Interpolation with Bounded Error

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Abstract

Given $n$ general points $p_1, p_2, \ldots, p_n \in \mathbb{P}^r$ it is natural to ask whether there is a curve of given degree $d$ and genus $g$ passing through them; by counting dimensions a natural conjecture is that such a curve exists if and only if

$$n \leq \left\lfloor \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} \right\rfloor.$$

The case of curves with nonspecial hyperplane section was recently studied in [1], where the above conjecture was shown to hold with exactly three exceptions.

In this paper, we prove a “bounded-error analog” for special linear series on general curves; more precisely we show that existence of such a curve subject to the stronger inequality

$$n \leq \left\lfloor \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} \right\rfloor - 3.$$

Note that the $-3$ cannot be replaced with $-2$ without introducing exceptions (as a canonical curve in $\mathbb{P}^3$ can only pass through 9 general points, while a naive dimension count predicts 12).

We also use the same technique to prove that the twist of the normal bundle $N_C(-1)$ satisfies interpolation for curves whose degree is sufficiently large relative to their genus, and deduce from this a bound on the number of general points contained in the hyperplane section of a general curve.

1 Introduction

If $C$ is a general curve, equipped with a general map $f: C \to \mathbb{P}^r$ of degree $d$, it is natural to ask how many general points are contained in $f(C)$. This problem has been studied in many cases, including for nonspecial curves [1], for space curves [7], and for canonical curves [6]. To state the problem precisely, we make the following definition:

Definition 1.1. We say a stable map $f: C \to \mathbb{P}^r$ of degree $d$ from a curve of genus $g$ is a Weak Brill-Noether curve (WBN-curve) if it is a limit of degree $d$ maps $C' \to \mathbb{P}^r$ with $[C'] \in \overline{M}_g$ of general moduli, which are either nonspecial or nondegenerate; in the latter case, we say $f$ is a Brill-Noether curve (BN-curve).

If $[f]$ lies in a unique component of $\overline{M}_g(\mathbb{P}^r, d)$, we say $f$ is an interior curve.
The celebrated Brill-Noether theorem then asserts that BN-curves exist if and only if

\[ \rho(d, g, r) := (r + 1)d - rg - r(r + 1) \geq 0. \]

Moreover, for \( \rho(d, g, r) \geq 0 \), there is only one component of \( \overline{M}_{g,n}(\mathbb{P}^r, d) \) (respectively \( \overline{M}_{g,n}(\mathbb{P}^r, d) \)) corresponding to BN-curves (respectively marked BN-curves); we write \( \overline{M}_g(\mathbb{P}^r, d) \) (respectively \( \overline{M}_g(\mathbb{P}^r, d) \)) for that component. The question posed at the beginning then amounts to asking when the natural map \( \overline{M}_{g,n}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^n \) is dominant. In order for this to happen, it is evidently necessary for

\[ (r + 1)d - (r - 3)(g - 1) + n = \dim \overline{M}_{g,n}(\mathbb{P}^r, d) \geq \dim(\mathbb{P}^r)^n = rn, \]
or equivalently,

\[ n \leq \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1}. \]

However, this is not sufficient: When \( (d, g, r) = (6, 4, 3) \), the above equation gives \( n \leq 12 \); but every canonical curve in \( \mathbb{P}^3 \) lies on a quadric, and so can only pass through 9 general points (three less than expected). Our main theorem implies that the above condition is “as close as possible to sufficient given the above example”:

**Theorem 1.2.** There exists a BN-curve of degree \( d \) and genus \( g \) in \( \mathbb{P}^r \) (with \( \rho(d, g, r) \geq 0 \)), passing through \( n \) general points, if

\[ (r - 1)n \leq (r + 1)d - (r - 3)(g - 1) - 2r. \]

In particular, such a curve exists so long as

\[ n \leq \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} - 3. \]

This theorem is proven by studying the normal bundle of the general marked BN-curve \( f: (C, p_1, p_2, \ldots, p_n) \to \mathbb{P}^r \). As long as \( f \) is unramified (respectively unramified and transverse to a hyperplane \( H \)), basic deformation theory implies the map

\[ f \mapsto (f(p_1), f(p_2), \ldots, f(p_n)) \]

(respectively the map \( f \mapsto (f(p_1), f(p_2), \ldots, f(p_n), f(C) \cap H) \)), from the corresponding Kontsevich space to \( (\mathbb{P}^r)^n \) (respectively to \( (\mathbb{P}^r)^n \times \text{Sym}^d H \)) is smooth at \( [f] \) if

\[ H^1(N_f(-p_1 - \cdots - p_n)) = 0. \]

(respectively if \( H^1(N_f(-1)(-p_1 - \cdots - p_n)) = 0 \)). Here, \( N_f = \ker(f^*\Omega_{\mathbb{P}^r} \to \Omega_C \vee) \) denotes the normal bundle of the map \( f: C \to \mathbb{P}^r \).

Since a map between reduced irreducible varieties is dominant if it is generically smooth, it suffices to check \( H^1(N_f(-p_1 - \cdots - p_n)) = 0 \) (respectively \( H^1(N_f(-1)(-p_1 - \cdots - p_n)) = 0 \)) for \( f: (C, p_1, p_2, \ldots, p_n) \to \mathbb{P}^r \) a general marked BN-curve. This condition is visibly open, so it suffices to exhibit an unramified marked BN-curve \( f: (C, p_1, p_2, \ldots, p_n) \to \mathbb{P}^r \) of each degree \( d \) and genus \( g \) satisfying the conditions of our theorem for which \( H^1(N_f(-1)(-p_1 - \cdots - p_n)) = 0 \) (respectively for which \( H^1(N_f(-1)(-p_1 - \cdots - p_n)) = 0 \)). This is closely related to the property of interpolation for the normal bundle \( N_f \):
**Definition 1.3.** We say that a vector bundle \( E \rightarrow C \) on a curve \( C \) satisfies interpolation if, for a general effective divisor \( D \) of any degree,

\[
H^0(E(-D)) = 0 \quad \text{or} \quad H^1(E(-D)) = 0.
\]

Note that if \( E \rightarrow C \) satisfies interpolation, then \( H^1(E(-p_1 - \cdots - p_n)) = 0 \) for general points \( p_1, p_2, \ldots, p_n \in C \) if and only if \( n \leq \chi(E)/\text{rk}(E) \). The above argument therefore shows:

- If \( N_f \) satisfies interpolation for \( f \) a general BN-curve of degree \( d \) and genus \( g \), then \( f(C) \) can pass through \( n \) general points if and only if
  \[
  n \leq \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1}.
  \]

- If \( N_f(-1) \) satisfies interpolation for \( f \) a general BN-curve of degree \( d \) and genus \( g \), then \( f(C) \) has a general hyperplane section, and passes through \( n \) additional general points (independent of its hyperplane section) if and only if
  \[
  n \leq \frac{2d - (r - 3)(g - 1)}{r - 1}.
  \]

Using similar techniques to Theorem 1.2, we prove a theorem on interpolation for the twist of the normal bundle:

**Theorem 1.4.** If \( C \subset \mathbb{P}^r \) is a general BN-curve of degree \( d \) and genus \( g \), then \( N_C(-1) \) satisfies interpolation provided that

\[
(2r - 3)d - (r - 2)^2g - 2r^2 + 3r - 9 \geq 0.
\]

