HMMT November 2019 Integration Bee Finals

Sponsored by Five Rings Capital

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Problem 1

Evaluate the following Integral:

$$\int \sqrt{\sec x} \tan x \, dx$$

Solution 1

Answer:

$$2\sqrt{\sec x} + C$$

Problem 2

Evaluate the following Integral:

$$\int_0^{\pi/2} \sin^{-1} \cos x \, dx$$

Solution 2

Answer:

$$\frac{\pi^2}{8}$$

Problem 3

Evaluate the following Integral:

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx$$

Solution 3

Answer: 0

Problem 4

Evaluate the following Integral:

$$\int e^{\sqrt{x}}\, dx$$

Solution 4

Answer:

$$2e^{\sqrt{x}}(\sqrt{x}-1)+C$$

Problem 5

Evaluate the following Integral:

$$\int_{-\infty}^{\infty} \operatorname{sech}(x)\, dx$$

Solution 5

Answer:

$$\pi$$

Problem 6

Evaluate the following Integral:

$$\int \frac{1}{x^6 - x}\, dx$$

Solution 6

Answer:

$$\frac{1}{5}\log\left(\frac{1}{x^5}-1\right)+C$$

Problem 7

Evaluate the following Integral:

$$\int_0^{10} \lceil x \rceil \lfloor x \rfloor \, dx$$

$\lfloor x \rfloor$ rounds x down to the nearest integer, while $\lceil x \rceil$ rounds x up to the nearest integer.

Solution 7

Answer:

330

Problem 8

Evaluate the following Integral:

$$\int_0^{2\pi} \left(\frac{\sin(3x)}{\sin(x)} \right)^3 dx$$

Solution 8

Answer:

$$14\pi$$

Problem 9

Evaluate the following Integral:

$$\int_0^\infty \frac{1}{1+x+x^2+x^3}\,dx$$

Solution 9

Answer: $\frac{\pi}{4}$

Problem 10

Evaluate the following Integral:

$$\int_{-1}^{1} \frac{\tan x}{x^4 + 2x^2 + 1}\, dx$$

Solution 10

Answer: 0

Problem 11

Evaluate the following Integral:

$$\int \sinh^2 x \, dx$$

Solution 11

Answer:

$$\frac{1}{4}\sinh(2x) - \frac{x}{2} + C$$

Problem 12

Evaluate the following Integral:

$$\int_0^{2\pi} \frac{\cos x}{\sin x} \sin(2^{2019} x)\, dx$$

Solution 12

Answer:

$$2\pi$$

Problem 13

Evaluate the following Integral:

$$\int \log(1+x^2)\,dx$$

Solution 13

Answer:

$$x \log(1 + x^2) - 2x + 2 \tan^{-1}(x) + C$$

Problem 14

Evaluate the following Integral:

$$\int \sec(x)\cosh(x)(\cosh(x)\tan(x) + 2\sinh(x))\,dx$$

Solution 14

Answer:

$$\sec x \cosh^2 x + C$$

Problem 15

Evaluate the following Integral:

$$\int_0^\infty \frac{x}{(x^2+1)(9x^2+1)}\,dx$$

Solution 15

Answer:

$$\frac{\log 3}{8}$$

Problem 16

Evaluate the following Integral:

$$\int \frac{\sin(1/x)}{x^3}\, dx$$

Solution 16

Answer:

$$\frac{\cos(1/x)}{x} - \sin(1/x) + C$$

Problem 17

Evaluate the following Integral:

$$\int_0^e W(x)\,dx$$

where $W(x)$ is the *Lambert-W function*, defined as the inverse of $f(x) = xe^x$ (i.e. $W(x)e^{W(x)} = x$).

Solution 17

Answer:

$$e - 1$$

Problem 18

Evaluate the following Integral:

$$\int_0^{2\pi} \cos^{10} x \, dx$$

Solution 18

Answer:

$$\frac{63}{128}\pi$$

Problem 19

Evaluate the following Integral:

$$\int \sqrt{1+x^2}\, dx$$

Solution 19

Answer:

$$\frac{1}{2}\sinh^{-1}(x) + \frac{1}{2}x\sqrt{1+x^2} + C$$

or

$$\frac{1}{2}\log(x+\sqrt{1+x^2}) + \frac{1}{2}x\sqrt{1+x^2} + C$$

Problem 20

Evaluate the following Integral:

$$\int_0^1 \left(\frac{1}{2} + \frac{x}{3} + \frac{x^2}{8} + \frac{x^3}{40} + \cdots + \frac{x^n}{n!(n+2)} + \cdots \right) dx$$

where the sum is infinite.

Solution 20

Answer:

$$e - 2$$

Problem 21

Evaluate the following Integral:

$$\int_0^{2\pi} \log(x + \sin(t))\, dt$$

Express your answer in terms of x.

Solution 21

Answer:

$$2\pi(\cosh^{-1}(x) - \log 2)$$

Problem 22

Evaluate the following Integral:

$$\int x^3\sqrt{1-x^2}\,dx$$

Solution 22

Answer:

$$-\frac{1}{15}(1-x^2)^{3/2}(3x^2+2)+C$$

Problem 23

Evaluate the following Integral:

$$\int \frac{\sin(x)e^{\sec(x)}}{\cos^2(x)}$$

Solution 23

Answer:

$$e^{\sec x} + C$$

Problem 24

Evaluate the following Integral:

$$\int_0^{\pi/2} \sin^7 x \cos^7 x \, dx$$

Solution 24

$$\frac{1}{280}$$