Integration and Summation

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Contents

Chapter 1

Preface

This text is designed to introduce various techniques in Integration and Summation, which are commonly seen in Integration Bees and other such contests. The text is designed to be accessible to those who have completed a standard single-variable calculus course. Examples, Exercises, and Solutions are presented in each section in order to help the reader become become acquainted with the techniques presented.

It is assumed that the reader is familiar with single-variable calculus methods of integration, including u-substitution, integration by parts, trigonometric substitution, and partial fractions. It is also assumed that the reader is familiar with trigonometric and logarithmic identities. Paul Lamar's Online Math Notes, accessible here: `http://tutorial.math.lamar.edu/` is a very comprehensive review of this content.

Note: In this text, log *denotes the natural (base-e) logarithm.*

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Chapter 2

Riemann Sums

2.1 Theory

A Riemann sum is a summation that can be used to approximate an integral, and is exact in the limit as $N \to \infty$. Thus, some limits of summations can be evaluated by converting the associated Riemann sum into a definite integral.

Formally, a Riemann sum is defined over a partition of an interval $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$ as the quantity

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx$$

where x_i^* is within the interval $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$. Various simplifications, such as taking the partitions to be equally spaced, can simplify the summation into something more recognizable. A common simplification is when the interval is $[0, 1]$ and the intervals are equally spaced, giving the summation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \int_0^1 f(x)\,dx.$$

Example 2.1.1

Evaluate $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$

We can rewrite the sum as one more reminiscent of a Riemann sum by factoring out n^2 in the denominator to yield that the limit is $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + \left(\frac{i}{n}\right)^2}$. But this is nothing more than $\displaystyle\int_0^1 \frac{dx}{1 + x^2}$ which is equal to $\dfrac{\pi}{4}$.

2.2 Exercises

1. $\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n-1}{n}\right)\left(1+\frac{n}{n}\right)\right)^{1/n}$
2. $\lim_{x\to\infty}\frac{e^{-1/x}+e^{-2/x}+\cdots+e^{-x/x}}{\sin(1/x)+\sin(2/x)+\cdots+\sin(x/x)}$
3. $\lim_{x\to 0^+}\sum_{k=1}^{\infty}\frac{2x}{k^2x^2+1}$
4. $\lim_{n\to\infty}\left(1+\frac{2}{\sqrt{n}}\right)\sum_{k=1}^{\infty}e^{-k^2/n}$

2.3 Solutions

1. Letting the limit be L, and after taking logarithms, we have that

$$\log L = \frac{1}{n}\sum \log\left(1+\frac{k}{n}\right) = \int_0^1 \log(1+x)\,dx = \log(4)-1 \implies L = \frac{4}{e}$$

2. Multiplying the numerator and denominator by $\frac{1}{x}$ gives that the limit is equal to

$$\frac{\int_0^1 e^{-x}\,dx}{\int_0^1 \sin(x)\,dx} = \frac{1-e^{-1}}{1-\cos(1)}$$

3. The substitution $u = \frac{1}{x}$ transforms the limit into

$$\lim_{u\to\infty} \frac{1}{u}\sum_{k=1}^{\infty} \frac{2}{1+(k/u)^2} = \int_0^{\infty} \frac{2}{1+x^2}\,dx = \pi$$

Note that the integral's bounds here are from 0 to infinity, since the Riemann sums' upper limit was ∞ rather than u.

4. First, note that the $(1\,+)$ part does not contribute to the value of the limit, since the limit of the sum is zero. So the limit we seek is, after substituting $x = \sqrt{k}$,

$$\lim_{x\to\infty} \frac{2}{x}\sum e^{-n^2/x^2} = 2\int_0^1 e^{-x^2}\,dx$$

This last integral is not elementary and hence this is the final form of the solution.

Chapter 3

Weierstrass Substitutions

3.1 Theory

The Weierstrass Substitution allows one to convert trigonometric integrals to integrals of rational functions. This is done by using the substitution $t = \tan(x/2)$.

With this substitution, the trigonometric functions are rational functions of t, as follows:

$$\sin(x) = \frac{2t}{1+t^2} \qquad \cos(x) = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}dt$$

From these identities, a function of trigonometric functions is completely reduced to one of a rational function.

Example 3.1.1
Evaluate $\displaystyle\int \frac{1}{2+\cos\theta}\,d\theta$

By enforcing the substitution $t = \tan(\theta/2)$, we obtain that the integral is equal to

$$\int \frac{1}{2+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt$$

which can be simplified to

$$\int \frac{2}{2(1+t^2)+(1-t^2)}\,dt = \int \frac{2}{3+t^2}\,dt$$

This latter integral is easily evaluated through a trigonometric substitution, from which yields that the integral is finally equal to $\dfrac{2\tan^{-1}\left(\frac{\tan(\theta)}{\sqrt{3}}\right)}{\sqrt{3}} + C$.