In particular, a general BN-curve of degree \( d \) and genus \( g \) in \( \mathbb{P}^r \) has general hyperplane section provided that

\[
(2r - 3)d - (r - 2)^2g - 2r^2 + 3r - 9 \geq 0.
\]

In fact, we show Theorem 1.4 can be leveraged to prove a generalization (and taking \( n = 0 \) a slight strengthening) of this consequence:

**Theorem 1.5.** The hyperplane section of a general BN-curve of degree \( d \) and genus \( g \) in \( \mathbb{P}^r \) contains \( d - n \) general points (with \( 0 \leq n \leq d \)) if

\[
(2r - 3)(d + 1) - (r - 2)^2(g - n) - 2r^2 + 3r - 9 \geq 0.
\]

More precise versions of Theorems 1.2, 1.4 and 1.5 are already known for \( r = 3 \) by work of Vogt [7], and for \( r = 4 \) by work of the author and Vogt [5]. We may therefore assume for simplicity that \( r \geq 5 \) for the remainder of the paper. (Although we note that with a bit more care, the techniques used here apply to lower values of \( r \) too; in particular, they cannot be used to prove a sharper version of Theorem 1.2 with the \(-3\) replaced by a \(-2\), as that would contradict the known counterexample with \( r = 3 \) mentioned above.)

The key idea to prove our main theorems is to degenerate \( f \) to a map \( f^\circ : C \cup D \rightarrow \mathbb{P}^r \) from a reducible curve, so that \( f^\circ|_C \) and \( f^\circ|_D \) are both nonspecial, and so that \( f^\circ|_D \) factors through a hyperplane \( H \). We then use a trick of [3] to reduce the desired statements to facts about the normal bundles of \( f^\circ|_C \) and \( f^\circ|_D \), which then follow from results of [1] on interpolation for nonspecial curves.
Note: Throughout this paper, we work over an algebraically closed field of characteristic zero.

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2 Degenerations for Theorem 1.2

The key input in our proof of Theorem 1.2 will be the following lemma from [3]:

Lemma 2.1 (Lemma 2.7 of [3]). Let $f : C \cup \Gamma D \rightarrow \mathbb{P}^r$ be an unramified map from a reducible curve, such that $f|_D$ factors as a composition of $f_D : D \rightarrow H$ with the inclusion of a hyperplane $\iota : H \subset \mathbb{P}^r$, while $f|_C$ is transverse to $H$ along $\Gamma$. Let $E$ and $F$ be divisors supported on $C \setminus \Gamma$ and $D \setminus \Gamma$ respectively. Suppose that, for some $i \in \{0,1\}$,

$$H^i(N_{f_D}(-\Gamma - F)) = H^i(\mathcal{O}_D(1)(\Gamma - F)) = H^i(N_{f|_C}(-E)) = 0.$$ 

Then we have

$$H^i(N_f(-E - F)) = 0.$$ 

We now consider a curve $f : C \cup \Gamma D \rightarrow \mathbb{P}^r$ of degree $d$ and genus $g$ (with $d < g + r$) of the above form; by the above lemma, it suffices to check

$$H^1(N_{f_D}(-\Gamma - F)) = H^1(\mathcal{O}_D(1)(\Gamma - F)) = H^1(N_{f|_C}(-E)) = 0$$

for some divisors $E$ and $F$ with

$$\deg E + \deg F = \left\lfloor \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} \right\rfloor - \frac{2r}{r - 1}.$$ 

Assume such an interior BN-curve $f$ exists with $\Gamma$ a set of $s + r$ general points in $H$, where $s = g + r - d$; and such that $f|_C$ and $f|_D$ are BN-curves. Suppose in addition that $D$ is of genus $(r - 2)t$ and $f|_D$ of degree $(r - 2)t + r - 1$, for some integer $t$. Note that this forces $C$ to be of genus $g + 1 - (r - 2)t - s - r = d - (r - 2)t - 2r + 1$, and $f|_C$ to be of degree $d - ((r - 2)t + r - 1) = d - (r - 2)t - r + 1$. In particular, $f|_C$ is nonspecial.

Since $f$ is an interior curve, and $\Gamma$ is a general set of points, we may deform $f$ to assume that $(f_D, \Gamma)$ is general in the component of $M_{(r-2)t,s+r}(H, (r - 2)t + r - 1)$ corresponding to BN-curves, and that $f|_C$ is general in the component of $M_{d-((r-2)t+r-1)}(\mathbb{P}^r, d - (r - 2)t - r + 1)$ corresponding to BN-curves.
By Corollary 1.4 of [1] for $f_D$, and for $f|_C$ (except when $r = 5$ and and $d - (r - 2)t - 2r + 1 = 2$), the above conditions reduce to:

$$\deg F + s + r \leq \frac{r \cdot ((r - 2)t + r - 1) - (r - 4)((r - 2)t - 1)}{r - 2} = r + 2 + 4t$$

$$\deg F - s - r \leq ((r - 2)t + r - 1) + 1 - (r - 2)t = r$$

$$\deg E \leq \frac{(r + 1)(d - (r - 2)t + r - 1) - (r - 3)(d - (r - 2)t + 2d)}{r - 1}$$

$$= \frac{4d - (4r - 8)t + r^2 - 6r + 1}{r - 1}.$$  

We will make the first inequality an equality by choosing $\deg F = 4t + 2 - s$; upon rearrangement, the second inequality becomes $2t \leq r + s + 1$. Finally, the third inequality becomes

$$\left[ \frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} - \frac{2r}{r - 1} \right] - (4t + 2 - s) \leq \frac{4d - (4r - 8)t + r^2 - 6r + 1}{r - 1};$$

this in turn follows from

$$\frac{(r + 1)d - (r - 3)(g - 1)}{r - 1} - \frac{2r}{r - 1} - (4t + 2 - s) \leq \frac{4d - (4r - 8)t + r^2 - 6r + 1}{r - 1},$$

or upon rearrangement, $2t \geq s - 1$. In other words, it suffices to find such a degeneration where

$$s - 1 \leq 2t \leq r + s + 1,$$

and in addition $d - (r - 2)t - 2r + 1 \neq 2$ if $r = 5$. To do this, we will first need the following lemma:

**Lemma 2.2.** Let $f : C \to \mathbb{P}^r$ be an unramified map from a curve with $H^1(N_f) = 0$. If $\Gamma \subset C$ is a set of $n \leq r + 2$ points with $f(\Gamma)$ in linear general position, then $H^1(N_f(-\Gamma)) = 0$.

**Proof.** The (long exact sequence in cohomology attached to the) short exact sequence of sheaves

$$0 \to N_f(-\Gamma) \to N_f \to N_f|_\Gamma \to 0$$

reduces our problem to showing $H^0(N_f) \to H^0(N_f|_\Gamma)$ is surjective. For this, we use the commutative diagram

$$
\begin{array}{ccc}
H^0(T_{\mathbb{P}^r}) & \longrightarrow & H^0(T_{\mathbb{P}^r}|_{f(\Gamma)}) \\
\downarrow & & \downarrow \\
H^0(N_f) & \longrightarrow & H^0(N_f|_\Gamma).
\end{array}
$$

The top horizontal map is surjective since $n \leq r + 2$ and $f(\Gamma)$ is in linear general position by assumption, and the right vertical map is always surjective. Consequently, the bottom horizontal map is surjective as desired.

