = 4$ or 5. A sketch of proof is as follows.

Let us suppose that the system $\dot{x} = f(x, u)$ is ISS. We choose a "robustness margin" $\rho \in \mathcal{K}_\infty$, i.e., a $\mathcal{K}_\infty$ function with the property that the closed-loop system $\dot{x} = f(x, d\rho(|x|))$ is GAS uniformly with respect to all disturbances such that $\|d\|_\infty \leq 1$. We next pick a smooth, proper, positive definite storage function $V$ so that

$$\nabla V(x) \cdot f(x, d\rho(|x|)) \;\leq\; -V(x) \quad \forall\, x, d$$

(such a function always exists, by the results already mentioned). Now suppose that we have been able to find a coordinate change so that $V(x) = |x|^2$, that is, a $T$ so that $W(z) := V(T^{-1}(z)) = |z|^2$ with $z = T(x)$. Then, whenever $|u| \leq \rho(|x|)$, we have

$$d\,|z|^2\,/dt = \dot{W}(z) = \dot{V}(x) \leq -V(x) = -\,|z|^2 \ .$$

It follows that, if $\chi \in \mathcal{K}_\infty$ is so that $|T(x)| \leq \chi(\rho(|x|))$, and

$$\alpha(r) \ := \ \max_{|u|\leq r, |z|\leq \chi(r)} d\,|z|^2\,/dt$$

then:

$$\frac{d\,|z|^2}{dt} \ \leq \ -\,|z|^2 + \alpha(|u|) = -\,|z|^2 + v$$

(where we denote by $v$ the input in new coordinates).

Integrating, one obtains $\int |z|^2 \leq |z^0|^2 + \int |v|^2$, and this gives the $L^2$ estimate as wanted. The critical technical step, thus, is to show that, up to coordinate changes, every Lyapunov function $V$ is quadratic. That fact is shown as follows. First notice that the level set $S := \{V(x) = 1\}$ is homotopically equivalent to $\mathbb{S}^{n-1}$ (this is well-known: $S \times \mathbb{R} \simeq S$ because $\mathbb{R}$ is contractible, and $S \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ via the flow of $\dot{x} = f(x,0)$). Thus, $\{V(x) = 1\}$ is diffeomorphic to $\mathbb{S}^{n-1}$, provided $n \neq 4, 5$. (In dimensions $n = 1, 2, 3$ this is proved directly; for $n \geq 6$ the sublevel set $\{V(x) < 1\}$ is a compact, connected smooth manifold with a simply connected boundary, and results on $h$-cobordism theory due to Smale and Milnor show the diffeomorphism to a ball. Observe that results on the generalized Poincaré conjecture would give a homeomorphism, for $n \neq 4$.) Finally, we consider the normed gradient flow:

$$\dot{x} \ = \ \frac{\nabla V(x)'}{|\nabla V(x)|^2}$$

and take the new variable

$$z \ := \ \sqrt{V(x)}\ \theta(x')$$

where $x'$ is the translate via the flow back into the level set, and $\theta : \{V = 1\} \simeq \{|z| = 1\}$ is the given diffeomorphism, see Fig. 28. (Actually, this sketch is not quite correct: one needs to make a slight adjustment in order to obtain also continuity and differentiability at the origin; the actual coordinate change is $z = \gamma(V(x))\theta(x')$, so $W(z) = \gamma(|z|)$, for a suitable $\gamma$.)

## 14 Two Sample Applications

For applications of ISS notions, the reader is encouraged to consult textbooks such as [27,43,44,58,60,66,96], as well as articles in journals as well as Proceedings of the various IEEE Conferences on Decision and Control. We highlight

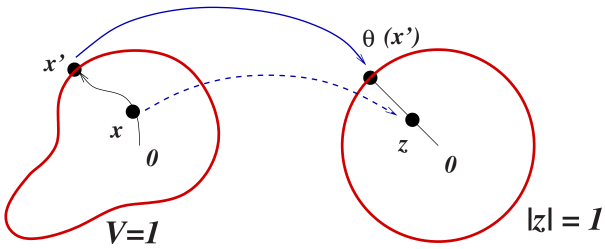


**Fig. 28.** Making level sets into spheres

next a couple of applications picked quite arbitrarily from the literature. They are chosen as illustrations of the range of possibilities afforded by the ISS viewpoint.

