The Art of Coordinate Bashing

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"All Geometry is Algebra"
-Anonymous Mathematician

1 Introduction

Coordinate bashing is a technique that allows one to solve problems in geometry by putting the geometric construction in question onto a coordinate plane, and using formulae from coordinate geometry to solve for the requested quantity. In this way, the technique can transform a synthetic geometry problem into one of algebra.

It's usually the most useful for finding intersections and lengths. However, it's not so good for these situations:

- Lots of circles. One or two is fine; many become messy.
- Problems where no concrete lengths are given (ex. problem only gives areas or angles)

2 Formulae

We give a list of formulae that are helpful in coordinate geometry, but is certainly not comprehensive. Also note that many concepts from synthetic geometry can also be applied to simplify coordinate bashes significantly.

- Equations of line connecting point (x_1, y_1) and point (x_2, y_2) :
  $y = m(x - x_1) + y_1$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line.
- Distance between point (x_1, y_1) and (x_2, y_2) :
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Midpoint between point (x_1, y_1) and (x_2, y_2) :
  $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- Equation of circle with radius r and center (a, b) :
  $(x - a)^2 + (y - b)^2 = r^2$.
- Formula for perpendicular line
  $m_1 m_2 = -1$, where m_1, m_2 are the slopes of the two lines that are perpendicular.
- Slope of a line that passes through the origin:
  $m = \tan\theta$, where θ is the angle the line makes with the x-axis.
- Area of a triangle from coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$:
  $$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left|x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_1y_3 - x_2y_1\right|$$

3 Examples

The best way to show this art is through examples. Hence, we present some below:
Note: It is highly recommended to follow along the solutions with diagrams of one's own, since the author was too lazy to include diagrams himself.

Example 3.1

(PUMAC 2016) Let $ABCD$ be a square with side length 8. Let M be the midpoint of BC and let ω be the circle passing through M, A, and D. Let O be the center of ω, X be the intersection point (besides A) of ω with AB, and Y be the intersection point of OX and AM. Find the length OY.

We put the whole construction on a coordinate plane. Let $A = (0, 0), B = (8, 0), C = (8, 8), D = (0, 8)$. Then the circle passes through the points $(0, 0), (0, 8)$, and $(8, 4)$. By symmetry, we know that O must be on the line $y = 4$ since it is equidistant to $(0, 0)$ and $(0, 8)$.

Let $O = (t, 4)$, and since it is the center, we know that $OA = OC$. Since $OC = 8 - t$ and $OA = \sqrt{t^2 + 4^2}$, we can solve for t to find that $t^2 + 16 = 64 - 16t + t^2 \implies t = 3$. Hence $O = (3, 4)$. This means that the radius of the circle is $\sqrt{3^2 + 4^2} = 5$, and hence it has equation $(x - 3)^2 + (y - 4)^2 = 25$.

The line AB is simply the line with $y = 0$, hence the point X can be found by substituting $y = 0$ into the equation of the circle. We then find $(x - 3)^2 + 16 = 25 \implies x = 3 \pm 3$. Since we know that $X \neq A$, we take the positive sign, which means $X = (6, 0)$.

The line OX is the line connecting $(3, 4)$ and $(6, 0)$, or $y = -\frac{4}{3}(x - 3) + 4 = 8 - \frac{4}{3}x$. The line AM is the line connecting $(0, 0)$ and $(8, 4)$, or $y = \frac{1}{2}x$. Setting these equal to find the intersection point, we find that $\frac{1}{2}x = 8 - \frac{4}{3}x \implies x = \frac{8}{\frac{1}{2}+\frac{4}{3}} = \frac{48}{11}$. This means that $y = \frac{1}{2} \cdot \frac{48}{11} = \frac{24}{11}$, hence $Y = (\frac{48}{11}, \frac{24}{11})$

Finally, we find $OY = \sqrt{(3 - \frac{48}{11})^2 + (4 - \frac{24}{11})^2} = \sqrt{\frac{1}{11^2}(15^2 + 20^2)} = \frac{25}{11}$.

Example 3.2

(AMC 2004 10B) In the right triangle $\triangle ACE$, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle DBF$ to that of $\triangle ACE$?

We put $A = (0, 12), C = (0, 0), E = (16, 0)$. By the conditions given, we also know that $B = (0, 9), D = (4, 0)$, and F is somewhere along EA. Since we know that the total length of EA equals 20 and EF is 5 units away, we can conclude that EF is $\frac{5}{20} = \frac{1}{4}$ the distance between A and E. Hence $F = \frac{3}{4}(16, 0) + \frac{1}{4}(0, 12) = (12, 3)$.

Since ACE is a right triangle, we can find its area as $\frac{12 \cdot 16}{2} = 96$. Now, we can use the area formula to find the area of DBF :

$$A = \frac{1}{2}\begin{vmatrix} 0 & 9 & 1 \\ 4 & 0 & 1 \\ 12 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left|0 + 9 \cdot 12 + 4 \cdot 3 - 0 - 0 - 36\right| = 42$$

Hence their ratio is $\frac{42}{96} = \frac{7}{16}$.