With this out of the way, the construction of $f$ can be done in most cases by the following lemma:
Lemma 2.3. There exists a reducible interior BN-curve \( f: C \cup_{\Gamma} D \to \mathbb{P}^r \) of the above form, with \( t = \lfloor s/2 \rfloor \).

Proof. We argue by induction on \( d \), for a stronger hypothesis: That such a curve \( f \) exists which, in addition, satisfies:

- \( f(C) \) passes through 2 (if \( s \) is odd) or 1 (if \( s \) is even) points in \( \mathbb{P}^r \) that are general, independently from \( f(\Gamma) \);
- \( f(D) \) passes through a point in \( H \) that is general, independently from \( f(\Gamma) \) and the above general point in \( f(C) \), provided that \( s \) is even;
- and \( H^1(N_f) = 0 \).

First we consider the case \( p(d, g, r) = 0 \), which implies \( d = r(s+1) \) and \( g = (r+1)s \).

When \( s = 1 \), we take \( f|_C \) to be a general elliptic normal curve — which has a general hyperplane section, and passes through 2 additional independently general points in \( \mathbb{P}^r \) as required, by Lemma 6.1 of [2]. We let \( f_D \) be a rational normal curve in \( H \) passing through all points of intersection of \( f|_C(C) \) with \( H \). The union is a BN-curve by Theorem 1.7 of [2], which is an interior curve satisfying \( H^1(N_f) = 0 \) by combining Lemmas 3.2, 3.3, and 3.4 of [2].

For the inductive argument, we assume the given statement for \( s - 1 \) and seek to verify it for \( s \). Let \( f_0: C_0 \cup_{\Gamma_0} D_0 \to \mathbb{P}^r \) be such a curve of degree \( d_0 = rs \) and genus \( g_0 = (r+1)(s-1) \).

If \( s \) is even: We pick general subsets of 3 points \( \Delta_C \subset C_0 \) and of \( r-1 \) points \( \Delta_D \subset D_0 \). Write \( \Lambda_C \simeq \mathbb{P}^2 \) and \( \Lambda_D \simeq \mathbb{P}^{r-2} \) for the linear spans of \( f_0(\Delta_C) \) and \( f_0(\Delta_D) \) respectively. Note that a line passing through two points of \( f_0(\Delta_C) \) is general (independent of \( f_0(\Gamma_0) \)) by our inductive hypothesis; in particular, \( \Lambda_C \cap H \) contains a point which is general in \( H \) (independent of \( f_0(\Gamma_0) \)), and \( \Lambda_C \) contains an additional independently general point in \( \mathbb{P}^r \).

Since \( \Lambda_D \subset H \) is a general hyperplane section, \( p = \Lambda_C \cap \Lambda_D \) is a general point of \( \Lambda_C \cap H \), and is thus general in \( H \) (independent of \( f_0(\Gamma_0) \)); moreover if \( q_1 \in \Lambda_C \) and \( q_2 \in \Lambda_D \) are general, then \( q_1 \) and \( q_2 \) are general in \( \mathbb{P}^r \) and \( H \) respectively (independent of \( f_0(\Gamma_0) \) and \( p \)). Let \( C' \subset \Lambda_C \) be a rational normal curve (i.e. of degree \( \text{dim} \Lambda_C = 2 \)) through \( f_0(\Delta_C) \cup \{p, q_1\} \), and \( D' \subset \Lambda_D \) be a rational normal curve through \( f_0(\Delta_D) \cup \{p, q_2\} \). Then we will show that

\[
 f: (C_0 \cup_{\Delta_C} C') \cup_{\Gamma_0 \cup \{p\}} (D_0 \cup_{\Delta_D} D') \to \mathbb{P}^r
\]

gives the required curve.

Writing \( f \) as \( (C_0 \cup_{\Gamma_0} D_0) \cup_{\Delta_C \cup \Delta_D} (C' \cup_p D') \to \mathbb{P}^r \), iteratively applying Theorem 1.6 of [1] shows it is a BN-curve. In addition, applying Theorem 1.6 of [1] shows \( f|_{C_0 \cup_{\Delta_C} C'} \) is a BN-curve to \( \mathbb{P}^r \), and \( f|_{D_0 \cup_{\Delta_D} D'} \) is a BN-curve to \( H \), as desired.

Moreover, Lemmas 3.2, 3.3 and 3.4 of [1] imply \( H^1(N_f|_{C'\cup_p D'}) = 0 \). Applying Lemma 2.2 we conclude \( H^1(N_f|_{C'\cup_p D'}(-\Delta_C - \Delta_D)) = 0 \). Together with our inductive hypothesis, using Lemmas 3.3 and 3.4 of [1], this implies \( H^1(N_f) = 0 \) as desired.

Finally, we note that \( f(C_0 \cup_{\Delta_C} C') \) passes through 1 point in \( \mathbb{P}^r \) that is general independent from \( f(\Gamma_0 \cup \{p\}) \), namely \( q_1 \); and \( f(D_0 \cup_{\Delta_D} D') \) passes through 1 point in \( H \) that is general independent from \( f(\Gamma_0 \cup \{p\}) \cup \{q_1\} \), namely \( q_2 \).
If $s$ is odd: We pick a general subset $\Delta \subset C_0$ of $r+1$ points, a general point $q_1 \in f(C_0)$, a general point $p \in D_0$, and a general point $q_2 \in \mathbb{P}^r$. By our inductive hypothesis, $q_1 \in \mathbb{P}^r$ is a general independent from $f_0(\Gamma_0)$, and $f_0(p) \in H$ is general independent from $f_0(\Gamma_0)$.

Let $C'$ be a rational normal curve through $f(\Delta) \cup \{p,q_2\}$. Then we will show that $f: (C_0 \cup \Delta C') \cup \Gamma \cup \{p\} \to \mathbb{P}^r$ gives the required curve.

Writing $f$ as $(C_0 \cup \Gamma \cup \{p\}) \to \mathbb{P}^r$, applying Theorem 1.6 of [1] shows it is a BN-curve. In addition, applying Theorem 1.6 of [1] shows $f|_{C_0 \cup \Delta C'}$ is a BN-curve to $\mathbb{P}^r$, as desired.

Moreover, Lemma 3.2 of [1] implies $H^1(Nf|_{C'}(-\Delta - p)) = 0$. Together with our inductive hypothesis, using Lemmas 3.3 and 3.4 of [1], this implies $H^1(Nf) = 0$ as desired.

Finally, we note that $f(C_0 \cup \Delta C')$ passes through 2 points in $\mathbb{P}^r$ that are general independent from $f(\Gamma_0 \cup \{p\})$, namely $\{q_1,q_2\}$.

It remains to verify Theorem 1.2 in the cases when $r = 5$ and $d - (2r + 1) = 2$; or upon rearrangement $r = 5$ and $d = 3 \cdot \lfloor s/2 \rfloor + 11$. Note that

$$s = g + 5 - d = \frac{d - 5 - \rho(d,g,r)}{5} \leq \frac{d - 5}{5};$$

consequently

$$d \leq 3 \cdot \frac{d - 5}{10} + 11 \Rightarrow d \leq \frac{95}{7} < 14,$$

and so

$$s \leq \frac{d - 5}{5} < \frac{9}{5} < 2 \Rightarrow s = 1,$$

which gives $d = 11$ and thus $g = 7$.

