Accurate computation of moist available potential energy with the Munkres algorithm

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The moist available potential energy (MAPE) of a domain of air is defined as the maximum amount of kinetic energy that can be released through reversible adiabatic motions of its air parcels. The MAPE can be calculated using a parcel-moving algorithm that finds the minimum enthalpy state for a given set of thermodynamic assumptions. However, the parcel-moving algorithms proposed previously do not always find the minimum enthalpy state. In this paper, we apply the Munkres algorithm to find the exact minimum enthalpy state, and we compare this exact algorithm with four inexact algorithms, including a new divide-and-conquer algorithm. The divide-and-conquer algorithm performs well in practice while being simpler and faster than the Munkres algorithm, and it is recommended for future calculation of MAPE when the exact result is not required.

Key Words: available potential energy; moist energetics; moist convection; latent heat release; assignment problem

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1. Introduction

Lorenz (1955) introduced the concept of available potential energy of a mass of dry air, which is the difference between its enthalpy and the minimum possible enthalpy of a reversible adiabatic rearrangement of the air. This is of interest as the total energy of a column of air is equal in the hydrostatic approximation to the sum of its enthalpy and kinetic energy. Therefore any decrease in enthalpy gives an increase in kinetic energy, and so the available potential energy of the air is the maximum amount of kinetic energy that could be released under adiabatic processes.

The (specific) enthalpy h of a parcel of dry air is fully determined by giving its (specific) entropy s and pressure p. When parcels are rearranged adiabatically and reversibly, they conserve their entropy s. The dependence of h on s and p for dry air is such that, for two different parcels at the same pressure level, raising the parcel with greater entropy and lowering the parcel with lesser entropy always decreases the total enthalpy of the two parcels. (This can be seen by examining equation (5) for

$\frac{\partial h}{\partial p}$, where virtual temperature is just temperature for dry air, and using that temperature increases with entropy at constant pressure.) As entropy is conserved under reversible, adiabatic rearrangements, the minimum enthalpy state can be found by sorting the parcels by their entropy. This matches our intuition that hot (high entropy) air rises. In particular, the only state that is stable to small perturbations is the minimum enthalpy state. Thus there is no computational difficulty to determining the adiabatic rearrangement of a mass of dry air that minimizes its enthalpy.

Subsequently, Lorenz (1978) introduced the moist available potential energy (MAPE or, per Lorenz, MAE), in which he removed the restriction that the air considered has no moisture. Because latent heating is now accounted for, the MAPE of a domain of air serves as an upper bound on the amount of kinetic energy that be produced in the absence of radiative or frictional heating. For hemispheric domains, accounting for moisture increases the available potential energy by $\sim 35\%$ in both hemispheres according to reanalysis data (table S2 of O'Gorman (2010)), and MAPE has been used to investigate the role of moisture in the response of the extratropical storm tracks to climate change (O'Gorman 2010, 2011). For moist convection, the release of the energy available for vertical motions is associated with phase changes of water and therefore cannot be represented with the dry available potential energy. Instead, MAPE has been used as a generalization of convective available potential energy (CAPE) with the advantage of accounting for both ascending and descending air (Randall and Wang (1992), Wang and Randall (1994)). MAPE is expected to be smaller in magnitude than the integrated CAPE of parcels in the boundary layer (Emanuel 1994). Lastly, alternative versions of MAPE that allow for latent heating but limit moist convective instability have also been considered for hemispheric domains (O'Gorman 2010) and for a tropical cyclone (Wong *et al.* 2015).

Computation of the MAPE of an air mass requires being able to determine its minimum enthalpy configuration, but this is substantially more challenging than in the dry case. The reason for this difficulty is that parcels now have two conserved quantities (entropy and water content) instead of one, so that sorting by the conserved quantities no longer produces an unambiguous ordering. Furthermore there can exist multiple states which are stable to small perturbations; an algorithm that performs a local search of the configuration space may find a local minimum but miss the global minimum.

Lorenz (1979) gave a "parcel swapping" algorithm for finding a low enthalpy configuration. Randall and Wang (1992) demonstrated that that algorithm did not always produce the minimal enthalpy configuration by exhibiting a case (similar to Case A below) in which Lorenz's algorithm found the impossible result of negative MAPE. Randall and Wang presented an algorithm that did not give negative MAPE for this case, but we will find below that it likewise does not always produce the minimal enthalpy configuration. The inability to exactly calculate the MAPE, or to rigorously estimate the magnitude of error associated with existing algorithms, is a drawback for the use of MAPE in studies of climate and weather.

