Let

$$S_1 = \left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 + \frac{1}{13}\right)\left(1 + \frac{1}{17}\right)\left(1 - \frac{1}{19}\right)\left(1 - \frac{1}{23}\right)\cdots,$$

where the product continues over all odd primes, and the positive sign is taken for those primes that are one more than a multiple of four. Our goal is to prove that

$$S_1 = \frac{2}{\pi}$$

Our proof uses only elementary math from calculus and earlier. Several times we require the formula for the sum of a geometric series, which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

This can easily be proven by multiplying the right side by $(1 - x)$ and finding that it equals 1. We will also need to know the fundamental theorem of arithmetic, which is that any positive number can be written uniquely as a product of prime numbers:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

This is the only property of prime numbers we will need. A very simple (but not obvious) consequence of this is what is known as the Euler product formula,

$$\sum_{n>0} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

To see why this is true, we expand the right side as a geometric series:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \prod_{p \text{ prime}} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots\right)$$

The right side is an infinite product of infinite sums. If we use the distributive law we get an infinite sum of terms of the form

$$p_1^{-a_1 s} p_2^{-a_2 s} p_3^{-a_3 s} \cdots = (p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots)^{-s}$$

But each integer n can be written as a product of primes exactly one way, so this is just the sum of n^{-s} over all integers, proving the Euler product formula.

Lemmas 1 and 2 are well known results; for completeness I provide proofs here.

By the way, the function

$$\zeta(s) = \sum_{n>0} n^{-s}$$

is called the *Riemann-zeta function*.

Lemma 1.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

Proof. Let

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Then

$$\frac{d}{dx} f(x) = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2}$$

by the geometric series formula. One can verify from properties of sine and cosine that $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$ and $f(0) = 0 = \arctan(0)$, so $f(x) = \arctan(x)$ and therefore

$$f(1) = \arctan(1) = \frac{\pi}{4}.$$

□

Lemma 2.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Proof. In other words, $\zeta(2) = \frac{\pi^2}{6}$.

This is a variation of a proof is due to Euler. Suppose $f(x)$ is function with $f(0) = 1$ and whose zeros are $z_1, z_2, z_3, \ldots$ (formally speaking we require that f is analytic near the origin and that we include complex zeros, if there are any); then

$$f(x) = \left(1 - \frac{x}{z_1}\right)\left(1 - \frac{x}{z_2}\right)\left(1 - \frac{x}{z_3}\right) \cdots .$$

This can be thought of as a generalization of factoring a polynomial as its roots $p(x) = (x - z_1)(x - z_2) \cdots (x - z_k)$. Now let

$$f(x) = \frac{\sin(\pi x)}{\pi x}.$$

Since the zeros of $\sin(x)$ are the integer multiples of π, then the zeros of $f(x)$ are just the nonzero integers $\pm 1, \pm 2, \pm 3, \ldots$. (0 is not a zero because of the πx in the denominator.) Then

$$\begin{aligned} f(x) &= \left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{1}\right)\left(1 - \frac{x}{2}\right)\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{3}\right)\left(1 + \frac{x}{3}\right) \cdots \\ &= \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right) \cdots \end{aligned}$$

We can use the distributive law to expand this; we are only interested in the coefficient of x^2. We get

$$= 1 - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) x^2 + \left(\sum_{a>b>0} \frac{1}{a^2 b^2}\right) x^4 - \cdots .$$

But also recall the Taylor series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

So

$$f(x) = \frac{\sin(\pi x)}{\pi x} = 1 - \frac{\pi^2}{3!}x^2 + \frac{\pi^4}{5!}x^4 - \cdots .$$

Since $3! = 6$, and the x^2 coefficients of the two expressions for $f(x)$ are equal, this gives the desired result.

Note that if we look at the x^4 coefficients we also get an interesting result, which is discussed after the main body of the text. □

Lemma 3. *Let*

$$S_2 = \left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{11}\right)\left(1 - \frac{1}{13}\right)\left(1 - \frac{1}{17}\right)\left(1 + \frac{1}{19}\right)\left(1 + \frac{1}{23}\right)\cdots,$$

then $S_2 = \frac{4}{\pi}$.

Proof. Here S_2 is defined so that the signs are the opposite as in S_1. We have

$$S_2 = \left(\prod_{p\equiv 1(4)} \left(1 - \frac{1}{p}\right)\right)\left(\prod_{p\equiv 3(4)} \left(1 + \frac{1}{p}\right)\right),$$

where the first product is over primes congruent to 1 modulo 4, and the latter over primes congruent to 3 modulo 4.