It thus remains to consider the case $(d,g,r) = (11,7,5)$, for which Theorem 1.2 asserts such a curve can pass through 11 general points. But a curve of degree 10 and genus 6 can pass through 11 general points by work of Stevens [9], and the union of a curve of degree 10 and genus 6 with a 2-secant line gives a curve of the required degree and genus, which is a BN-curve by Theorem 1.6 of [2].

3 The Twist

We now turn to studying interpolation for the twist $N_C(-1)$. In greater generality, we make the following definition:

Definition 3.1. Let $d,g,r,n$ be nonnegative integers with $n \leq d$ and $\rho(d,g,r) \geq 0$; take $f: C \to \mathbb{P}^r$ to be a general BN-curve of degree $d$ and genus $g$.

We say $(d,g,r,n)$ is good if the general hyperplane section $f(C) \cap H$ contains $d-n$ general points in $H$.

We say $(d,g,r,n)$ is excellent if $N_f(-D)$ satisfies interpolation, where $D \subset C$ is a divisor of degree $d-n$ supported in a general hyperplane section.
The twist $N_f(-1)$ then satisfies interpolation if and only if $(d, g, r, 0)$ is excellent. Mirroring our previous argument, we will begin from knowledge that some range of degrees and genera are excellent:

**Proposition 3.2.** Let $d, g, r, n$ be nonnegative integers with $n \leq d$ and $\rho(d, g, r) \geq 0$. Then $(d, g, r, n)$ is excellent provided that:

\[
\begin{align*}
  d &\geq g + r, \\
  (d, g, r) &\notin \{(5, 2, 3), (6, 2, 4), (7, 2, 5)\}, \quad \text{and} \\
  2d + (r - 1)n &\geq (2r - 4)g - r + 3.
\end{align*}
\]

**Proof.** This follows from combining Theorem 1.3 and Proposition 4.12 of [1].

Our goal in this section is to show $N_f(-1)$ satisfies interpolation subject to the inequality

\[
(2r - 3)d - (r - 2)^2g - 2r^2 + 3r - 9 \geq 0. \tag{1}
\]

Since we are assuming $r \geq 5$, note that [1] implies $d \geq g + r$; in addition, (1) is not satisfied for $(d, g, r) = (7, 2, 5)$. In particular, if $2d \geq (2r - 4)g - r + 3$, then we are done. We therefore assume for the remainder of this section that

\[
2d \leq (2r - 4)g - r + 2. \tag{2}
\]

Note that, using (1), this implies

\[
g \geq 3 + \frac{5r + 12}{2r^2 - 6r + 4} \quad \Rightarrow \quad g \geq \begin{cases} 5 & \text{if } r \in \{5, 6\} \\ 4 & \text{otherwise.} \end{cases} \tag{3}
\]

Returning to the situation of Lemma 2.1, we suppose there exists such an interior BN-curve $f : C \cup_D \rightarrow \mathbb{P}^r$, with $f|_C$ and $f_D$ BN-curves too, so that $D$ is of genus $t \geq 1$, and is of degree $(r - 3)t + 1$, and $\Gamma$ to be a set of $t$ general points in $H$, and $F = O_D(1)(p)$ for $p \in D$ a general point.

As in the previous section, we may deform $f$ to assume that $(f_D, \Gamma)$ is general in the component of $M_{t,t}(H, (r - 3)t + 1)$ corresponding to BN-curves, and that $f|_C$ is general in the component of $M_{g-2t+1}(\mathbb{P}^r, d - (r - 3)t - 1)$ corresponding to BN-curves.

Note with these choices that

\[
O_D(1)(\Gamma - F) = O_D(\Gamma)(-p) \quad \text{and} \quad N_{f_D}(-\Gamma - F) = N_{f_D}(-1)(-\Gamma - p)
\]

both have Euler characteristic zero. Moreover, $N_{f_d}(-1)$ satisfies interpolation by the above, provided that

\[
(r - 3)t + 1 \geq t + r - 1 \\
((r - 3)t + 1, t, r - 1) \notin \{(5, 2, 3), (6, 2, 4), (7, 2, 5)\} \\
2((r - 3)t + 1) \geq (2r - 6)t - r + 4.
\]
The last of these conditions is immediate for \( r \geq 5 \), while the first two follow from

\[
\begin{align*}
  t &\geq 2 \\
  (r, t) &\notin \{(5, 2), (6, 2)\}. 
\end{align*}
\]

(4)

When these inequalities are satisfied, we conclude

\[
H^0(O_D(1)(\Gamma - F)) = H^1(O_D(1)(\Gamma - F)) = H^0(N_{f_D}(-\Gamma - F)) = H^1(N_{f_D}(-\Gamma - F)) = 0.
\]

In particular, applying Lemma 3.3, we see that \( N_f(-1) \) satisfies interpolation provided that \( N_{f|c}(-1) \) does, which in turn (by the above) follows from:

\[
\begin{align*}
  g - 2t + 1 &\geq 0 \\
  d - (r - 3)t - 1 &\geq g - 2t + 1 + r, \\
  (d - (r - 3)t - 1, g - 2t + 1, r) &\notin \{(5, 2, 3), (6, 2, 4), (7, 2, 5)\}, \quad \text{and} \\
  2(d - (r - 3)t - 1) &\geq (2r - 4)(g - 2t + 1) - r + 3;
\end{align*}
\]

or upon rearrangement (again using \( r \geq 5 \)):

\[
\begin{align*}
  g &\geq 2t - 1 \\
  d &\geq g + r + 2 + 3r, \\
  (d - (r - 3)t, g - 2t, r) &\notin (8, 1, 5), \quad \text{and} \\
  2d - (2r - 4)g + (2r - 2)t &\geq r + 1.
\end{align*}
\]

(5)

**Lemma 3.3.** If \( d \geq (r - 2)t + 1 \) in addition to the inequalities (4) and (5), there exists a reducible interior BN-curve \( f : C \cupr D \to \mathbb{P}^r \) of the above form.

**Proof.** Note first that (4) implies there is a nonspecial curve \( f_D : D \to H \) of degree \((r - 3)t + 1\) and genus \( t \), for which \( N_{f_D} \) satisfies interpolation; as \( r[(r-3)t+1]-(t-4)(t-1) \geq (t-2)(t+2) \), the general such curve passes through at least \( t + 2 \) general points.

We argue by induction on \( t \) for a stronger hypothesis: That such a curve \( f \) exists which, in addition, satisfies \( H^1(N_f) = 0 \), and for which \( f|_D \) passes through \( 2 \) additional points in \( H \) which are general independent of \( \Gamma \).

