# 1 Convexity explains SVMs

- The convex hull of a set is the collection of linear combinations of points in the set where the coefficients are nonnegative and sum to one.
- Two sets are linearly separable if and only if their convex hulls don't intersect.
- To find the SVM boundary between two sets, pick from their convex hulls a pair of points $(u, v)$ that are closest together, and draw the boundary that (1) passes through their midpoint, (2) is normal to the line joining them.
- The points $u$ and $v$ can be expressed as linear combinations of their respective sets. The normal vector $n$ is their difference $u - v$, i.e. the difference between those linear combinations. That's where this familiar formula comes from:

$$n = \sum_{\text{positive}} \alpha_i \vec{x}_i - \sum_{\text{negative}} \alpha_j \vec{x}_j$$

- As long as the convex hulls don't intersect, the points $(u, v)$ that are closest together will be on the *surface* of their hulls. The surface of the hull corresponds to linear combinations of points where some of the points have zero coefficients. (Equivalently, the "faces" of the convex hull are the linear combinations of the points at the face's vertices, with zero coefficients for all other points. Edges have zero coefficients wherever either of its neighboring faces do, and each vertex has all zero coefficients—except a 1 for itself.)
- Each point in a convex hull is a linear combination whose coefficients sum to one. That's why when you add up the coefficients of $u$ and subtract the coefficients of $v$, you get zero[1].

$$\sum_{\text{positive}} \alpha_i = \sum_{\text{negative}} \alpha_j = 1$$

[1] The sum of the positive alphas is 1 in this paper (they're convex coefficients), but not in 6.034. In this paper, the normal vector $n$ is geometric; it's the displacement between the two gutters. In 6.034, the normal vector $w$ has a length that satisfies different conditions. In that setting, the right equation is $\sum_{\text{positive}} \alpha_i = \sum_{\text{negative}} \alpha_j = \frac{1}{2}||w||^2$

## 2 Results

**1 Convention** In the following discussion, we assume some particular $\mathbf{R}^d$ as our underlying inner product space.[2] The inner product of $x$ and $y$ is written $\langle x, y \rangle$.

**2 Definition** Fix two points $x$ and $y$. The *path* from $x$ to $y$ is the function $g : [0,1] \to \mathbf{R}^d$ sending $t \mapsto (1-t)x + ty$, and the *segment* joining $x$ and $y$ is its image.

**3 Definition** The *convex hull* of a finite set $X \subseteq \mathbf{R}^d$ is the set $\mathcal{H}(X)$ consisting of all linear combinations of members of $X$ where the coefficients are nonnegative and sum to one.

**4 Remark** Every convex hull is closed and compact. After all, the set that generates the hull is presumed finite.

**5 Definition** Given $n, p \in \mathbf{R}^d$ with $n \neq 0$, we define the *boundary* $\mathcal{B}(n, p)$ to be the set

$$\mathcal{B}(n, p) = \{x \in \mathbf{R}^d : \langle n, x - p \rangle = 0\}.$$

**6 Remark** *(Properties of boundaries.)*

1. Every boundary is closed.
2. For any boundary, $p \in \mathcal{B}(n, p)$.
3. The boundaries $\mathcal{B}(n, p)$ and $\mathcal{B}(m, q)$ are equal whenever $q \in \mathcal{B}(n, p)$, and $m = \alpha n$ for some real number $\alpha$.

**7 Definition** A boundary *separates* sets $X$ and $Y$ if these two conditions hold:

1. For each $x \in X$ and each $y \in Y$, the segment joining $x$ and $y$ intersects the boundary.
2. Neither $X$ nor $Y$ intersects the boundary

Relatedly, we say that $X$ and $Y$ are *separable* if there exists a boundary which separates them.

[2] The proofs will work in all of them!

**8 Lemma** A boundary $\mathcal{B}(n, p)$ separates sets $X$ and $Y$ if and only if the function $f(z) = \langle n, z - p \rangle$ is nonzero throughout $X$ and $Y$, having the same sign for all points in $X$, and the opposite sign for all points in $Y$.

*Proof.* First, observe that a point is on the boundary if and only if $f$ sends it to zero; hence, $f$ is nonzero throughout $X$ and $Y$ if and only if neither $X$ nor $Y$ intersects the boundary.

Second, fix points $x \in X$, $y \in Y$, and let $g$ be the path between them. Observe that the segment joining $x$ and $y$ intersects the boundary if and only if $f$ is zero at some point on the segment, which is true if and only if one of the following holds: $f$ has opposite signs at $x$ and $y$ (by the intermediate value theorem applied to $f \circ g$) or $f$ is zero at $x$, or $f$ is zero at $y$ (because segments contain their endpoints).

Third, observe that $f$ has the same sign throughout $X$ and the opposite sign throughout $Y$ if and only if for each pair of points $x \in X$ and $y \in Y$, $f(x)$ and $f(y)$ have opposite signs.