The paper [63] provides a new observer suitable for output-feedback stabilization, and applies the design to the stabilization of surge and the magnitude of the first stall mode, in the single-mode approximation of the Moore–Greitzer PDE axial compressor model used in jet engine studies. The equations are as follows:

$$\begin{aligned}\dot{\phi} &= -\psi + \tfrac{3}{2}\phi + \tfrac{1}{2} - \tfrac{1}{2}(\phi+1)^3 - 3(\phi+1)R \\ \dot{\psi} &= \frac{1}{\beta^2}(\phi + 1 - u) \\ \dot{R} &= \sigma R(-2\phi - \phi^2 - R) \qquad (R \geq 0)\end{aligned}$$

where $\phi$ denotes the mass flow relative to a setpoint, $\psi$ the pressure rise relative to the setpoint, and $R$ the magnitude of the first stall mode. The objective is to stabilize this system, using only $y = \psi$.

The systematic use of ISS-type properties is central to the analysis: taking the magnitude of the first stall mode as evolving through uncertain dynamics, the authors require that their estimator have an error that is ISS with respect to this unmodeled dynamics, and that the first mode be IOS with respect to mass flow deviation from its setpoint; an ISS small-gain theorem is then used to complete the design. Abstractly, their general framework in [63] is roughly as follows. One is given a system with the block structure:

$$\begin{aligned}\dot{x} &= f(x, z, u) \\ \dot{z} &= g(x, z)\end{aligned}$$

and only an output $y = h(x)$ is available for stabilization. The $z$-subsystem ($R$ in the application) is unknown (robust design). The authors construct a state-feedback $u = k(x)$ and a reduced-order observer that produces an estimate $\widehat{x}$ so that:

- The error $e = x - \widehat{x}$ is ISS with respect to $z$

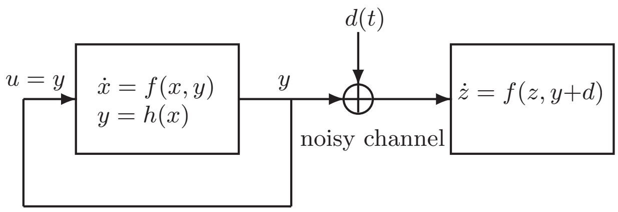


**Fig. 29.** Synchronized systems

- The system $\dot{x} = f(x, z, k(\widehat{x})) = F(x, z, e)$ is ISS with respect to both $e$ and $z$
- The system $\dot{z} = g(x, z)$ is ISS with respect to $x$

Combining with a small-gain condition, the stability of the entire system is guaranteed.

A completely different application, in signal processing, can be found in [4], dealing with the topic of synchronized chaotic systems, which arises in the study of secure communications. A "master-slave" configuration is studied, where a second copy of the system (receiver) is driven by an output from the first (Fig. 29). The main objective is to show that states synchronize:

$$|x(t) - z(t)| \leq \max\{\beta(|x^0 - z^0|, t), \|d\|\}$$

This can be shown, provided that the system is *incrementally* ISS, in the sense discussed in Sect. 8.6.

One particular example is given by the Lorentz attractor:

$$\begin{aligned}\dot{x}_1 &= -\beta x_1 + \text{sat}(x_2)\text{sat}(x_3),\\ \dot{x}_2 &= \sigma(x_3 - x_2),\\ \dot{x}_3 &= -x_3 + u,\\ y &= \rho x_2 - x_1 x_2,\end{aligned}$$

where $\beta = 8/3$, $\sigma = 10$, $\rho = 28$ (the saturation function, $\text{sat}(r) = r$ for $|r| < 1$ and $\text{sat}(r) = \text{sign}(r)$ otherwise, is inserted for technical reasons and does not affect the application). Preservation of the i-ISS property under cascades implies that this system (easily to be seen a cascade of i-ISS subsystems) is i-ISS. The paper [4] provides simulations of the impressive behavior of this algorithm.