If \( t \leq r + 2 \), we let \( f_C : C \to \mathbb{P}^r \) be a general BN-curve of degree \( d = (r - 3)t - 1 \) and genus \( g = 2t + 1 \); by (5), and then our assumption that \( d \geq (r - 2)t + 1 \), we see that \( f_C(C) \cap H \) consists of \( d - (r - 3)t - 1 \geq t \) general points. By the first paragraph of this proof, there is a nonspecial curve \( f_D : D \to H \) of degree \( (r - 3)t + 1 \) and genus \( t \), passing through a subset \( \Gamma \) of \( t \) points of \( f_C(C) \cap H \), and passing through \( 2 \) additional general points in \( H \). Gluing \( f_C \) to \( f_D \) along \( \Gamma \), there exists such a curve \( f : C \cupr D \to \mathbb{P}^r \), with \( f|_C \) and \( f_D \) general BN-curves, meeting at a general set of \( t \) points \( \Gamma \subset H \). The curve \( f \) is a BN-curve by Theorem 1.9 of \( [2] \), and is an interior curve with \( H^1(N_f) = 0 \) by combining Lemmas 3.2, 3.3, and 3.4 of \( [2] \); by construction, \( f|_D \) passes through \( 2 \) additional points in \( H \) which are general independent of \( \Gamma \).

For the inductive step, we suppose \( t \geq r + 3 \), and let \((d_0, g_0, t_0) = (d - r + 3, g - 2, t - 1)\); note that (4) is satisfied for \((d_0, g_0, t_0)\) by our assumption that \( t \geq r + 3 \), and that (5) and \( d_0 \geq (r - 2)t_0 + 1 \) are immediate. We may therefore let \( f_0 : C_0 \cupr D_0 \to \mathbb{P}^r \) be such a reducible BN-curve of degree \( d_0 \) and genus \( g_0 \) with \( \# \Gamma_0 = t_0 \), which satisfies \( H^1(N_{f_0}) = 0 \), and such that
\( f_0(D_0) \) passes through two additional points that are general independent of \( \Gamma_0 \). Let \( \{q_1, q_2\} \) and \( \{q'_1, q'_2\} \) be two distinct such sets of points, so both are general independent of \( \Gamma_0 \) (although of course not necessarily independent from \( \Gamma_0 \) and eachother).

Note that \( \deg f_0|_{C_0} = (d - r + 3) - (r - 3)(t - 1) - 1 \geq t \) by assumption; we may therefore pick a point \( p \in (f_0(C_0) \cap H) \setminus \Gamma_0 \). By (5), the hyperplane section \( f_0(C_0) \cap H \) is general; thus \( p \) is general, independent of \( \Gamma_0 \), and thus independent of \( \Gamma_0 \cup \{q_1, q_2\} \). Pick a linear subspace \( \Lambda \simeq \mathbb{P}^{r-3} \subset H \) passing through \( \{p, q_1, q_2\} \), and let \( D' \subset \Lambda \) be a rational normal curve through \( \{p, q_1, q_2\} \). We then claim

\[
\begin{align*}
\text{Let } f_0 &: C_0 \cup \Gamma_0 \cup \{p\} (D_0 \cup \{q_1, q_2\} D') \to \mathbb{P}^r
\end{align*}
\]

gives the required curve.

Writing \( f \) as \( (C_0 \cup \Gamma_0 D_0) \cup \{p, q_1, q_2\} D' \to \mathbb{P}^r \), applying Theorem 1.6 of [2] shows it is a BN-curve.

Moreover, Lemmas 3.2 of [2] implies \( H^1(N_{f|_D}(-p - q_1 - q_2)) = 0 \). Together with our inductive hypothesis, using Lemmas 3.3 and 3.4 of [2], this implies \( H^1(N_f) = 0 \) as desired.

Finally, we note that \( f(D_0 \cup \{q_1, q_2\} D') \) passes through two general points in \( H \), indepdent of \( \Gamma_0 \cup \{p\} \), namely \( \{q'_1, q'_2\} \).

In conclusion, it suffices to show that, subject to \( r \geq 5 \) and (1) and (2), there exists an integer \( t \) with:

\[
\begin{align*}
d &\geq (r - 2)t + 1 \\
g &\geq 2t - 1 \\
t &\geq 2 \\
(r, t) &\notin \{(5, 2), (6, 2)\} \\
d &\geq g + r + 2 + (r - 5)t, \\
(d - (r - 3)t, g - 2t, r) &\neq (8, 1, 5), \text{ and} \\
2d - (2r - 4)g + (2r - 2)t &\geq r + 1.
\end{align*}
\]

We shall take \( t \in \{s, s + 1\} \) where

\[
s = \left\lceil \frac{(2r - 4)g - 2d + r + 1}{2r - 2} \right\rceil.
\]

By construction,

\[
t \geq s \geq \frac{(2r - 4)g - 2d + r + 1}{2r - 2},
\]

which implies the final inequality. Moreover, rearranging (2), we obtain

\[
\frac{(2r - 4)g - 2d + r + 1}{2r - 2} \geq \frac{2r - 1}{2r - 2} > 1 \quad \Rightarrow \quad s \geq 2,
\]

which, since \( t \geq s \), implies the third inequality. In addition,

\[
s \leq \frac{(2r - 4)g - 2d + r + 1}{2r - 2} + 1 - \frac{1}{2r - 2}.
\]
which together with (1) implies the first inequality when \( t = s \) for \( r \) arbitrary, and when \( t = s + 1 \) for \( r \in \{5, 6\} \).

For the second and fifth inequalities, the obvious upper bound (7) will not suffice; instead we rearrange (1) to produce

\[
\frac{(2r - 4)g - 2d + r + 1}{2r - 2} \leq \frac{g - 1}{2} - \frac{(r - 1)g + 18}{4r^2 - 10r + 6},
\]

which in turn implies \( s \leq g/2 \). If \( r \in \{5, 6\} \), then (3) gives \( g \geq 5 \). Note that when \( g = 5 \), our bound \( s \leq 5/2 \) immediately gives \( s \leq 4/2 = 2 \). Moreover, if \( r \in \{5, 6\} \) and \( g \geq 6 \), then \( \frac{(r-1)g+18}{4r^2-10r+6} > \frac{1}{2} \). We conclude that

\[
s \leq \begin{cases} \frac{g-1}{2} & \text{if } r \in \{5, 6\}; \\ \frac{g}{2} & \text{otherwise.} \end{cases}
\]

(8)

This bound implies the second inequality when \( t = s \) for \( r \) arbitrary, and when \( t = s + 1 \) for \( r \in \{5, 6\} \).

Moreover, when \( g \geq 8 \), it also implies the fifth inequality when \( t = s \) for \( r \) arbitrary, and when \( t = s + 1 \) for \( r \in \{5, 6\} \); in light of (3), it thus remains to verify the fifth inequality for \( g \in \{4, 5, 6, 7\} \). In these cases, (1) becomes

\[
d \geq \begin{cases} \frac{3r - 5 + \frac{10}{2r-3}}{7r-13} + \frac{r+19}{4r-6} & \text{if } g = 4; \\ \frac{4r-8 + \frac{r+9}{2r-3}}{3r-18} + \frac{r+20}{4r-6} & \text{if } g = 5; \\ \frac{9r-18}{2} + \frac{r+20}{4r-6} & \text{if } g = 6. \end{cases}
\]

\[
\Rightarrow \quad d \geq \begin{cases} \frac{3r-4}{2} & \text{if } g = 4; \\ \frac{7r-12}{2} & \text{if } g = 5; \\ 4r-7 & \text{if } g = 6; \\ \frac{9r-17}{2} & \text{if } g = 7. \end{cases}
\]

And (8) becomes

\[
s \leq \begin{cases} 2 & \text{if } g \in \{4, 5\}; \\ 3 & \text{if } g \in \{6, 7\}. \end{cases}
\]