In section 2 we formally state the problem of finding the minimum enthalpy configuration. We discuss in section 3 the Munkres algorithm (popularly known as the "Hungarian method"), and attributed to Kuhn (1955) and Munkres (1957) among others, which always finds the minimum enthalpy configuration. A recent independent study by Hieronymus and Nycander (2015) uses the Munkres algorithm for an analogous problem in available potential energy for the ocean. Further in section 3, we discuss the algorithms of Lorenz (1979) and Randall and Wang (1992), as well as a greedy algorithm which is equivalent to the top-down sorting approach of Wong *et al.* (2015), and a new algorithm based on divide-and-conquer. In section 4 we compare the accuracy and speed of these algorithms on several test cases, and in section 5 we discuss the implications of our results.

2. Problem definition

Consider a domain of air with a fixed horizontal cross-section, independent of height, with upper and lower boundaries of the domain at some fixed pressures $p_{\min}$ and $p_{\max}$, respectively. Within this domain, the entropy s and total water mixing ratio w (kilograms of water per kilogram of dry air) are known at each position. We consider reversible adiabatic motions, which are those that conserve s and w. Our goal is to find the adiabatic rearrangement of the air in the domain that minimizes its enthalpy;

then the MAPE of the air in the domain is defined as its enthalpy minus the enthalpy of the least enthalpy rearrangement.

We are interested in a discretized version of the problem, where the domain is divided into a finite number n of parcels, each of which has a state $(s_i, w_i), i = 1, \ldots, n$, and each of which has the same mass δM. Lorenz (1979) has shown that a staggered grid may be used with the hydrostatic approximation to decompose the atmosphere into a set of parcels of equal mass that are linearly spaced in pressure. In this case, the difference in pressure from one parcel to the next in a vertically stratified configuration is

$$\delta p = g\delta M/A \tag{1}$$

where g is the gravitational acceleration, and A is the cross-sectional area of the column. We define the pressures $p_1, \ldots, p_n$ of the parcels by

$$p_j = p_{\text{min}} + (j - (1/2))\delta p. \tag{2}$$

Then a configuration can be fully described by specifying which pressure level p_j the parcel i is at for each i, or equivalently which parcel i is in pressure level p_j for each j. This configuration can be specified by giving a permutation σ of $\{1, \ldots, n\}$, where for each $i \in \{1, \ldots, n\}$, the parcel i is at pressure level $p_{\sigma(i)}$.

Now let $h_{i,j}$ be the enthalpy of a parcel of air with entropy s_i, total water mixing ratio w_i, and pressure p_j. Suppose the configuration of the domain is given by σ. Then the enthalpy of parcel i is $h_{i,\sigma(i)}$, and the total enthalpy of the domain per unit mass is

$$H = \sum_{i=1}^{n} h_{i,\sigma(i)}. \tag{3}$$

We wish to find the permutation σ that minimizes the value of H.

3. Algorithms

We developed multiple algorithms whose intention is to compute a low enthalpy configuration of an atmospheric domain; we discuss several of them here. In addition we discuss Lorenz's parcel swapping algorithm and Randall and Wang's variant.

We can compute the value of $h_{i,j}$ for every i and j using the known thermodynamic properties of moist air (see appendix). Our thermodynamic equations neglect ice for simplicity, but it could be accounted for following Wang and Randall (1994).

In this section $\partial_p h$ means the continuous derivative of h with respect to pressure and $\Delta_p h$ the discrete derivative, in both cases holding s and w constant:

$$\Delta_p h = \frac{h_{i,j+1} - h_{i,j}}{p_{j+1} - p_j} = \frac{h_{i,j+1} - h_{i,j}}{\delta p}. \tag{4}$$

In the limit $n \to \infty$, Δ_p converges to ∂_p.

The first law of thermodynamics and the ideal gas law gives that $dh = Tds + R_* T_v dp/p$, where R_* is the specific gas constant for dry air, and T_v is the virtual temperature (including the effect of condensate), such that

$$\partial_p h = R_* T_v/p. \tag{5}$$

For an alternative derivation of this, see equation (22). Thus, $\partial_p h$ is equal to specific volume, with units of m^3 kg^{-1}.