## 15 Additional Discussion and References

The paper [99] presented the definition of ISS, established the result on feedback redefinition to obtain ISS with respect to actuator errors, and provided the sufficiency test in terms of ISS-Lyapunov functions. The necessity of this Lyapunov-like characterization is from [109], which also introduced the "small gain" connection to margins of robustness; the existence of Lyapunov functions then followed from the general result in [73]. The asymptotic gain characterizations of ISS are from [110]. (Generalizations to finite-dimensional and infinite-dimensional differential inclusions result in new relaxation theorems, see [41] and [39], as well as [81] for applications to switched systems.) Asymptotic gain notions appeared also in [20, 114]. Small-gain theorems for ISS and IOS notions originated with [53]. See [40] for an abstract version of such results.

The notion of ISS for time-varying systems appears in the context of asymptotic tracking problems, see, e.g., [124]. In [24], one can find further results on Lyapunov characterizations of the ISS property for time-varying (and in particular periodic) systems, as well as a small-gain theorem based on these ideas. See also [78].

Coprime factorizations are the basis of the parameterization of controllers in the Youla approach. As a matter of fact, as the paper's title indicates, their study was the original motivation for the introduction of the notion of ISS in [99]. Some further work can be found in [98], see also [28], but much remains to be done.

One may of course also study the notion of ISS for discrete-time systems. Many ISS results for continuous time systems, and in particular the Lyapunov characterization and ISS small gain theorems, can be extended to the discrete time case; see [52, 55, 56, 61, 67]. Discrete-time iISS systems are the subject of [2], who proves the very surprising result that, in the discrete-time case, iISS is actually no different than global asymptotic stability of the unforced system (this is very far from true in the continuous-time case, of course); see also [74].

Questions of sampling, relating ISS properties of continuous and discrete-time systems, have been also studied, see [119] which shows that ISS is recovered under sufficiently fast sampling, as well as the papers [86, 87, 90].

The paper [5] introduces a notion of ISS where one merely requires good behavior on a generic (open dense) subset of the state space. Properties of this type are of interest in "almost-global" stabilization problems, where there are topological constraints on what may be achieved by controllers. The area is still largely undeveloped, and there are several open problems mentioned in that reference.

More generally than the question of actuator ISS, one can ask when, given a system $\dot{x} = f(x, d, u)$, is there a feedback law $u = k(x)$ such that the system $\dot{x} = f(x, d, k(x))$ becomes ISS (or iISS, etc) with respect to $d$. One approach to this problem is in terms of control-Lyapunov function ("cLf") methods, and concerns necessary and sufficient cLf conditions, for the existence of such (possibly dynamic) feedback laws. See for example [120], which deals primarily with systems of the form $\dot{x} = f(x, d) + g(x)u$ (affine in control, and control vector fields are independent of disturbances) and with assigning precise upper bounds to the "nonlinear gain" obtained in terms of $d$.

A problem of decentralized robust output-feedback control with disturbance attenuation for a class of large-scale dynamic systems, achieving ISS and iISS properties, is studied in [49].

Partial asymptotic stability for differential equations is a particular case of output stability (IOS when there are no inputs) in our sense; see [125] for a survey of the area, as well as the book [94], which contains a converse theorem for a restricted type of output stability. The subject of IOS is also related to the topic of "stability in two measures" (see, e.g., [68]), in the sense that one asks for stability of one "measure" of the state ($h(x)$) relative to initial conditions measured in another one (the norm of the state).

A useful variation of the notion of ISS is obtained when one studies stability with respect to a closed subset $K$ of the state space $\mathbb{R}^n$, but not necessarily $K = \{0\}$. One may generalize the various definitions of ISS, IOS, IOSS, etc. For instance, the definition of ISS becomes

$$|x(t, x^0, u)|_K \;\leq\; \beta(|x^0|_K\,, t) \;+\; \gamma(\|u\|_\infty)\,,$$

where $|x|_K$ denotes the distance from $x$ to the set $K$. (The special case when $u \equiv 0$ implies in particular that the set $K$ must be invariant for the unforced system.) The equivalence of various alternative definitions can be given in much the same way as the equivalence for the particular case $K = \{0\}$ (at least for compact $K$), since the general results in [73] are already formulated for set stability; see [108] for details. The interest in ISS with respect to sets arises in various contexts, such as the design of robust control laws, where the set $K$ might correspond to equilibria for different parameter values, or problems of so-called "practical stabilization," concerned with stabilization to a prescribed neighborhood of the origin. See [107] for a theorem relating practical stabilization and ISS with respect to compact attractors.