(9)

This implies the fifth inequality when \( t = s \) for \( r \) arbitrary, and when \( t = s + 1 \) for \( r \in \{5, 6\} \), except for the cases \((r, g) \in \{(5, 5), (6, 4), (6, 5), (6, 6), (6, 7)\}\). In those cases, (1) becomes

\[
d \geq \begin{cases} \frac{89}{7} & \text{if } (r, g) = (5, 5); \\ \frac{127}{9} & \text{if } (r, g) = (6, 4); \\ \frac{143}{9} & \text{if } (r, g) = (6, 5); \\ \frac{53}{7} & \text{if } (r, g) = (6, 6); \\ \frac{175}{9} & \text{if } (r, g) = (6, 7). \end{cases}
\]

\[
\Rightarrow \quad d \geq \begin{cases} 13 & \text{if } (r, g) = (5, 5); \\ 15 & \text{if } (r, g) = (6, 4); \\ 16 & \text{if } (r, g) = (6, 5); \\ 18 & \text{if } (r, g) = (6, 6); \\ 20 & \text{if } (r, g) = (6, 7). \end{cases}
\]

Together with (9) this implies the fifth inequality in these cases, with \( t = s \) or \( t = s + 1 \).

If the sixth inequality does not hold, then \((d, g, r) = (2t + 8, 2t + 1, 5)\); in this case, (1) becomes upon rearrangement \( t \leq \frac{3}{4} \), contradicting the third inequality (which we have already verified). We can therefore satisfy all inequalities (including the fourth) by taking

\[
t = \begin{cases} s + 1 & \text{if } (r, s) \in \{(5, 2), (6, 2)\}; \\ s & \text{otherwise.} \end{cases}
\]
4 General Points in a Hyperplane Section

In this section, we investigate the number of general points contained in the hyperplane section of a general BN-curve. For the remainder of this section, we let \((d, g, r, n)\) denote nonnegative integers with \(\rho(d, g, r) \geq 0\) and \(n \leq d\) and \(r \geq 5\); our goal is to prove Theorem 1.5 which asserts that \((d, g, r, n)\) is good (c.f. Definition 3.1) provided that

\[(2r - 3)(d + 1) - (r - 2)^2(g - n) - 2r^2 + 3r - 9 \geq 0.\]

Our argument will be via induction, using the results of the preceding section as a base case. The various inductive arguments we shall use are as follows:

**Lemma 4.1.** Let \((d, g, r, n)\) be nonnegative integers with \(\rho(d, g, r) \geq 0\) and \(n \leq d - 4\). Suppose that \(g \geq 2r\) with strict inequality when \(r = 5\), and that \(n \geq 2r - 6\). Then \((d, g, r, n)\) is good if \((d - 2r + 2, g', r, n - 2r + 6)\) is good, where

\[g' = \begin{cases} g - 2r & \text{if } r \geq 6; \\ g - 11 & \text{if } r = 5. \end{cases}\]

**Proof.** Note that our assumptions imply \(g' \geq 0\), and that \(n' := n - 2r + 6\) and \(d' := d - 2r + 2\) satisfy \(0 \leq n' \leq d'\). Moreover,

\[\rho(d', g', r) \geq (r + 1)(d - 2r + 2) - r(g - 2r) - r(r + 1) = (r + 1)d - rg - r(r + 1) + 2 \geq 0.\]

In particular, \((d', g', r, n')\) are nonnegative integers with \(0 \leq n' \leq d'\) and \(\rho(d', g', r) \geq 0\). By assumption, \((d', g', r, n')\) is good.

So let \(f_1 : C \to \mathbb{P}^r\) be a general BN-curve of degree \(d'\) and genus \(g'\), whose hyperplane section \(f_1(C) \cap H\) contains a set \(S\) of \(d' - n' = d - n - 4\) general points. Pick a set \(T\) of 4 independantly general points in \(H\), and let \(H'\) be a general hyperplane containing \(T\) (in particular \(H'\) is independently general from \(C\), and from \(H'\) since \(r \geq 5\)).

Since \(\rho(d, g, r) \geq 0\), we have

\[d \geq \frac{rg + r(r + 1)}{r + 1} \geq \frac{r \cdot 2r + r(r + 1)}{r + 1} > 3r - 2 \implies d \geq 3r - 1 \implies d' \geq r + 1.\]

Similarly, when \(r = 5\), we have

\[d \geq \frac{5g + 30}{6} \geq \frac{5 \cdot 11 + 30}{6} > 14 \implies d \geq 15 \implies d' \geq 7.\]

Putting these together, we conclude \(d' \geq c\), where we define

\[c := \begin{cases} r + 1 & \text{if } r > 5; \\ 7 & \text{if } r = 5. \end{cases}\]

By Lemma 6.1 of [2], the hyperplane section \(f_1(C) \cap H'\) contains a set \(\Gamma\) of \(c\) general points. By [6], there is a canonical curve \(f_2 : D \to H'\) (of genus \(r\)) passing through \(\Gamma\). We may then construct \((f_1 \cup f_2) : C \cup T \to \mathbb{P}^r\), which is a BN-curve, by Theorem 1.9 of [2]: and is of degree \(d\) and genus \(g\) passing through the set \(S \cup T \subset H\) of \(d - n\) general points as desired. \(\square\)
Lemma 4.2. Write
\[ a = \left\lfloor \frac{r - 2}{2} \right\rfloor, \]
and suppose \( g \geq a + 1 \) and \( n \geq a \). Then \((d, g, r, n)\) is good if \((d - a, g - a - 1, r, n - a)\) is good.

Proof. Note that our assumptions imply \( g' := g - a - 1 \geq 0 \), and that \( n' := n - a \) and \( d' := d - a \) satisfy \( 0 \leq n' \leq d' \). Moreover,
\[
\rho(d', g', r) = (r + 1)(d - a) - r(g - a - 1) - r(r + 1) = (r + 1)d - rg - r(r + 1) + r - a \geq r - a \geq 0.
\]
In particular, \((d', g', r, n')\) are nonnegative integers with \( 0 \leq n' \leq d' \) and \( \rho(d', g', r) \geq 0 \). By assumption, \((d', g', r, n')\) is good.

So let \( f : C \to \mathbb{P}^r \) be a general BN-curve of degree \( d' \) and genus \( g' \), whose hyperplane section \( f(C) \cap H \) contains \( d' - n' = d - n \) general points.

By Theorem 1.8 of [2], there exists a BN-curve \( \hat{f} : C' \cup H \to \mathbb{P}^r \) with \( \# \Gamma = a + 2 \) and \( \hat{f}|_{\mathbb{P}^1} \) of degree \( a \), such that \( f|_C = f \). In particular, \( \hat{f} \) is a BN-curve of degree \( d' + a = d \) and genus \( g' + a + 1 = g \) whose hyperplane section contains the hyperplane section of \( f \), and thus contains \( d - n \) general points as desired. \( \square \)

Lemma 4.3. Suppose that \( g \geq 1 \), and \( n \geq 1 \), and \( \rho(d, g, r) \geq 1 \). Then \((d, g, r, n)\) is good if \((d - 1, g - 1, r, n - 1)\) is good.