3.2 Exercises

With the Weierstrass substitution, evaluate the following:

1. $\displaystyle\int \frac{1}{\cos x}\,dx$
2. $\displaystyle\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,d\theta$
3. $\displaystyle\int \frac{1}{a+b\cos\theta+c\sin\theta}\,d\theta$

3.3 Solutions

1. $$\int \frac{dx}{\cos(x)} = \int \frac{1+t^2}{1-t^2}\frac{2dt}{1+t^2} = 2t - 2\frac{t^3}{3} + C = 2\tan(x/2) - \frac{2}{3}\tan^3(x/2) + C$$

2. Since $\tan(x/2)$ is not injective over this domain, we transform the domain by noting that $\cos(x) = \cos(2\pi - x)$ to yield that the integral is equal to

$$2\int_0^{\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{4}{a+b}\int_0^{\infty} \frac{dt}{1 + \frac{a-b}{a+b}t^2}$$

Doing a trigonometric substitution $\sqrt{\frac{a-b}{a+b}}t = z$ yields that the integral is equal to

$$\frac{4}{\sqrt{(a+b)(a-b)}}\int_0^{\infty} \frac{dz}{1+z^2} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

[This result is worth memorizing due to its prevalence.]

3. The Weierstrass substitution gives that the integral is equal to

$$\int \frac{2\,dt}{(c-a)t^2 + 2bt + (a+c)}$$

Completing the square and using a standard trigonometric substitution yields that the integral is equal to

$$\frac{2}{\sqrt{c^2-(a^2+b^2)}}\tan^{-1}\left(\frac{(c-a)\tan(\theta/2)+b}{\sqrt{c^2-(a^2+b^2)}}\right) + C$$

Chapter 4

Clever Substitutions

4.1 Theory

Oftentimes an integral can be simplified through the use of a clever u substitution that can take many forms. The most common ones are $u = x \pm \dfrac{1}{x}$ (particularly suitable for symmetric integrands), $u = \dfrac{1 \pm x}{1 \mp x}$, or $u = \sqrt[n]{f(x)}$, where the whole root is substituted inside the integrand. Of course, deciding which substitution to use requires practice.

Example 4.1.1

Evaluate $\displaystyle\int_0^1 \frac{1-x^2}{1+x^4}\,dx$

At first glance, one may think of substituting $u = x^2$, but unfortunately that introduces a square root term which is hard to deal with. Instead, by dividing both the numerator and denominator by x^2, one finds that the integral is equal to

$$\int_0^1 \frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

using the identity $\left(x+\frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2}$. Since $\dfrac{d}{dx}\left[x+\dfrac{1}{x}\right] = 1 - \dfrac{1}{x^2}$, it makes sense to use the substitution $u = x + \dfrac{1}{x}$. The integral then simplifies to

$$\int_2^\infty \frac{du}{u^2-2} = \boxed{\frac{\log(\sqrt{2}+1)}{\sqrt{2}}}$$

4.2 Exercises

1. $\displaystyle\int_0^1 \frac{dx}{1+x^6}$. (Hint: $2 = (1 - x^2 + x^4) + x^2 + (1 - x^4)$)
2. $\displaystyle\int_0^1 \sqrt{\frac{1 \pm x}{1 \mp x}}\, dx$ (do both cases)
3. (MIT Integration Bee) $\displaystyle\int_{-\infty}^{\infty} (x^3+1)^2 x^2 e^{-x^6-2x^3}\, dx$
4. $\displaystyle\int \sqrt{\tan(x)}\, dx$.

4.2.1 Solutions

1.
$$\begin{aligned}\int_0^1 \frac{dx}{1+x^6} &= \frac{1}{2}\int_0^1 \frac{(1-x^2+x^4)+x^2+(1-x^4)}{(1+x^2)(1-x^2+x^4)}\,dx\\ &= \frac{1}{2}\left[\frac{\pi}{4}+\int_0^1 \frac{x^2}{1+x^6}\,dx+\int_0^1 \frac{1-x^2}{1-x^2+x^4}\,dx\right]\end{aligned}$$

The first integral can be tackled with the substitution $u = x^3$, while the second can be solved with the substitution $u = x + \frac{1}{x}$. These substitutions ultimately yield the result that the integral is equal to $\frac{\pi+\sqrt{3}\log(2+\sqrt{3})}{6}$.

2. The substitution
$$u = \frac{1+x}{1-x} \implies x = \frac{1-u}{1+u}$$
yields that the (+) integral is equal to
$$\int_1^\infty \frac{2u^{1/2}}{(1+u)^2}\,du$$
Further substituting $z^2 = u$ yields that the integral is equal to
$$\int_1^\infty \frac{4z^2}{(1+z^2)^2}\,dz$$
which can be evaluated by a standard trigonometric substitution to yield $\frac{\pi}{2}+1$.
The (-) integral can be evaluated in a similar manner to yield $\frac{\pi}{2}-1$.

3.
$$\int_{-\infty}^\infty (x^3+1)^2x^2e^{-x^6-2x^3}\,dx = \frac{e}{3}\int_{-\infty}^\infty (x^3+1)^2\cdot 3x^2\cdot e^{-(x^3+1)^2}\,dx$$
from which the substitution $u = x^3+1$ is obvious. This remaining integral can be evaluated similarly to that in Exercise 1.2.2 to be $\frac{e\sqrt{\pi}}{6}$.

4. Substitute away the radical with $u^2 = \tan x$. We have that $2u\,du = \sec^2 x\,dx \implies dx = \frac{2u\,du}{u^4+1}$. Then, the integral is equal to
$$\begin{aligned}I &= \int \frac{2u^2}{1+u^4}\,dx\\ &= \int_0^\infty \frac{\left(1+\frac{1}{u^2}\right)+\left(1-\frac{1}{u^2}\right)}{u^2+\frac{1}{u^2}}\,du\\ &= \int \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2}\,du+\int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2}\,du\end{aligned}$$
We can now note that the numerator is the derivative of the squared quantity in both integrals, from which it is not too hard to finish the computation.