3.1. *Munkres algorithm*

Given the values of the $h_{i,j}$, the problem of finding the permutation σ that minimizes the sum $\sum_{i=1}^{n} h_{i,\sigma(i)}$ is known in computer science as the assignment problem, and can be solved by the Munkres algorithm. This algorithm is proven to always find the optimal permutation for any values for the $h_{i,j}$, and therefore makes no assumptions about the thermodynamic properties of the air; indeed this algorithm was originally designed in the context of transportation theory. The essential idea of the Munkres algorithm is to try to assign parcels of air to pressure levels at which they would have relatively low enthalpy, while keeping track of the difficulty of finding a low enthalpy match for each parcel of air and each pressure level. As these difficulties are updated the algorithm will adjust assignments previously made.

A slower version of the algorithm was introduced in Kuhn (1955); the O(n^3) version used here was introduced in Munkres (1957). A modern presentation of the algorithm can be found in Lawler (2001).

3.2. *Greedy algorithm*

The greedy algorithm proceeds from low pressure p_1 to high pressure p_n; at each pressure p_j, it chooses a parcel i to assign to that pressure among those that have not been assigned already. This assignment is done by choosing the remaining parcel which maximizes $\partial_p h$ at p_j. The aim of this choice is to ensure that a lower enthalpy state would not result from swapping parcels that are adjacent in pressure, but it doesn't guarantee that the lowest possible enthalpy state is found globally. The greedy algorithm is the same as the top-down sorting approach discussed in Wong *et al.* (2015).

3.3. *Lorenz's algorithm*

We reproduce here Lorenz's parcel swapping algorithm, proposed in Lorenz (1979). As in the greedy algorithm, Lorenz's algorithm proceeds from low pressure p_1 to high pressure p_n, at each step choosing a parcel from those that remain. To choose a parcel to assign to pressure level p_j, Lorenz first identifies the parcels a and b among those that remain such that parcel a maximizes the virtual temperature T_v at p_1, and parcel b maximizes the virtual temperature T_v at p_n. Then the algorithm chooses to assign either parcel a or parcel b to pressure level p_j according to which one maximizes $\Delta_p h$ at p_j.

As Lorenz observed, maximizing T_v at a fixed pressure is the same as maximizing $\partial_p h$ (see equation (5)). Therefore Lorenz's algorithm can be rephrased in terms of only $\partial_p h$ and $\Delta_p h$ (which are equal in the limit of large n).

3.4. *Divide and conquer algorithm*

Our divide and conquer algorithm proceeds recursively by dividing the domain into smaller subdomains. If there is only a single parcel ($n = 1$), there is only one configuration possible. Otherwise we divide the domain into two smaller parts. Let $m = \text{floor}((n+1)/2)$, so that the pressure p_m is half-way between p_1 and p_n. Then we calculate $\partial_p h$ for all n parcels at the pressure p_m. The m parcels with the highest values of $\partial_p h$ at p_m we assign to the low-pressure subdomain from p_1 to p_m, and the other $n-m$ parcels we assign to the high-pressure subdomain p_{m+1} to p_n. Then we recursively perform the algorithm on the two subdomains p_1 to p_m and p_{m+1} to p_n.

3.5. *Randall and Wang's algorithm*

Randall and Wang (1992) proposed a variation of Lorenz's algorithm. This algorithm is designed to never produce a configuration with higher enthalpy than the given initial configuration. The algorithm starts with a given configuration and successively modifies it by lifting parcels to the top of the subdomain under consideration, shifting the intervening parcels downwards; therefore the algorithm can give different minimum enthalpy states depending on what the initial configuration is (unlike the other algorithms). Further details can be found in Appendix B of Randall and Wang (1992). (Note that we do not use the mass flux approach discussed in their section 3).

4. Comparison of algorithms

We discuss the configurations produced by these algorithms on three test cases. Test cases A and B are taken from real-world data, and test case C is designed to illustrate the performance in the face of multiple stable states. For each case we ran each algorithm and computed the enthalpy of the configuration found by the algorithm. The difference between that enthalpy and the true minimum possible enthalpy is termed the residual MAPE and is displayed in Table 1. A perfect algorithm will always find the true minimum and therefore have a value of 0 for each case. The maximal possible residual MAPE is also listed for comparison.