Example 3.3

(AMC 2005 10B) Equilateral $\triangle ABC$ has side length 2, M is the midpoint of $\overline{AC}$, and C is the midpoint of $\overline{BD}$. What is the area of $\triangle CDM$?

Like always, we place the triangle onto the coordinate plane. Let $B = (-1, 0), C = (1, 0)$. Since the triangle is equilateral, we can find that the height is $\sqrt{3}$. Hence $A = (0, \sqrt{3})$. This means that $M = \frac{1}{2}(0, \sqrt{3}) + \frac{1}{2}(1, 0) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Also, since C is the midpoint of BD, we know that $\frac{1}{2}(-1, 0) + \frac{1}{2}D = (1, 0) \implies D = (3, 0)$.

Now we use the area formula to find that

$$A = \frac{1}{2}\begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ 1 & 0 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \frac{1}{2}\left|\frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2}$$

Example 3.4

(AMC 2004 10B) A triangle with sides of $5, 12$, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

We use the following facts from synthetic geometry:

- For a right triangle, the circumcircle has center at the midpoint of the hypotenuse.
- The area of any triangle is given by the formula $A = r \cdot \frac{P}{2}$, where r is the inradius and P is the perimeter.

We put $A = (0, 0), B = (5, 0), C = (0, 12)$. By the first fact, we know that the circumcircle's center is the midpoint of BC, i.e. $(\frac{5}{2}, 6)$.

We can also calculate the area of the triangle as $\frac{1}{2} \cdot 5 \cdot 12 = 30$ since the triangle is right. But we also know that $A = r \cdot \frac{P}{2} \implies r = \frac{2A}{P}$. We can calculate $P = 5 + 12 + 13 = 30$, hence $r = \frac{2 \cdot 30}{30} = 2$.

Now, since the incircle is tangent to the two legs of the triangle, we must have that the incircle's center is $(2, 2)$ such that it only intersects AB and AC once.

Hence the distance we seek is $\sqrt{(\frac{5}{2} - 2)^2 + (6 - 2)^2} = \frac{\sqrt{65}}{2}$.

Example 3.5

(AMC 2008 10A) A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C. What is the area of quadrilateral $ABCD$?

Let $A = (0, 0, 0), B = (0, 1, \frac{1}{2}), C = (1, 1, 1), D = (1, 0, \frac{1}{2})$. By symmetry, we know that the area of $ABCD$ is twice the area of ABD. By the distance formula, we know that $BD = \sqrt{1^2 + 0^2 + 1^2}$. The height of this triangle is the length from A to the midpoint of BD, since the triangle is isosceles. This length is $\sqrt{\frac{1}{2}^2 + \frac{1}{2}^2 + \frac{1}{2}^2} = \frac{\sqrt{3}}{2}$. Hence the area of $ABCD$ is equal to $\frac{\sqrt{3}}{2} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}$.

4 Exercises

1. Let ABC be a triangle with D on BC. Suppose $AB = \sqrt{2}, AC = \sqrt{3}, \angle BAD = 30°$, $\angle CAD = 45°$. Find AD.

2. (AMC 2009 10B) Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Point M is the midpoint of diagonal $\overline{AC}$, and E is on AB with $\overline{ME} \perp \overline{AC}$. What is the area of $\triangle AME$?

3. (AMC 2018 10B) Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

4. (PUMAC 2017) Triangle ABC has $AB = BC = 10$ and $CA = 16$. The circle Ω is drawn with diameter BC. Ω meets AC at points C and D. Find the area of triangle ABD.

5. (PUMAC 2012) Two circles centered at O and P have radii of length 5 and 6 respectively. Circle O passes through point P. Let the intersection points of circles O and P be M and N. Find the area of triangle MNP.

6. (PUMAC 2018) Triangle ABC has $\angle A = 90°, \angle C = 30°$, and $AC = 12$. Let the circumcircle of this triangle be ω. Define D to be the point on arc BC not containing A so that $\angle CAD = 60°$. Define points E and F to be the foots of the perpendiculars from D to lines AB and AC, respectively. Let J be the intersection of line EF with ω, where J is on the minor arc AC. The line DF intersects W at H other than at D. Find the area of the triangle FHJ.

7. (PUMAC 2018) Consider rectangle $ABCD$ with $AB = 30$ and $BC = 60$. Construct circle T whose diameter is AD. Construct circle S whose diameter is AB. Let circles T and S intersect at P, so that $P \neq A$. Let AP intersect BC at E. Let F be the point on AB so that EF is tangent to the circle with diameter AD. Find the area of triangle AEF.

Numeric Answers

1. $\frac{\sqrt{6}}{2}$
2. $\frac{75}{8}$
3. $\frac{15}{32}\sqrt{3}$
4. 24
5. $\frac{432}{25}$
6. $\frac{2}{3}(\sqrt{15} - \sqrt{3})$
7. 75 or 225, depending on interpretation.