Chapter 5

Integral and Summation Substitutions

5.1 Theory

Oftentimes an integral or summation can be simplified by artificially introducing an integral or summation representation of the function being integrated or summed. The order of integration/summation can then be reversed, allowing for an easier integration.

Remark 5.1.1. *There are some cases in which interchanging the order is actually not allowed. Formally, to do so, we would need to invoke the Monotone Convergence Theorem or Dominated Convergence Theorem.*

Example 5.1.2
Evaluate $\displaystyle\int_0^1 \frac{\log(1-x)}{x}\,dx$

By utilizing the MacLaurin series representation of $\log(1-x)$ and substituting this in, the integral is equal to

$$\int_0^1 \left(-1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \cdots\right) dx$$

Switching the order of integration and summation we find that the whole integral is equal to

$$-\frac{x}{1} - \frac{x^2}{4} - \frac{x^3}{9} - \cdots \Big|_0^1 = \boxed{-\frac{\pi^2}{6}}$$

where we used $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example 5.1.3
Evaluate $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx$.

We use the integral representation $\displaystyle\frac{1}{x^2} = \int_0^\infty te^{-tx}\,dt$. Substituting this into our integral, we find that it is equal to

$$\int_0^\infty (1-\cos x)\int_0^\infty te^{-tx}\,dt\,dx = \int_0^\infty\int_0^\infty te^{-tx} - \cos x\cdot te^{-tx}\,dx\,dt = \int_0^\infty 1-\frac{t^2}{t^2+1}\,dt = \boxed{\frac{\pi}{2}}$$

Example 5.1.4
Evaluate $\displaystyle\sum_{n\geq 0}\frac{(-1)^n}{n+1}$

This sum can be simplified by noting that $\int_0^1 x^n\,dx = \frac{1}{n+1}$. Substituting this into the sum and switching the order we obtain that the sum can be represented as

$$\int_0^1 \sum_{n\geq 0}(-x)^n\,dx.$$

The sum is simply a geometric series, evaluating to $\displaystyle\frac{1}{1+x}$, and the integral can be evaluated with standard techniques to obtain a result of $\boxed{\log 2}$

Remark 5.1.5. *This series is known as the alternating harmonic series and its value is commonly given without proof.*

Example 5.1.6
Evaluate $\displaystyle\int_0^\infty \frac{x}{1+e^x}\,dx$.

Note that the integral can be written as $\displaystyle\int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx$. We can use the familiar geometric series $\displaystyle\sum_{n\geq 0} t^n = \frac{1}{1-t}$ with $t=-e^{-x}$ in the integral. Substituting this into our integral, we find that it is equal to $\displaystyle\sum_{k=0}^\infty(-1)^k\int_0^\infty xe^{-(k+1)x}\,dx = \sum_{k=0}^\infty\frac{(-1)^k}{(k+1)^2}$. This series is well known to equal $\boxed{\frac{\pi^2}{12}}$, which can be derived from the sum in Example 5.1.2.

5.2 Exercises

1. Find $\displaystyle\int_0^1 \frac{\log(1-x^n)}{x}\,dx$ for arbitrary integral n.
2. (Putnam 1969) Prove $\displaystyle\int_0^1 x^x\,dx = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-n}$.

   Can you derive a similar identity for $\displaystyle\int_0^1 x^{-x}\,dx$?
3. If f is continuous and differentiable, monotone, and $\lim_{x\to\infty} f(x) = 0$, then prove that $\displaystyle\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\,dx = f(0)\log(b/a)$. Hint: Consider $\displaystyle\int_a^b f'(tx)\,dt$
4. Prove $\displaystyle\int_0^1 \frac{\log(x)}{1+x^2} = -G$, where G is defined as $G = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$
5. Prove $\displaystyle\int_0^1 \frac{\log^2(x)}{1-x} = 2\zeta(3)$. $\zeta(x)$ is defined as $\zeta(x) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{x^n}$
6. Prove $\displaystyle\int_0^{\infty} \frac{t^{x-1}}{e^t-1}\,dt = \Gamma(x)\cdot\zeta(x)$. $\Gamma(x)$ is defined in section 7.
7. Find $\displaystyle\sum_{n\geq 1}\frac{1}{nk^n}$ for $k=2$, and then general k.
8. Evaluate $\displaystyle\int_0^1 \frac{x^a-1}{\log(x)}\,dx$
9. (Putnam 2016) Evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=0}^{\infty}\frac{1}{k2^n+1}$.