Case A is based on a GATE sounding found in Randall and Wang (1992) which had 37 parcels in the range 100 to 1000 hPa. Randall and Wang used this GATE sounding to demonstrate that their algorithm can produce better (i.e., lower enthalpy) results than Lorenz's. We linearly interpolated this sounding to 1000 parcels to give more robust results. * The true MAPE found by the Munkres algorithm is 10.813 J kg^{-1}, and a similar value is found by the Randall and Wang algorithm, while the divide and conquer algorithm finds slightly less. By contrast, the Lorenz and

*Our implementations of Randall and Wang's algorithm and Lorenz's algorithm gave results for the original 37 parcels which qualitatively agree with the results found by the implementations of these algorithms due to Randall and Wang. However the small number of parcels made the results very sensitive to the input and choice of thermodynamic constants, and so case A was generated by linearly interpolating w and s with respect to pressure to give 1000 equal-mass parcels. The divide and conquer, Lorenz, and greedy algorithms all produced significantly better results on the interpolated data than the non-interpolated data; we generally do not expect that interpolating other data sets would have this same effect. Above 150 parcels, our results were not very sensitive to the number of parcels used.

greedy algorithms perform poorly and only find roughly half of the MAPE.

Case B is taken from NCEP2 reanalysis (Kanamitsu (2002)), and is a zonal and time average over June-July-August 1981-2000. The temperature and humidity are sampled at 40 latitudes poleward of 20N, and the latitudes are spaced to give equal surface area in each latitude band. Each latitude is sampled at 40 pressures from 50 hPa to 1000 hPa, staggered with respect to other latitudes following the approach of Lorenz (1979); in total there are 1600 parcels. This case was previously considered in O'Gorman (2010) but taking into account the effect of ice.[†] The minimal enthalpy configuration is shown in Figure 1. We see that there is a discontinuity in the pressure levels in the minimum state, as some very moist parcels appear above 400 hPa while some slightly less moist parcels appear below 700 hPa. The convection of these moist parcels upwards relative to the initial configuration lowers the enthalpy of the system substantially. All of the algorithms find a similar discontinuity in pressure and are able to liberate most of the initial MAPE, but disagree slightly on the exact number of parcels that should be lifted and thus have small differences in MAPE.

Case C is a minimal example to illustrate the difficulty with multiple states that are stable to small perturbations. This case involves 900 dry parcels and 100 wet parcels in the range 400 to 650 hPa with $\delta p = 0.25$ hPa (all parcels have equal mass). The dry parcels contain no water and have a temperature at 10^5 Pa of 375 K; the wet parcels are one third water by molar fraction ($w = 0.311$ kg / kg) and have a temperature at 10^5 Pa of 335 K. Note that all dry parcels have the same s and w, and all wet parcels have the same s and w. Figure 2 displays $\partial_p h$ as a function of pressure for the dry and wet parcels and can be used to infer the parcel configuration of the minimal enthalpy state. As parcels with higher $\partial_p h$ than their environment tend to rise, we see that wet parcels above 450 hPa will tend to rise relative to dry parcels, while those below 450 hPa will tend to fall relative to dry parcels. Therefore any state with the wet parcels at the extreme top and bottom and the dry parcels in between is stable against small perturbations, and there are 101 such stable states. To determine which of these has the least enthalpy, notice that the change of enthalpy of switching a wet and dry parcel at different pressure levels is given by the area between the curves and between those pressure levels in Figure 2, so the maximal reduction of enthalpy is achieved by moving the wet parcels to the bottom (650 hPa to 625 hPa) where the curves are farthest apart. The Lorenz, Randall and Wang, and greedy algorithms all instead find the configuration with the wet parcels at the top, which has the highest enthalpy of any state that is stable to small perturbations, nearly the highest enthalpy of any state at all, and much higher than would be found by randomly arranging the parcels. (Random arrangements in case C tend to have an enthalpy near the average of the minimum and maximum enthalpies.) Indeed if these algorithms were used to estimate the MAPE of the initial configuration of case C, the result would be off by a factor of 21.3. (The initial configuration for case C was decided to be the configuration with maximal enthalpy. This choice only matters for running the Randall and Wang algorithm, and has no effect on the other algorithms.)