Perhaps the most interesting set of open problems concerns the construction of feedback laws that provide ISS stability with respect to observation errors. Actuator errors are far better understood (cf. [99]), but save for the case of special structures studied in [27], the one-dimensional case (see, e.g., [25]) and the counterexample [26], little is known of this fundamental question. Recent work analyzing the effect of small observation errors (see [103]) might provide good pointers to useful directions of research (indeed, see [69] for some preliminary remarks in that direction). For special classes of systems, even output feedback ISS with respect to observation errors is possible, cf. [88].

A stochastic counterpart of the problem of ISS stabilization is proposed and solved in [22], formulated as a question of stochastic disturbance attenuation with respect to noise covariance. The paper [21], for a class of systems that can be put in output-feedback form (controller canonical form with an added stochastic output injection term), produces, via appropriate clf's, stochastic ISS behavior ("NSS" = noise to state stability, meaning that solutions converge in probability to a residual set whose radius is proportional to bounds on covariances). Stochastic ISS properties are treated in [123].

For a class of block strict-feedback systems including output feedback form systems, the paper [48] provided a global regulation result via nonlinear output feedback, assuming that the zero dynamics are iISS, thus generalizing the ISS-like minimum-phase condition in the previous [47], which in turn had removed the more restrictive assumption that system nonlinearities depend only on the output. See also [45] for iISS and ISS-stabilizing state and output feedback controllers for systems on strict-feedback or output-feedback forms.

For a class of systems including "Euler-Lagrange" models, the paper [17] provides a general result on global output feedback stabilization with a disturbance attenuation property. The notion of OSS and the results on unbounded observability both play a key role in the proof of correctness of the design.

An ISS result for the feedback interconnection of a linear block and a nonlinear element ("Lurie systems") is provided in [14], and an example is worked out concerning boundedness for negative resistance oscillators, such as the van der Pol oscillator.

The authors of [15] obtain robust tracking controllers with disturbance attenuation for a class of systems in strict-feedback form with structurally (non-parametric) unknown dynamics, using neural-network based approximations. One of the key assumptions is an ISS minimum phase condition, when external disturbances are included as inputs to the zero dynamics.

Output-feedback robust stabilization both in the ISS and iISS sense is studied, for large-scale systems with strongly nonlinear interconnections, in [51], using decentralized controllers.

Both ISS and iISS properties have been featured in the analysis of the performance of switching controllers, cf. [36]. The paper [35] dealt with hybrid control strategies for nonlinear systems with large-scale uncertainty, using a logic-based switching among a family of lower-level controllers, each of which is designed by finding an iISS-stabilizing control law for an appropriate system with disturbance inputs. The authors provide a result on stability and finite time switching termination for their controllers. The dissipation characterizations of ISS and of iISS were extended to a class of hybrid switched systems in [80].

A nonstandard application of IOSS, or more precisely of an MES property for turbulent kinetic energy and dissipation, was the method for *de*stabilization of pipe flows (to enhance mixing) studied in [1]. The authors used the wall velocity as inputs (blowing/suction actuators are assumed distributed on the pipe wall) and pressure differences across the pipe as outputs (using pressure sensors to measure). Detectability in the sense of IOSS provided a useful way to express the energy estimates required by the controller.

The papers [115, 116] introduced the notion of "formation input-to-state stability" in order to characterize the internal stability of leader-follower vehicle formations. There, and in related papers by other authors (e.g., [91]), ISS is used as a framework in which to systematically quantify the performance of swarm formations under time-varying signals (leader or enemy to be followed, noise in observation, actuator errors); in this context, the state $x$ in the ISS estimate is in reality a measure of formation error. Thus, in terms of the original data of the problem, this *formation ISS* is an instance not of ISS itself, but rather of input/output stability (IOS), in which a function of state variables is used in estimates.

For results concerning averaging for ISS systems, see [89], and see [19] for singular perturbation issues in this context. See [97] for a notion which is in some sense close to IMES. Neural-net control techniques using ISS are mentioned in [95]. There are ISS-small gain theorems for certain infinite dimensional classes of systems such as delay systems, see [117].

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