5.3 Solutions

1. Expand $\log(1-x^n) = -\sum_{k\geq 1} \frac{x^{kn}}{k}$. The procedure is similar to that of the first example in the section, and the final result should be equal to $-\dfrac{\pi^2}{6n}$.
2. See `https://en.wikipedia.org/wiki/Sophomore%27s_dream`.
3. Note that the integrand is equal to the integral given in the hint. Then, we can interchange the order of integration to show that
$$\int_0^\infty \int_a^b f'(tx)\,dx\,dt = \int_a^b \int_0^\infty f'(tx)\,dt\,dx = \int_a^b \frac{1}{x} f(0)\,dx = f(0)\log(b/a).$$
4. First we need to show that $\int_0^1 x^n \log(x)\,dx = -\frac{1}{(1+n)^2}$ This can be done by either differentiating under the integral sign (sec. 8) or by integration by parts. Then, we can series expand $\frac{1}{1+x^2} = \sum(-x^2)^n$, to show the desired result.
5. We follow a similar path as above. First, we show that $\int_0^1 x^n \log^2(x)\,dx = \frac{2}{(1+n)^3}$, which can be done in the same manner, and then series expand $\frac{1}{1-x} = \sum x^n$.
6. First multiplying both the numerator and denominator by e^{-x}, then expanding the denominator as a geometric series yields that the integral is equal to
$$\sum_{n\geq 1} \int_0^\infty x^{s-1} e^{-nx}\,dx$$
Applying the substitution $z = nx$, then applying the definition of the Gamma function, and finally applying the definition of the Zeta function, yields the desired result.
7. $k = 2$: Note that $\frac{1}{n} = \int_0^1 x^{n-1}\,dx$. Substitution of this identity, followed by interchanging the integral and the sum, gives the answer $\log(2)$.
The case for general k is not much different, ultimately arriving at an answer of $\log\left(\frac{k}{k-1}\right)$.
8. Note that $\int_1^a x^u\,du = \frac{x^a-1}{\log x}$, so we can substitute this into the integral, interchange the order of integration, to finally yield that the integral is equal to $\log(a+1)$.
9. Substitute $\int_0^1 x^{k2^n}\,dx = \frac{1}{k2^n+1}$ into the summation, then sum over k to obtain that the summation is equal to
$$\int_0^1 \sum_{n=0}^\infty \log(1+x^{2^n})\,dx$$
Since $(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = 1+x+x^2+x^3+\cdots = \frac{1}{1-x}$, taking logarithms, we see that the sum in question is equal to $-\log(1-x)$. Integrating this thus yields that the summation is equal to 1.

Chapter 6

Substitutions About the Domain

6.1 Theory

Sometimes it is convenient to make a substitution about the domain in order to reduce the integral into a more manageable form, or even to cancel out the integral altogether. The most effective substitutions usually take place by reversing the domain completely. For example, in integrals of the form $\int_0^a f(x)\,dx$, it makes sense to substituting $u = a - x$. When working with logarithms or improper integrals, the substitution $u = \frac{1}{x}$ sometimes helps.

Example 6.1.1
Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(\sin x) + \cos^2(\cos x)\,dx$

Let I denote the desired integral. By using the substitution $u = \dfrac{\pi}{2} - x$ we obtain that

$$I = \int_0^{\frac{\pi}{2}} \sin^2(\cos u) + \cos^2(\sin u)\,du.$$

Adding this with the original integral and the new one finds that $2I = \displaystyle\int_0^{\frac{\pi}{2}} 2\,dx$. Thus, we know that $I = \boxed{\dfrac{\pi}{2}}$.

Example 6.1.2

(Putnam 1953) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

By using the substitution $u = \frac{\pi}{2} - x$ we obtain that the desired integral is equal to $\displaystyle\int_0^{\frac{\pi}{2}} \log \cos x$. Summing the original and new integral, we have that

$$2I = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x) = \int_0^{\frac{\pi}{2}} \log(1/2) + \log(\sin 2x) \, dx = -\frac{\pi}{2}\log(2) + \int_0^{\frac{\pi}{2}} \log(\sin 2x) \, dx.$$

Now, note that $\displaystyle\int_0^{\frac{\pi}{2}} \log(\sin 2x) \, dx = \frac{1}{2}\int_0^{\pi} \log(\sin x) \, dx$. As $\sin x$ is symmetric about $(0, \pi)$, the integral is simply $2I$, and therefore we have that $2I = -\frac{\pi}{2}\log(2) + \frac{1}{2} \cdot 2I \implies I = \boxed{-\frac{\pi}{2}\log(2)}$.

Remark 6.1.3. *This is a relatively common result and is worth memorizing.*

Example 6.1.4

Evaluate $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\frac{1}{1+x^n} \, dx$

Split the integral up into $\int_0^1$ and $\int_1^{\infty}$. If we use the substitution $u = 1/x$ on the latter integral, the value becomes

$$I = \int_1^0 \frac{1}{1+\frac{1}{x^2}} \cdot \frac{1}{1+\frac{1}{x^n}} \cdot \left(-\frac{1}{x^2}\right) dx = \int_0^1 \frac{1}{1+x^2} \cdot \frac{1}{1+\frac{1}{x^n}} \, dx$$

Summing these two representations, we have that

$$2I = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \implies I = \boxed{\frac{\pi}{8}}$$

Example 6.1.5

Evaluate $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2} \, dx$

Use the standard trigonometric substitution $x = \tan\theta$ in order to get that $I = \displaystyle\int_0^{\pi/4} \log(1+\tan\theta) \, d\theta$. Now, enforce the substitution $u = \frac{\pi}{4} - \theta$ and use the tangent subtraction formula, to obtain that

$$I = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta.$$

Summing these two representations we see that $2I = \frac{\pi \log 2}{4}$ and so $I = \boxed{\frac{\pi \log 2}{8}}$.