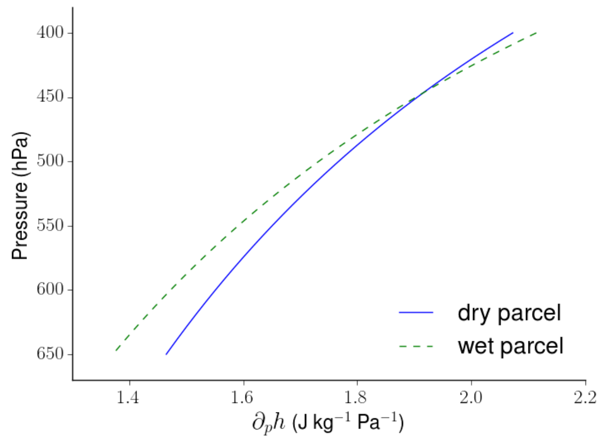
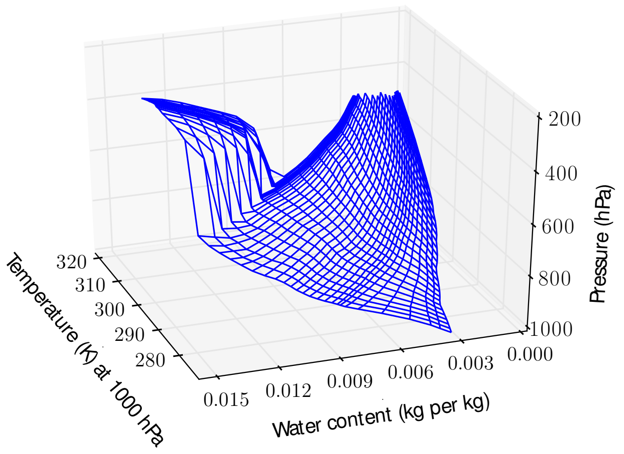


Figure 2. $\partial_p h$ as a function of pressure for dry (solid) and wet (dashed) parcels in case C, which is a minimal example of a case with multiple stable states. Note that $\partial_p h$ equals specific volume. The values of $\partial_p h$ coincide near 450 hPa, so at lower pressures wet parcels will rise relative to dry parcels, and at higher pressures they will fall. Therefore there are multiple stable configurations as wet parcels can either go to the top or bottom of the domain. The configuration with all wet parcels at the bottom has the least enthalpy, but the Lorenz, greedy, and Randall and Wang algorithms prefer to put the wet parcels at the top.

The computation times for the different algorithms are also given in Table 1. In every case examined, Lorenz's algorithm and divide and conquer are substantially faster than the other algorithms. Asymptotically, the running time of Lorenz's algorithm is $O(n)$; of divide and conquer is $O(n \log n)$; of the

[†] O'Gorman (2010) reported that the Lorenz algorithm gave better results than the Randall and Wang algorithm for case B with ice, but this was because of an error in implementation of the Randall and Wang algorithm, and in fact both algorithms give very similar results for case B with ice. Using the divide and conquer algorithm introduced here does not substantially alter the results in O'Gorman (2010, 2011).



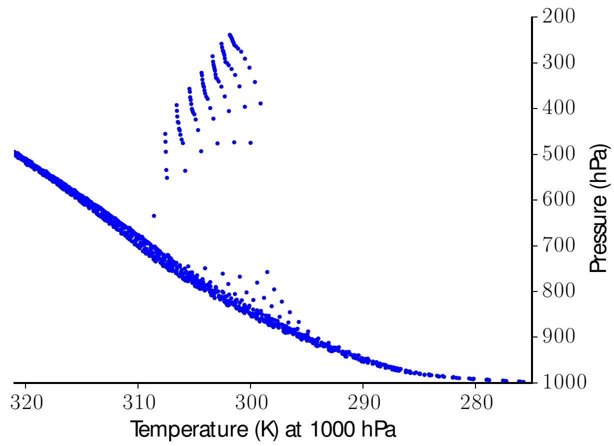


Figure 1. Minimal enthalpy configuration of case B, which is based on a zonal and time mean poleward of 20N in the NCEP reanalysis during JJA. The configuration is shown in (a) as the relationship between pressure, water content (w), and the temperature each air parcel would have at 10^5 Pa, and similarly in (b) but omitting water content. Hot, dry parcels with temperatures from 320 to 500 K are not shown; these parcels correspond to upper-tropospheric and stratospheric air. In the initial configuration there is no discontinuity in pressure as a function of water content and temperature at 10^5 Pa, but in the minimal enthalpy configuration there is a discontinuity due to a section of particularly wet, warm air parcels convecting upwards. Except for these parcels, (b) shows that the pressure in the minimum enthalpy state is almost a monotonic function of the temperature that the parcels would have at 10^5 Pa. The exact configuration shown is calculated using the Munkres algorithm, but all algorithms tested produce a qualitatively similar result on this case.