Combining these three observations yields: "$f$ is nonzero throughout $X$ and $Y$, and $f$ has the same sign for all $x \in X$ and the opposite sign for all $y \in Y$" if and only if "$X$ and $Y$ don't intersect the boundary, but every segment between them does."

□

**9 Theorem** A boundary separates *finite* sets $X$ and $Y$ if and only if it separates their hulls.

*Proof.* ($\Leftarrow$). Hulls contain their finite sets.

($\Rightarrow$). Let $f$ be as in the previous theorem, and observe that if $\zeta$ is a convex combination of points in a finite set $Z$, then $f(\zeta)$ is that same convex combination of the $f(z)$. Indeed,

$$\begin{aligned} f(\zeta) &= f\left(\sum_{z \in Z} \zeta_z \cdot z\right) \\ &= \langle n, -p + \sum_{z \in Z} \zeta_z \cdot z \rangle \\ &= \langle n, \sum_{z \in Z} \zeta_z \cdot (-p + z) \rangle \\ &= \sum_{z \in Z} \zeta_z \cdot \langle n, -p + z \rangle \\ &= \sum_{z \in Z} \zeta_z \cdot f(z). \end{aligned}$$

Therefore, since each point in $\mathcal{H}(X)$ is a convex combination of points in $X$, their images under $f$ have the same sign as the $f(x)$; the analogous statement holds for each point in $\mathcal{H}(Y)$. Hence, by the intermediate value theorem, each line between the hulls intersects the boundary.

□

**10 Definition** The *distance* between a closed set and a compact set is the smallest possible value of $||x-y||$, where $x$ is a point in the first set and $y$ is a point in the second set.

**11 Lemma** *(Gutters.)* If finite sets $X$ and $Y$ have nonintersecting hulls, and $u \in \mathcal{H}(X)$, $v \in \mathcal{H}(Y)$ are as close as possible, then

1. For each $z \in \mathcal{H}(X)$, $\langle u-v, z-u\rangle \geq 0$
2. For each $z \in \mathcal{H}(Y)$, $\langle v-u, z-v\rangle \geq 0$

*Proof.* We prove the first statement; by a symmetric argument (exchanging $X$ and $Y$, $u$ and $v$), the second statement follows.

Suppose $z$ lies in the hull of $X$. If $z = u$, the result is immediate. Otherwise, consider $z \neq u$ in the hull of $X$, put $f : \square \mapsto \langle u-v, \square - u\rangle$, and let $g$ be the path between $u$ and $z$.

Now every point on the segment $g$ is in the hull of $X$ since its endpoints are. Therefore, since $(u, v)$ is minimal, we have $||g(t) - v|| \geq ||u - v||$. From this, we can derive

$$\begin{aligned}
0 &\geq ||u-v||^2 - ||g(t)-v||^2 \\
&= \langle u-v, u-v\rangle - \langle (1-t)u + tz - v,\ (1-t)u + tz - v\rangle \\
&= \langle u-v, u-v\rangle - \langle (u-v) + t(z-u),\ (u-v) + t(z-u)\rangle \\
&= \langle u-v, u-v\rangle - \langle u-v, u-v\rangle - 2t\langle u-v, z-u\rangle - t^2\langle z-u, z-u\rangle \\
&= -2t\langle u-v, z-u\rangle - t^2\langle z-u, z-u\rangle \\
0 &\leq t^2\langle z-u, z-u\rangle + 2t\langle u-v, z-u\rangle
\end{aligned}$$

Let's consider the case where $t \neq 0$, so we can divide by $t$, yielding

$$t\langle z-u, z-u\rangle + 2\langle u-v, z-u\rangle \quad \geq \quad 0$$

Taking the limit as $t \downarrow 0$ gives our desired result:

$$\langle u-v, z-u\rangle \geq 0.$$

□

**12 Theorem** If finite sets $X$ and $Y$ have nonintersecting hulls, and $u \in \mathcal{H}(X)$, $v \in \mathcal{H}(Y)$ are as close as possible, then the boundary

$$\mathcal{B}\left(u - v, \frac{u+v}{2}\right)$$

separates $X$ and $Y$.