6.2 Exercises

1. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^\pi}\,dx$
2. (Putnam 1987) $\displaystyle\int_2^4 \frac{\sqrt{\log(9-x)}}{\sqrt{\log(9-x)}+\sqrt{\log(x+3)}}\,dx$
3. Prove that $\displaystyle\int_1^\infty \frac{\log(x)}{1+x^2}\,dx = G$, where G is defined in 5.2.4. Hint: It may help to consider $\displaystyle\int_0^\infty \frac{\log(x)}{1+x^2}\,dx$
4. (BMT 2015) Compute $\displaystyle\int_{1/2}^2 \frac{x^2+1}{x^2(x^{2015}+1)}\,dx$
5. (BMT 2018) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{-x}}\,dx$

6.3 Solutions

1. Enforce the substitution $u = \frac{\pi}{2} - x$. Then, when we add the original integral and this one, the denominator cancels out, and we obtain that $2I = \int_0^{\pi/2} dx$. Thus, $I = \frac{\pi}{4}$.
2. Substitute $9 - x = u + 3$ in order to take advantage of symmetry in the problem. Adding the resulting integrals, the fractions cancel out to give that $2I = \int_2^4 dx \implies I = 1$.
3. After enforcing the substitution $u = \frac{1}{x}$ on the integral given in the hint, the integral is equal to the negative of itself, hence it is equal to zero. Since we know from the Ex. 5.2.4 that the integral from 0 to 1 is $-G$, the integral from 1 to infinity must be G.
4. Enforcing the substitution $u = \frac{1}{x}$, the integral simplifies in a similar manner to example 3 in the section. The denominator cancels and we find that the integral in question is equal to $\frac{3}{2}$.
5. Enforce the substitution $u = -x$, then add the resulting integrals. The denominators once again cancel out, and the answer turns out to be 2.

Chapter 7

Gaussian Integral; Gamma/Beta Functions

7.1 Gamma Function

The Pi Function is defined by the following integral:

$$\Pi(x) = \int_0^\infty t^x e^{-t}\, dt$$

$\Pi(x)$ has the special property that $\Pi(x+1) = (x+1)\Pi(x)$, and $\Pi(0) = 1$ which implies that for each natural x, $\Pi(x) = x!$ where the ! denotes the factorial. This can be proven through repeated integration by parts and induction. $\Pi(x)$ is undefined at negative integers x.

However, the Pi function is not well-known within the mathematical literature. Instead, the Gamma function is used for extending the factorial. $\Gamma(x)$ is defined as follows:

$$\Gamma(x) = \Pi(x-1) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

This implies that $\Gamma(x) = (x-1)!$ for natural x and that $\Gamma(x+1) = x\Gamma(x)$. $\Gamma(x)$ is undefined at nonpositive integer x.

7.2 Beta Function

The Beta Function is defined as the following integral:

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

It has the special property that $\beta(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, which can be proven with theorems from multivariate calculus. The Beta Function is also symmetric: $\beta(x, y) = \beta(y, x)$.

Note that from the definition, we have that $\beta(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)} = \Gamma^2\left(\frac{1}{2}\right)$. We can also evaluate the integral on the left directly to see that $\Gamma^2\left(\frac{1}{2}\right) = \sqrt{\pi}$. This value can be used, along with the previous recurrence relation, to find $\Gamma(\frac{n}{2})$.

Example 7.2.1

Find $\Gamma(-\frac{3}{2})$.

$-\frac{3}{2}\Gamma(-\frac{3}{2}) = \Gamma(-\frac{1}{2})$ and $-\frac{1}{2}\Gamma(-\frac{1}{2}) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, so $\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi}$.

Example 7.2.2

(BMT 2012) Evaluate $\dfrac{1}{\sum_{n\geq 1} \frac{2012}{n(n+1)(n+2)\cdots(n+2012)}}$

Let us focus solely on the sum in the denominator. It can be rewritten as $\displaystyle\sum_{n\geq 1} \frac{2012\Gamma(n)}{\Gamma(n+2013)} = \frac{2012}{\Gamma(2013)} \sum_{n\geq 1} \frac{\Gamma(n)\Gamma(2013)}{\Gamma(n+2013)} = \frac{2012}{\Gamma(2013)} \sum_{n\geq 1} \beta(2013, n) = \frac{2012}{\Gamma(2013)} \int_0^1 \sum_{n\geq 1} x^{2012}(1-x)^{n-1}\,dx = \frac{2012}{\Gamma(2013)} \int_0^1 x^{2011}\,dx = \frac{1}{\Gamma(2013)} = \frac{1}{2012!}$ Thus, the requested value is $\boxed{2012!}$

7.3 Gaussian Integral

The Gaussian Integral is the famous integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Though the integrand does not have an elementary antiderivative, the definite integral can still be evaluated, and shows up fairly often that its result is expected to be known. This result can be derived from substituting $u = x^2$, which results in equality with $\Gamma(\frac{1}{2})$.

Remark 7.3.1. *Throughout this text, we will see examples of integrals that have no elementary antiderivative, but nevertheless have definite integrals that can be evaluated in closed form.*

Now let's do some examples with the Gaussian integral and probability distributions, which are related to the Gaussian.

Example 7.3.2

Evaluate $\displaystyle\int_{-\infty}^{\infty} x^4 e^{-x^2}\, dx$.