	residual MAPE (J kg^{-1})			computation time (s)		
	A	B	C	A	B	C
Munkres	0	0	0	23.16	70.99	14.85
divide and conquer	0.391	0.000414	0	0.10	0.18	0.07
Randall and Wang	$< 10^{-4}$	0.00645	95.3	17.22	47.55	6.29
greedy	5.90	0.00645	95.3	12.49	30.62	2.36
Lorenz	5.90	0.510	95.3	0.10	0.15	0.02
minimal enthalpy	0	0	0			
initial configuration	10.8	200	100			
maximal enthalpy	2250	8690	100			

Table 1. Residual MAPE is defined as the enthalpy of the configuration found by each algorithm minus the true minimum possible enthalpy; it is the amount of MAPE left over after each algorithm extracted as much as it could. Lower values indicate that the algorithm found a lower enthalpy configuration, with zero the best possible result. Shown for comparison are the residual MAPE for the minimal, maximal, and initial configurations, the first of which is 0 by definition.

greedy algorithm and Randall and Wang's algorithm is $O(n^2)$; and of the Munkres algorithm is $O(n^3)$.

5. Discussion

Of these five, which algorithm should be used to compute the moist available potential energy of a domain of air? When accuracy is necessary, the Munkres algorithm should be used as it is the only one of these algorithms that is proven to always compute the minimal enthalpy configuration; in all other cases, the divide and conquer algorithm should be used on account of its speed, simplicity, ease of coding, and generally high accuracy.

The difference in accuracy between the Munkres and divide and conquer algorithms relative to the others considered can be related to whether the algorithm immediately begins assigning parcels to pressure levels or defers all assignment until the end of the algorithm. The latter is necessary for two reasons. First, assigning a parcel to some pressure level prevents other parcels from being assigned to that same pressure level, so that all parcels must be considered together. Second, due to the thermodynamics of moist parcels, which exhibit different behavior above and below their condensation level and therefore may have multiple stable levels, all pressure levels must be considered together. Algorithms which begin assigning parcels to pressure levels before having considered the whole scenario in its entirety may, like the Lorenz, greedy, and Randall and Wang algorithms, be vulnerable to overlooking low enthalpy arrangements.

The availability of the Munkres algorithm to calculate the exact MAPE puts the use of MAPE for studies of the atmosphere on a firmer footing. Our results for case B suggest that previous calculations of MAPE for hemispheric-scale domains were not seriously compromised by the use of the approximate parcel moving algorithms, although cases A and C clearly show that problems can arise, and that the divide and conquer algorithm seems to be the best available fast algorithm at present.

The Munkres algorithm cannot be adapted for cases where the parcels have different masses. In this situation we recommend either using another algorithm or regridding (using, e.g. linear interpolation) the data so that the parcels have the same mass. The regridded parcel size should be not much bigger than the smallest starting parcel size, so the regridding process may increase the number of parcels if the initial parcels were of widely varying sizes.

Recently, Hieronymus and Nycander (2015) applied the Munkres algorithm to find the minimum potential energy of a domain in the ocean and compared it to several approximate methods. What they call Huang's algorithm (Huang (2005)) is analogous to the greedy algorithm. Hieronymus and Nycander find that the Munkres algorithm is too slow to use on their largest data sets, and it would be worth investigating if the divide and conquer algorithm introduced here may be adapted to large oceanic data sets.

We thank a reviewer for bringing to our attention the recent work of Su and Ingersoll (2016) which discusses an exact algorithm which they find runs faster than the Munkres algorithm when run on 3D grids with few distinct pressure levels. It may be possible to use a similar approach to speed up calculation of MAPE for 2D or 3D domains in the atmosphere.