Proof. Note that our assumptions imply \( g' := g - 1 \geq 0 \), and that \( n' := n - 1 \) and \( d' := d - 1 \) satisfy \( 0 \leq n' \leq d' \). Moreover,
\[
\rho(d', g', r) = (r + 1)(d - 1) - r(g - 1) - r(r + 1) = (r + 1)d - rg - r(r + 1) - 1 \geq 0.
\]
In particular, \((d', g', r, n')\) are nonnegative integers with \( 0 \leq n' \leq d' \) and \( \rho(d', g', r) \geq 0 \). By assumption, \((d', g', r, n')\) is good.

So let \( f : C \to \mathbb{P}^r \) be a general BN-curve of degree \( d' \) and genus \( g' \), whose hyperplane section \( f(C) \cap H \) contains \( d' - n' = d - n \) general points.

Pick \( \{p, q\} \subset C \) general. By Theorem 1.6 of [2], the curve \( \hat{f} : C' \cup \{p, q\} \mathbb{P}^1 \to \mathbb{P}^r \), where \( \hat{f}|_{\mathbb{P}^1} \) is a line, is a BN-curve. It is evidently of degree \( d' + 1 = d \) and genus \( g' + 1 = g \), and its hyperplane section contains the hyperplane section of \( f \), and thus contains \( d - n \) general points as desired. \( \square \)

Lemma 4.4. If \((d, g, r, n)\) is good, then so is \((d + 1, g, r, n)\).

Proof. Let \( f : C \to \mathbb{P}^r \) be a general BN-curve of degree \( d \) and genus \( g \), whose hyperplane section \( f(C) \cap H \) contains \( d - n \) general points.

Pick a general point \( p \in C \). By Theorem 1.6 of [2], the curve \( \hat{f} : C' \cup \{p\} \mathbb{P}^1 \to \mathbb{P}^r \), where \( \hat{f}|_{\mathbb{P}^1} \) is a line, is a BN-curve. It is evidently of degree \( d + 1 \) and genus \( g \), and its hyperplane section contains the hyperplane section of \( f \), plus the independently general point \( \hat{f}(\mathbb{P}^1) \cap H \), and thus contains \( d + 1 - n \) general points as desired. \( \square \)

Lemma 4.5. Suppose that, for some integer \( b \geq 0 \), we have \( b \leq d - n \), and \( b \leq g \), and \( \rho(d, g, r) \geq b \), and that
\[
2d + (r - 1)n - (r - 3)g - 4b - 2 \geq 0.
\]
Then \((d, g, r, n)\) is good if \((d - b, g - b, r, n)\) is excellent.
Proof. If \( b = 0 \), the result is obvious; we thus suppose \( b \geq 1 \). Note that our assumptions imply \( g' := g - b \geq 0 \), and that \( n \) and \( d' := d - b \) satisfy \( 0 \leq n \leq d' \). Moreover,

\[
\rho(d', g', r) = (r + 1)(d - 1) - r(g - 1) - r(r + 1) = (r + 1)d - rg - r(r + 1) - 1 \geq 0.
\]

In particular, \((d', g', r, n)\) are nonnegative integers with \( 0 \leq n \leq d' \) and \( \rho(d', g', r) \geq 0 \). By assumption, \((d', g', r, n)\) is excellent.

So let \( f : C \to \mathbb{P}^r \) be a general BN-curve of degree \( d' \) and genus \( g' \), whose hyperplane section contains a set \( D \) of \( d' - n' = d - n - b \) general points. By assumption,

\[
\chi(N_f(-D)) = (r + 1)(d - b) - (r - 3)(g - b - 1) - (r - 1)(d - b - n) \geq (r - 1)(b + 1),
\]

and so \( f \) passes through a set \( S \) of \( b + 1 \) points that are general, independent of \( D \).

Let \( C' \) be a rational curve of degree \( b \) through \( S \). Then \( \hat{f} : C \cup_S C' \to \mathbb{P}^r \) is a BN-curve by Theorem 1.6 of [2]. It is evidently of degree \( d' + b = d \) and genus \( g' + b = g \), and its hyperplane section is the union of the hyperplane sections of \( \hat{f}|_C \) and \( \hat{f}|_{C'} \), which contain independently general sets of \( d - b - n \) and \( b \) points by construction, for \( d - n \) general points in total. \( \Box \)

Lemma 4.6. Suppose that nonnegative integers \( d_1, g_1, d_2, g_2, n_1, n_2, k \) with \( \rho(d_i, g_i, r) \geq 0 \) and \( n_i \leq d_i \) for \( i \in \{1, 2\} \), and with \( k \geq 1 \), satisfy

\[
(r + 1)d_i - rg_i + r \geq rk, \\
2d_2 - (r - 3)(g_2 - 1) \geq (r - 1)(k - n).
\]

Then \((d_1 + d_2, g_1 + g_2 + k - 1, r, n_1 + n_2)\) is good if \((d_1, g_1, r, n_1)\) is good and \((d_2, g_2, r, n_2)\) is excellent.

Proof. Since \((d_1, g_1, r, n_1)\) is good by assumption, we may let \( f_1 : C_1 \to \mathbb{P}^r \) be a general BN-curve of degree \( d_1 \) and genus \( g_1 \), whose hyperplane section contains a set \( D_1 \) of \( d_1 - n_1 \) general points. Since \((r + 1)d_1 - rg_1 + r \geq rk\), Corollary 1.3 of [3] implies that \( f_1 \) passes through a set \( S \) of \( k \) general points.

Since by assumption \((d_2, g_2, r, n_2)\) is excellent and \( 2d_2 - (r - 3)(g_2 - 1) \geq (r - 1)(k - n) \), we may let \( f_2 : C_2 \to \mathbb{P}^r \) be a general BN-curve of degree \( d_2 \) and genus \( g_2 \), passing through \( S \) and an independantly general set \( D_2 \) of \( d_2 - n_2 \) general points.

By Theorem 1.6 of [2], the curve \( C_1 \cup_S C_2 \to \mathbb{P}^r \) is a BN-curve; by inspection, its hyperplane section contains a set \( D_1 \cup D_2 \) of \( d_1 + d_2 - n_1 - n_2 \) general points. \( \Box \)

If \((d, g, r, n)\) satisfies the assumptions of Theorem 1.5 and of Lemma 4.1 then we note that

\[
(2r - 3)((d - 2r + 2) + 1) - (r - 2)^2(g' - (n - 2r + 6)) - 2r^2 + 3r - 9
\]

\[
= (2r - 3)(d + 1) - (r - 2)^2(g - n) - 2r^2 + 3r - 9 + \begin{cases}
2r^2 - 14r + 18 & \text{if } r \geq 6; \\
7 & \text{if } r = 5.
\end{cases}
\]

In particular, it suffices by induction to show \((d - 2r + 2, g', r, n - 2r + 6)\) is good. Consequently, it suffices to verify Theorem 1.5 in the cases \( g \leq 2r \) (with strict inequality for \( r \geq 6 \)), and \( n \leq 2r - 7 \).
In these cases, we apply Lemma 4.2 \(x\) times where

\[ x = \min \left( \left\lfloor \frac{g}{a+1} \right\rfloor, \left\lfloor \frac{n}{a} \right\rfloor \right), \]

followed by Lemma 4.3 \(y\) times where

\[ y = \min(g - (a + 1)x, n - ax), \]

followed by Lemma 4.5 with \(b = z\) where

\[ z = \min(g - (a + 1)x - y, d - n, 10). \]

This can be done so long as

\[ \rho(d - ax, g - (a + 1)x, r) \geq y + z \]

\[ 2(d - ax - y) + (r - 1)(n - ax - y) - (r - 3)(g - (a + 1)x - y) - 4z - 2 \geq 0; \]

if these inequalities hold, then we are reduced to showing that

\((d', g', r, n') := (d - ax - y - z, g - (a + 1)x - y - z, r, n - ax - y)\)

is excellent.