We take the reciprocal and find

$$\begin{aligned} S_2^{-1} &= \left(\prod_{p\equiv 1(4)} \frac{1}{1 - \frac{1}{p}}\right)\left(\prod_{p\equiv 3(4)} \frac{1}{1 + \frac{1}{p}}\right), \\ &= \left(\prod_{p\equiv 1(4)} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots\right)\right)\left(\prod_{p\equiv 3(4)} \left(1 - \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3} + \frac{1}{p^4} - \cdots\right)\right) \end{aligned}$$

Now we distribute this like in the Euler product formula. Since the prime 2 is not included in this product, only the odd integers appear. We get the sum

$$= \sum_{\text{odd } n>0} \pm\frac{1}{n}$$

where some work remains to figure out whether each sign is positive or negative. The ones that are positive are for those odd integers n which are divisible by an even number of primes congruent to 3 modulo 4 (in which case the negative signs all cancel out), which is exactly the same as the n congruent to 1 modulo 4 (because if you multiply together an even number of numbers congruent to 3 modulo 4, you get 1 modulo 4). Thus

$$\begin{aligned} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \\ &= \frac{\pi}{4}. \end{aligned}$$

□

Theorem 4. $S_1 = \frac{2}{\pi}$.

Proof. The technique used for calculating S_2 does not work for S_1. However,

$$\frac{3}{4} S_1 S_2 = \left(1 - \frac{1}{2^2}\right) \cdot \prod_{p \text{ odd prime}} \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)$$
$$= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right),$$

However

$$\prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^2}} = \sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$$

by the Euler product formula, so

$$S_1 \cdot \frac{3}{\pi} = \frac{3}{4} S_1 S_2 = \frac{6}{\pi^2}$$

therefore $S_1 = \frac{2}{\pi}$. □

Note that we can use the distributive law to expand the expression S_1 into an infinite sum,

$$S_1 = \sum_{n \text{ odd}} (-1)^{(n-1)/2} \frac{|\mu(n)|}{n}$$

where $\mu(n)$ is the Möbius function, and $|\mu(n)|$ is 1 when n is square-free (not divisible by a perfect square larger than 1) and 0 otherwise. This gives a very interesting comparison with the series in Lemma 1:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} - \frac{1}{23} + \frac{1}{25} - \frac{1}{27} + \frac{1}{29} \cdots = \frac{\pi}{4}$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \qquad - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} - \frac{1}{23} \qquad + \frac{1}{29} \cdots = \frac{2}{\pi}$$

In the second expression, terms which are not square-free (9, 25, 27, 45, 49, 63, 75, ...) are omitted.

As a bonus we compute $\sum_{n>0} \frac{d(n)}{n^2}$ where $d(n)$ is the *divisor function* which counts the number of positive divisors of n (for example, $d(12) = 6$ because 1, 2, 3, 4, 6, and 12 divide 12).

We have

$$\begin{aligned}\sum_{n>0} \frac{d(n)}{n^2} &= \sum_{n>0} \sum_{a \cdot b = n} \frac{1}{n^2} \\ &= \sum_{n>0} \sum_{a \cdot b = n} \left(\frac{1}{a^2}\right) \left(\frac{1}{b^2}\right) \\ &= \left(\sum_{a>0} \frac{1}{a^2}\right) \left(\sum_{b>0} \frac{1}{b^2}\right) \\ &= \left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{36}.\end{aligned}$$

When n can be factored into a product of two numbers (a, b), there are three possibilities: either $a < b$, $a = b$, or $a > b$. Since $n = a \cdot b$ is the same as $n = b \cdot a$, exactly half of the factorizations have $a > b$ (and half have $a < b$), unless n is a perfect square, in which case there is an additional factorization of $n = a \cdot a$ for some a. Therefore

$$\sum_{n>0} \frac{d(n)}{n^2} = 2 \cdot \sum_{a>b>0} \frac{1}{(ab)^2} + \sum_{a>0} \frac{1}{(a^2)^2}.$$

However by comparing the x^4 coefficients at the end of lemma 2 we see that

$$\sum_{a>b>0} \frac{1}{(ab)^2} = \frac{\pi^4}{120}.$$

Since we just found $\sum_{n>0} \frac{d(n)}{n^2} = \frac{\pi^4}{36}$, and $1/36 - 2/120 = 1/90$, therefore

$$\zeta(4) = \sum_{n>0} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Continuing in this way with higher and higher coefficients of the function $\frac{\sin(\pi x)}{\pi x}$ from lemma 2, we can find a general formula for $\zeta(k)$ for any even $k \geq 2$. There is no known formula for odd $k > 2$.