All code and data needed to duplicate our results are found at `https://github.com/estansifer/mape/`.

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A. Thermodynamic equations

We define constants (from Haynes *et al.* (2013))

$$R = 8.3145 \text{ J mol}^{-1} \text{ K}^{-1} \quad \text{universal gas constant} \tag{6}$$

$$c_d = 29.162 \text{ J mol}^{-1} \text{ K}^{-1} \quad \text{heat capacity of dry air} \tag{7}$$

$$c_v = 33.689 \text{ J mol}^{-1} \text{ K}^{-1} \quad \text{heat capacity of water vapor} \tag{8}$$

$$c_l = 75.522 \text{ J mol}^{-1} \text{ K}^{-1} \quad \text{heat capacity of liquid water} \tag{9}$$

$$M_d = 0.028959 \text{ kg mol}^{-1} \quad \text{molecular weight of dry air} \tag{10}$$

$$M_w = 0.018015 \text{ kg mol}^{-1} \quad \text{molecular weight of water} \tag{11}$$

and $\epsilon = M_w/M_d$. All heat capacities are at constant pressure.

Let $e_s(T)$ be the saturation vapor pressure of water at the temperature T. Then we assume the Clausius-Clapeyron relation, which states that

$$e_s(T) = p_c \exp(((c_v - c_l)/R)\ln(T/T_c) - L_c/(RT)) \quad (12)$$

where the constants have values

$$p_c = 3.2238 \cdot 10^{13} \text{ Pa}, \quad (13)$$

$$T_c = 283.15 \text{ K}, \quad (14)$$

$$L_c = 56481 \text{ J mol}^{-1}. \quad (15)$$

Let w be the water content of a parcel of air, in kilograms of water per kilogram of dry air. Define

$$w_s = \epsilon \frac{e_s}{p - e_s},$$
$$L = \frac{RT^2}{e_s}\frac{de_s}{dT} = (c_v - c_l)T + L_c.$$

For a saturated parcel, w_s measures the water vapor content, in kilograms of water vapor per kilogram of dry air. Note that L is the latent heat of condensation of water per mole.

Now for unsaturated and for saturated parcels, respectively, we define the specific entropy s as

$$\begin{aligned} M_d(1+w)s = {} & (c_d + (w/\epsilon)c_v)\ln(T/T_0) \\ & - R(1 + (w/\epsilon))\ln(p/p_0) \\ & + R(1 + (w/\epsilon))\ln(1 + (w/\epsilon)) \\ & - R(w/\epsilon)\ln(w/\epsilon), \end{aligned} \quad (16)$$

$$\begin{aligned} M_d(1+w)s = {} & (c_d + (w/\epsilon)c_v)\ln(T/T_0) \\ & - R(1 + (w/\epsilon))\ln(e_s/p_0) \\ & + R\ln(w_s/\epsilon) + (L/T)(w_s - w)/\epsilon. \end{aligned} \quad (17)$$

T_0 and p_0 are arbitrary reference values for temperature and pressure. The condition for whether a parcel is saturated is whether $w \geq w_s$. The equations agree on the value of s at saturation, that is, at $w_s = w$.

Given the values of s, p, and w for a parcel, these above equations can be solved for temperature T; the solution is unique.

Now for unsaturated and saturated parcels, respectively, we define the specific enthalpy h as

$$M_d(1+w)h = (c_d + (w/\epsilon)c_v)T, \quad (18)$$

$$M_d(1+w)h = (c_d + (w/\epsilon)c_v)T + L(w_s - w)/\epsilon. \quad (19)$$

Again these agree on the value of h at saturation. Note that equations 16 - 19 are consistent with equations 11-14 of Lorenz (1979).

With some work we are able to compute $\partial_p h$. For unsaturated parcels,

$$M_d(1+w)\partial_p h = R(1 + (w/\epsilon))T/p, \quad (20)$$

and for saturated parcels,

$$M_d(1+w)\partial_p h = R(1 + (w_s/\epsilon))T/p. \quad (21)$$

Thus in both cases,

$$\partial_p h = \frac{RT_v}{M_d p} \quad (22)$$

where T_v is virtual temperature, and thus $\partial_p h$ is equal to specific volume. Recall that R/M_d is the specific gas constant of dry air, and so this agrees with Lorenz's equations 17 and 18.