Since we have a power times an exponential, let's try to integrate by parts. Integrating by parts once reveals that the integral is equal to

$$\frac{x^5}{5} \cdot e^{-x^2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^5}{5} \cdot (-2x) \cdot e^{-x^2}\, dx$$

Oh no, the exponent increased by 2! So, let's integrate by parts with $\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx$ instead:

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{x^3}{3} \cdot e^{-x^2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^3}{3} \cdot (-2x) \cdot e^{-x^2}\, dx = \frac{2}{3} \int_{-\infty}^{\infty} x^4 e^{-x^2}\, dx$$

Now we can integrate the original Gaussian integral by parts in order to relate it with the previous integral. We get that

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, dx = x \cdot e^{-x^2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \cdot (-2x) \cdot e^{-x^2}\, dx = 2 \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx$$

and so our desired integral is equal to $\boxed{\dfrac{3}{4}\sqrt{\pi}}$

7.4 Exercises

1. A way to represent continuous probability distributions of some random variable X is through a **probability density function**. Then, the integral from a to b of this probability density function gives $\mathbb{P}(a \le X \le b)$. A commonly used distribution in many fields is the normal distribution, given by
$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$
where μ is the mean and σ is the standard deviation of the distribution.
A necessary condition on a probability density function is that its integral over the real line must be equal to 1, or that $\int_{-\infty}^{\infty} P(x)\,dx = 1$. The mean, or expectation value, of a random variable X with probability distribution $P(x)$ is given by $E[X] = \int_{-\infty}^{\infty} xP(x)\,dx$. The standard deviation can be defined as $\sqrt{E[X^2] - (E[X])^2}$.
   (a) Prove that the normal distribution has integral 1 over the real line.
   (b) Prove that if X is normally distributed, that $E[X] = \mu$ and that the standard deviation of X is indeed equal to σ, as defined.
2. Show that $\displaystyle\int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x}\,dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right)$. This is an integral commonly seen in physics.
3. The Maxwell Boltzmann distribution for gases states that a gas's velocity's probability density function is $P(v) = Kv^2 e^{-Mv^2/(2RT)}$ where M is the molar mass in kg, R is the gas constant, and T is the temperature.
   (a) Find K in terms of M, R, and T.
   (b) Find the mean speed of a gas molecule following the Maxwell-Boltzmann distribution.
   (c) The **root-mean square** of a random variable X is given by $\sqrt{E[X^2]}$. Prove that the root-mean square speed of gas molecules following the Maxwell Boltzmann Distribution is equivalent to the result through classical chemistry, that is, $v_{rms} = \sqrt{3RT/M}$.
4. (BMT 2017) Evaluate $\displaystyle\int_0^1 (-x^2 + x)^k \cdot \lfloor kx \rfloor\,dx$ where $k = 2017$. Try to find a formula for general k.
5. Find the normalization constant A in the hydrogen atom 3s-orbital wavefunction $\psi(r) = A\left(3 - \frac{2r}{a_0} + \frac{2r^2}{9{a_0}^2}\right) e^{-r/(3a_0)}$ such that $\displaystyle\int_0^{\infty} \psi^2(r)\,dr = 1$. ($a_0$ is the Bohr radius, a constant.)
6. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}}$.
7. Use the reflection formula $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ to evaluate $\int_0^{\pi/2} \sqrt{\tan x}\,dx$, an integral whose antiderivative was found earlier.

7.5 Solutions

1. To first show that the integral over the real line is 1, rescale the variable with the substitution $u = \frac{x-\mu}{\sqrt{2}\sigma}$. This nicely gets rid of a factor of $\frac{1}{\sigma\sqrt{2}}$ and leaves just the integral $\int_{-\infty}^{\infty} e^{-u^2}\,du$.
   To calculate the mean, we need to find $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-(x-\mu)^2/(2\sigma^2)}\,dx$. Once again, we can use the same substitution as above to rescale the integral and get rid of a factor of $\frac{1}{\sigma\sqrt{2}}$. The odd integral equals zero, and the remaining integral is equal to $\mu\sqrt{\pi}$, so the mean is indeed μ.
   A similar computation yields the standard deviation. One would need to integrate by parts in order to find the integral of the form $\int x^2 e^{-\alpha x}$, similar to the example problem.
2. We complete the square in the exponent, and then shift the integration domain:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\,dx = \int_{-\infty}^{\infty} e^{-\alpha(x-\beta/2\alpha)^2+\beta^2/4\alpha}\,dx = e^{\beta^2/4\alpha}\cdot\int_{-\infty}^{\infty} e^{-\alpha u^2}\,du = \sqrt{\frac{\pi}{\alpha}}\cdot\exp\left(\frac{\beta^2}{4\alpha}\right)$$

3. Let $k = \frac{M}{2RT}$. Then, we need to find

$$K = \left(\int_0^{\infty} v^2 e^{-kv^2}\,dv\right)^{-1}$$

   First, enforce the substitution $u = \sqrt{k}v$, then the substitution $x = u^2$. We have that the integral is then equal to

$$I = \frac{1}{k^{3/2}}\int_0^{\infty} u^2 e^{-u^2}\,du = \frac{1}{2k^{3/2}}\int_0^{\infty}\sqrt{x}e^{-x}\,dx$$

   This last integral is equal to $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ by definition, and so the integral is equal to $\frac{\sqrt{\pi}}{4k^{3/2}}$.
   The constant we want is the inverse of this, or $4\sqrt{\frac{(\frac{M}{2RT})^3}{\pi}} = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2}$.
   The average speed of the gas particles is

$$v = \int_0^{\infty} Kv^3 e^{-kv^2}\,dv$$

   Using the substitution $x = v^2$, the integral is equal to

$$\frac{K}{2}\int_0^{\infty} xe^{-kx}\,dx = \frac{K}{2k^2} = \frac{2}{\sqrt{\pi}k^{1/2}} = \sqrt{\frac{8RT}{M\pi}}$$

   The root-mean square speed can be evaluated similarly to those above, with the square-speed being equal to

$$K\int_0^{\infty} v^4 e^{-kv^2}\,dv$$

   This integral is similar to the one in the examples earlier, and the final result is $\sqrt{\frac{3RT}{M}}$ as desired.