By definition of \(y\), either \(n' = n - ax - y = 0\), or \(g - (a + 1)x - y = 0\); in the second case, by definition of \(z\), we also have \(z = 0\) and so \(g' = g - (a + 1)x - y - z = 0\).

Next, by definition of \(z\), either \(z = d - n\) (which gives \(d' = n'\)), or \(z = 10\), or \(z = g - (a + 1)x - y\) (which gives \(g' = 0\)). Since \((d', g', r, n')\) is automatically excellent when \(g' = 0\) by Proposition 3.2, and we cannot have \(d' = n'\) if \(n' = 0\), it remains to consider the case where \(n' = 0\) and \(z = 10\) but \(g' \neq 0\). Note that this forces \(x = \lfloor n/a \rfloor\). In this case, we can invoke Theorem 1.4 provided that

\[(2r - 3)(d - ax - y - z) - (r - 2)^2(g - (a + 1)x - y - z) - 2r^2 + 3r - 9 \geq 0.\]

Combining all these inequalities and substituting \(y = n - ax\) and \(z = 10\), we are reduced to showing

\[ rx + (r + 1)d - rg - n - r^2 - r - 10 \geq 0 \]

\[ (r - 3)x + 2d - (r - 3)g + (r - 5)n - 42 \geq 0 \]

\[ (2r - 3)d + (r - 2)^2(x - g) + (r^2 - 6r + 7)n + 8r^2 - 57r + 61 \geq 0. \]

Using the hypothesis of Theorem 1.5 to bound \(d\) from below, and recalling that \(g' = g - n - x - 10\), these inequalities reduce to

\[ \frac{r^3 - 5r^2 + 3r + 4}{r + 1}g' + 10r^2 - 62r + 82 \geq (2r - 3)n - (r - 2)^2x \]

\[ \frac{r - 1}{2}g' + 2r^2 - 42r + 70 \geq (2r - 3)n - (r - 2)^2x \]

\[ 10r^2 - 62r + 73 \geq (2r - 3)n - (r - 2)^2x \]
Note that \((r - 2)^2 / (2r - 3) \leq (r - 2) / 2 \leq a\), and that \(n \leq 2r - 7 < 4 \cdot (r - 2) / 2 \leq 4a\). Since \(x = \lfloor n / a \rfloor\), this implies

\[
(2r - 3)n - (r - 2)^2 x \leq (2r - 3) \cdot (4a - 1) - (r - 2)^2 \cdot 3 \leq (2r - 3) \cdot (2r - 3) - (r - 2)^2 \cdot 3 = r^2 - 3.
\]

Since \(g' \geq 1\), this implies the first inequality for \(r \geq 5\), the middle inequality for \(r \geq 40\), and the last inequality for \(r \geq 6\). When \(r = 5\), we have \(a = 2\) and \(n \leq 3\); for each of these four values of \(n\), we easily verify the final inequality. We are thus done unless \(r \leq 39\) and

\[
\frac{r - 1}{2} g' + 2r^2 - 42r + 70 < (2r - 3)n - (r - 2)^2 x;
\]

or, upon rearrangement (using \(g' = g - n - x - 10\)):

\[
(r - 1)g < (5r - 7)n - (2r^2 - 9r + 9)x - 4r^2 + 94r - 150,
\]

or equivalently

\[
g \leq \frac{(5r - 7)n - (2r^2 - 9r + 9)x - 4r^2 + 94r - 151}{r - 1}.
\]

In other words, to verify Theorem 1.5, it remains to check the finitely many values of \((g, r, n)\) with \(5 \leq r \leq 39\) and \(n \leq 2r - 7\) and \(g\) bounded as above. And for each such triple, by Lemma 4.4 we just have to check the minimal value of \(d\) satisfying the hypotheses of Theorem 1.5; all that remains in the proof of Theorem 1.5 is thus a finite computation, which is done in Appendix A.

### A Code for Theorem 1.5

In this section, we give python code to do the finite computations described in Section 4; running this code produces no output, thus verifying Theorem 1.5 in these cases.

```python
def excellent(d, g, r, n):
    if (2 * r - 3) * d - (r - 2)**2 * g - 2 * r**2 + 3 * r - 9 >= 0:
        return True
    return ((d >= g + r) and ((d, g, r) not in ((5, 2, 3), (6, 2, 4), (7, 2, 5))) and (2 * d - (r - 3) * (g - 1) >= (r - 1) * (g - n))

GOOD = {}
def good(d, g, r, n):
    dgrn = (d, g, r, n)
    if GOOD.has_key(dgrn):
        return GOOD[dgrn]
    if excellent(d, g, r, n):
        GOOD[dgrn] = True
        return True
    a = (r - 1) / 2
    if (g >= a + 1) and (n >= a):
```

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if good(d - a, g - a - 1, r, n - a):
    GOOD[dgrn] = True
    return True

if (g >= 1) and (n >= 1) and ((r + 1) * d - r * g - r * (r + 1) >= 1):
    if good(d - 1, g - 1, r, n - 1):
        GOOD[dgrn] = True
        return True

for i in xrange(1, min(d - n, g, (r + 1) * d - r * g - r * (r + 1)) + 1):
    if 2 * d + (r - 1) * n - (r - 3) * g - 4 * i - 2 >= 0:
        if excellent(d - i, g - i, r, n):
            GOOD[dgrn] = True
            return True

for g1 in xrange(g + 1):
    for g2 in xrange(g + 1 - g1):
        k = g + 1 - g1 - g2

        for n1 in xrange(n):
            n2 = n - n1
            d1min = max(n1, r + (r * g1 + r) / (r + 1))
            d2min = max(n2, r + (r * g2 + r) / (r + 1))

            for d1 in xrange(d1min, d + 1 - d2min):
                d2 = d - d1

                if (r + 1) * d1 - r * g1 + r >= r * k:
                    if 2 * d2 - (r - 3) * (g2 - 1) >= (r - 1) * (k - n):
                        if good(d1, g1, r, n1) and excellent(d2, g2, r, n2):
                            GOOD[dgrn] = True
                            return True

GOOD[dgrn] = False
return False

for r in xrange(5, 40):
    a = (r - 1) / 2
    nmax = 2 * r - 7
    for n in xrange(nmax + 1):
        x = n / a
        gmax = ((5 * r - 7) * n - (2 * r**2 - 9 * r + 9) * x - 4 * r**2 + 94 * r - 151) / (r - 1)

        for g in xrange(gmax + 1):
            dmin1 = r + (r * g + r) / (r + 1)
            dmin2 = ((r - 2)**2 * (g - n) + 2 * r**2 - 3 * r + 9 + (2 * r - 4)) / (2*r - 3) - 1
            d = max(dmin1, dmin2, n)

            if not good(d, g, r, n):
                print d, g, r, n
References