*Proof.* Let $u \in \mathcal{H}(X)$, $v \in \mathcal{H}(Y)$ be as close as possible, and put $f : \Box \mapsto \langle u - v, \Box - (u+v)/2\rangle$. The above lemma establishes that $f(x) \geq f(u)$ for any $x \in \mathcal{H}(X)$, and that $f(y) \leq f(v)$ for any $y \in \mathcal{H}(Y)$. Indeed,

$$\begin{aligned} \langle u - v, x - u\rangle \geq 0 \quad &\iff \quad \langle u - v, x\rangle \geq \langle u - v, z\rangle \\ &\iff \quad \langle u - v, x - \frac{u+v}{2}\rangle \geq \langle u - v, z - \frac{u+v}{2}\rangle \\ &\iff \quad f(x) \geq f(u) \end{aligned}$$

and symmetrically by exchanging $u$ and $v$, $x$ and $y$. Therefore, it is enough to prove that $f(u)$ is positive and $f(v)$ is negative. But indeed,

$$\begin{aligned} f(u) &= \langle u + v, u - \frac{u+v}{2}\rangle = \frac{1}{2}||u - v||^2 \\ f(v) &= \langle u + v, v - \frac{u+v}{2}\rangle = -\frac{1}{2}||u - v||^2 \end{aligned}$$

Since $u \neq v$, $||u - v||^2$ is strictly positive.

□

**13 Corollary** Finite sets are separable *if and only if* their hulls don't intersect.

*Proof.* ($\Leftarrow$) (Theorem 12).

($\Rightarrow$). Finite sets are separable if and only if their hulls are (Theorem 9). Suppose their hulls intersect at $w$ and that some boundary separates them. Then in particular the segment from $w$ to $w$ intersects the boundary and so $w$ belongs to the boundary and so the boundary does *not* separate the hulls—a contradiction. □

**14 Theorem** *(Optimal boundary.)* Let $X$ and $Y$ be finite separable sets, and find $u \in \mathcal{H}(X)$, $v \in \mathcal{H}(Y)$ which are as close as possible.

The boundary $B = \mathcal{B}(u - v, (u+v)/2)$, which separates $X$ and $Y$, is optimal in the following sense: any other boundary $C$ which separates $X$ and $Y$ is strictly closer to $\mathcal{H}(X)$ or $\mathcal{H}(Y)$ than $B$ is.

*Proof.* Fix a boundary $C = \mathcal{B}(m, q)$. Since $C$ separates $X$ and $Y$, it must intersect the line between $u$ and $v$ somewhere.

First suppose that $C$ does not intersect at $\frac{u+v}{2}$. Then it must intersect at some other point $w = (1-t)u + tv$ where $t \neq \frac{1}{2}$. But $||w-u|| + ||w-v|| = ||u-v||$, so when $t \neq \frac{1}{2}$, we either have $||w-u|| < \frac{1}{2}||u-v||$ or $||w-v|| < \frac{1}{2}||u-v||$. Since the distance from $B$ to either hull is $\frac{1}{2}||u-v||$, we've established the result for this case.

Next, suppose that $C$ *does* intersect at $\frac{u+v}{2}$. This means that $C$ is equivalent to $\mathcal{B}(m, \frac{u+v}{2})$ (Remark 6). Observe that the distance between $B$ and either hull is $\frac{1}{2}||u-v||$, and that this minimum is attained by $u$ and by $v$. We will show that $C$ must be closer to $u$ than $B$ is, which will establish the desired result.

Define

$$u' = u - \frac{\frac{1}{2}\langle u-v, m\rangle}{\langle m, m\rangle} m$$

(so that $u'$ is the "shadow of $u$ cast on C"). We have that $u' \in C$, since $\langle m, u' - \frac{u+v}{2}\rangle = 0$. It follows that the distance between $u$ and $u'$ is an upper bound for the distance between $C$ and $\mathcal{H}(X)$.

By the Pythagorean theorem, the distance between $u$ and $u'$ is related to the distance between $u$ and $(u+v)/2$ as follows:

$$\begin{aligned} ||u-u'||^2 &= \left\| u - \frac{u+v}{2} \right\|^2 + \left\| u' - \frac{u+v}{2} \right\|^2 \\ ||u-u'|| &\leq \left\| u - \frac{u+v}{2} \right\| \\ &= \left\| \frac{u-v}{2} \right\|. \end{aligned}$$

If the inequality is strict in the second line, we have our desired result—$C$ is closer to $u$ than $B$ is; hence (since $u$ is the closest point in the hull to $B$) closer to the hull of $X$.

Otherwise, equality holds in the second line and so the term $||u' - (u +$

$v)/2||^2$ from the first line vanishes. We observe

$$\begin{aligned}
\left\| u' - \frac{u+v}{2} \right\|^2 = 0 \quad &\iff \quad u' = \frac{u+v}{2} \\
&\iff \quad u - \frac{\frac{1}{2}\langle u-v, m\rangle}{\langle m, m\rangle} m = \frac{u+v}{2} \\
&\iff \quad \frac{u-v}{2} = \frac{\frac{1}{2}\langle u-v, m\rangle}{\langle m, m\rangle} m \\
&\iff \quad u - v = \frac{\langle u-v, m\rangle}{\langle m, m\rangle} m \\
&\iff \quad m \text{ is a nonzero multiple of } (u-v)
\end{aligned}$$

But if $m$ is a nonzero multiple of $u - v$, and $(u + v)/2 \in C$, then $B = C$ (Remark 6).

□