4. First enforce the substitution $u = 1 - x$, and add this integral and the original, to obtain that

$$2I = \int_0^1 x^k(1-x)^k(\lfloor kx \rfloor + \lfloor k(1-x) \rfloor)\,dx$$

The term in the parentheses is constant for almost all x over the integration domain and is equal to $k-1$. Hence $I = \frac{(k!)^2}{(2k+1)!} \cdot \frac{k-1}{2}$.

5. An exercise with Gamma function integral computations. Let $u = \frac{2r}{3a_0}$, then the integral equals

$$\frac{3a_0A^2}{2}\int_0^\infty \left(3 - 3u + \frac{u^2}{2}\right)^2 e^{-u}\,du$$

After this this is just no more than an exercise in computation and substitution with the gamma function, ultimately yielding $A = \frac{1}{9\sqrt{3\pi}a_0^{3/2}}$.

6.
$$\sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}} = \sum_{n\geq 0}(2n+1)\beta(n+1,n+1) = \int_0^1 \sum_{n\geq 0}(2n+1)(x(1-x))^n\,dx$$

Now we can invoke the standard geometric series and the result that $\sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$ (see sec. 9.2) to yield that the integral is equal to (after simplification)

$$\int_0^1 -\frac{1}{1-x+x^2} + \frac{2}{(1-x+x^2)^2}\,dx$$

These integrals are both elementary and can be evaluated with a trigonometric substitution, with a final value of $\frac{4}{3} + \frac{2\sqrt{3}\pi}{27}$.

7. To evaluate this integral, we perform the substitution $t = \sin^2 x$. Then, we have that

$$\begin{aligned} I &= \frac{1}{2}\int_0^1 t^{-1/4}(1-t)^{-3/4}\,dt \\ &= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right) \\ &= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \\ &= \frac{1}{2}\pi\csc\left(\frac{3}{4}\pi\right) \\ &= \frac{\sqrt{2}}{2}\pi. \end{aligned}$$

Chapter 8

Differentiation Under the Integral Sign

8.1 Theory

Differentiation under the integral sign is a technique where an artificial parameter is introduced, then the integral is differentiated with respect to that parameter, often simplifying it significantly. Then, it is re-integrated with respect to the parameter, and using known values, the constant term in the integration can be found.

Also, note that the technique can be applied in reverse, for example, in evaluating $\int_0^1 x(\log x)^n \, dx$

Example 8.1.1

Evaluate $\displaystyle\int_0^\pi \log\left(1 + \frac{1}{2}\cos x\right)\, dx$

We can convert this integral into one involving a parameter as such: $I(\mu) = \displaystyle\int_0^\pi \log(1+\mu\cos x)\, dx$. Differentiating we find that $I'(\mu) = \displaystyle\frac{1}{\mu}\int_0^\pi \frac{\cos x}{\frac{1}{\mu} + \cos x} = \frac{\pi}{\mu} - \frac{1}{\mu}\int_0^\pi \frac{1}{1+\mu\cos x}$. This second integral can be evaluated to be $\displaystyle\frac{\pi}{\sqrt{1-\mu^2}}$, using 3.2.2 and symmetry.

Integrating with respect to μ implies that $I(\mu) = \pi(\log(\mu) + \operatorname{sech}^{-1}(\mu) + C)$.

From the original integral, we know that $I(0) = 0$. So, we take the limit $\mu \to 0$ in the above form and find $C = \log 2$. Hence, the integral equals $\pi(-\log(2) + \operatorname{sech}^{-1}(1/2) + \log(2)) = \boxed{\pi\log(2+\sqrt{3})}$

8.2 Exercises

1. Show that $\displaystyle\int_0^1 \frac{x^5-1}{\log(x)}\,dx = \log(6)$. (This can also be done by introducing an integral, see Ex. 5.3.8.)
2. Find x such that $\int_0^{2\pi} \log(x+\sin(t))\,dt = 0$. You may want to find the integral in terms of x directly.
3. $\displaystyle\int_0^\infty \frac{\arctan x}{x(x^2+1)}\mathrm{d}x$
4. (Putnam) $\displaystyle\int_0^\infty \frac{\arctan(\pi x)-\arctan(x)}{x}\,dx$
5. (Putnam) $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$ (This can be done with a substitution about the domain, as seen in Section 5. Try finding a different solution.)

8.3 Solutions

1. Let
$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$$
Note that
$$I'(\alpha) = \int_0^1 x^\alpha\, dx = \frac{1}{\alpha + 1}$$
hence $I(\alpha) = \log(\alpha + 1) + C$. Now, note that $I(0) = 0$, hence $C = 0$. Thus $I = I(5) = \log 6$.

2. If we let $I(x) = \displaystyle\int_0^{2\pi} \log(x + \sin(t))\, dt$, then $I'(x) = \displaystyle\int_0^{2\pi} \frac{1}{x + \sin(t)}\, dt = \frac{2\pi}{\sqrt{x^2 - 1}}$. Hence $I(x) = 2\pi \cosh^{-1}(x) + C$. Note that $I(1) =$ $\displaystyle\int_0^{2\pi} \log(1 + \sin x)\, dx = \int_0^{2\pi} \log(1 + \cos x)\, dx = 2\pi \log(2) + \int_0^{2\pi} \log(\cos^2(x/2))\, dx$ $= 2\pi \log(2) + \displaystyle\int_0^{2\pi} \log(\sin^2(x/2))\, dx = 2\pi \log(2) + 4\int_0^{\pi} \log(\sin(x))\, dx = 2\pi \log 2$, hence $C = -2\pi \log 2$. Since $\cosh^{-1}(5/4) = \log 2$, the integral is equal to zero at $x = 5/4$.

3. Let $I(a) = \displaystyle\int_0^\infty \frac{\arctan(ax)}{(1 + x^2)x}\, dx$, then $I'(a) = \displaystyle\int_0^\infty \frac{dx}{(x^2 + 1)(a^2x^2 + 1)}$. This integral is elementary and can be evaluated through partial fractions, ultimately yielding that it is equal to $\dfrac{\pi}{2}\dfrac{1}{1 + a}$. Integration and setting $a = 0$ yields that $I(a) = \dfrac{\pi}{2}\log(1 + a)$, hence the integral is equal to $\dfrac{\pi}{2}\log(2)$.

4. Let $I(a) = \displaystyle\int_0^\infty \frac{\arctan(ax) - \arctan(x)}{x}\, dx$, then $I'(a) = \displaystyle\int_0^\infty \frac{dx}{1 + a^2x^2} = \frac{\pi}{2a}$. Integration yields that $I(a) = \frac{\pi}{2}\log(a)$, hence the total integral is $\dfrac{\pi}{2}\log(\pi)$.

5. We parameterize the integral as follows: $I(a) = \displaystyle\int_0^1 \frac{\log(1 + ax)}{1 + x^2}\, dx$. Differentiation with respect to a yields $I'(a) = \dfrac{\pi a + 2\log 2 - 4\log(a + 1)}{4(a^2 + 1)}$ Hence $I(a) = \dfrac{\log(2)\arctan(a)}{2} +$ $\dfrac{\pi \log(a^2 + 1)}{8} - \displaystyle\int_0^a \frac{\log(1 + t)}{1 + t^2}\, dt$, and thus $I(1) = \dfrac{\pi \log 2}{4} - I(1)$. Solving for $I(1)$ we find that it is equal to $\dfrac{\pi \log 2}{8}$.

Chapter 9

Frullani's Theorem

Frullani's Theorem is a generalized version of Exercise 5.2.3. It states that if f is continuous and differentiable, monotone, and $\lim_{x\to\infty} f(x) = L$, then $\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = (f(0) - L)\log(b/a)$. Exercise 5.2.3 is simply the case with $L = 0$.

9.1 Exercises

1. Prove Frullani's Theorem.
2. Do exercise 8.2.4 with Frullani's Theorem.
3. We previous saw that with differentiation under the integral, that $\int_0^1 \frac{x^a - 1}{\log(x)}\, dx = \log(a+1)$. Instead, use the substitution $x = e^{-t}$ and Frullani's to arrive at the same result.

9.2 Solutions

1. Let $u(x) = f(x) - L$. Then, we can apply Ex. 5.2.3 to $u(x)$, and back substitution yields the theorem.
2. We have $f(x) = \arctan(x), a = \pi, b = 1, L = \pi/2$, hence plugging in values we obtain that the integral is equal to $\frac{\pi}{2}\log\pi$.
3. The integral is equal to $-\displaystyle\int_0^\infty \frac{e^{-(a+1)x} - e^{-x}}{t}\,dt$, which upon applying Frullani's theorem, yields that the integral is equal to $\log(a+1)$.

Chapter 10

Further Exercises

These integrals will use various combinations of the aforementioned techniques. Some integrals may even fall to standard methods. It is left to the reader to decide which method would be most suitable.

1. (Putnam 1968) Show $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi > 0$, and thus prove that $\pi \neq \dfrac{22}{7}$
2. (BMT) Evaluate $\displaystyle\int_0^\infty xI_0(2x)e^{-x^2}\,dx$ where $I_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi e^{x\cos\theta}\,d\theta$
3. Show that $\displaystyle\int_0^\infty \frac{1}{(1+x^\phi)^\phi}\,dx = 1$ where $\phi := \dfrac{1+\sqrt{5}}{2}$ is the golden ratio.
4. (Putnam 1985) $\displaystyle\int_0^\infty x^{-1/2}e^{-1985(x+1/x)}\,dx$.
5. (BMT 2013) Prove that $\displaystyle\sum_{0\le k\le 2n} \frac{(-1)^k}{m+k}\binom{2n}{k} > 0$ for all positive integers m, n.
6. (BMT 2017) Find $\displaystyle\sum_{n=1}^{k} \binom{k}{n} H_n(-1)^n$, where $H_n = \displaystyle\sum_{j=1}^{n} \frac{1}{j}$, and $k = 2017$. Try to find a formula for general k.
7. (BMT) Find $\displaystyle\sum_{n\ge 1} \frac{1}{(3n-1)(3n-2)}$
8. Find $\displaystyle\int_0^{2\pi} \log(1-2a\cos x + a^2)\,dx$ for $a > 1$.
9. (SMT) Find $\displaystyle\int_0^\infty \frac{\log x}{x^2+4}\,dx$
10. (SMT) Find $\displaystyle\int_{1/2}^2 \frac{\arctan x}{x^2-x+1}\,dx$