A Quasi-Convex Optimization Approach to Parameterized Model Order Reduction

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***Abstract*—In this paper, an optimization-based model order reduction (MOR) framework is proposed. The method involves setting up a quasi-convex program that solves a relaxation of the optimal $\mathcal{H}_\infty$ norm MOR problem. The method can generate guaranteed stable and passive reduced models and is very flexible in imposing additional constraints such as exact matching of specific frequency response samples. The proposed optimization-based approach is also extended to solve the parameterized model-reduction problem (PMOR). The proposed method is compared to existing moment matching and optimization-based MOR methods in several examples. PMOR models for large RF inductors over substrate and power-distribution grid are also constructed.**

***Index Terms*—Parameterized model order reduction (PMOR), quasi-convex optimization, RF inductor.**

I. Introduction

DEVELOPING parameterized model order reduction (PMOR) algorithms would allow digital, mixed-signal, and RF analog designers to promptly instantiate field solver accurate small models for their parasitic dominated components (interconnect, RF inductors, microelectromechanical resonators, etc.). The existing PMOR techniques are based either on statistical performance analysis [1]–[5] or on moment matching [6]–[14]. Some non-PMOR or identification techniques based on an optimization approach are present in literature. References [15] and [16] identify systems from sampled data by essentially solving the Yule–Walker equation derived from a linear least squares problem. However, these methods might not be satisfactory, since the objective of their minimization is not the norm of the difference between the original and reduced transfer functions but rather the same quantity multiplied by the denominator of the reduced model. References [17] and [18] directly formulate the model reduction problem as a rational fit, minimizing the $\mathcal{H}_2$ norm error, and therefore, they solve a nonlinear least squares problem, which is not convex. To address the problem, those papers propose solving linear least squares iteratively, but it is not clear whether the procedure will converge and whether they can handle additional constraints such as positive realness passivity. In order to reduce positive real systems, the authors of [19] propose the use of the KYP lemma/semidefinite programming relationship [20] and show that the reduction problem can be cast into a semidefinite program, if the poles of the reduced models are given *a priori*. Reference [21] uses a different result derived from [22] to check positive realness. In that procedure, a set of scalar inequalities evaluated at some frequency points are checked. Reference [21] then suggests an iterative scheme that minimizes the $\mathcal{H}_2$ norm of the error system for the frequency points given in the previous iteration. However, this scheme does not necessarily generate optimal reduced models since, in order to do that, both the system model and the frequency points should be considered as decision variables. In short, the available methods lack one or more of the following desirable properties: rational fit, guaranteed stability, and passivity, convexity, optimality, or flexibility to impose constraints.

In principle, the method proposed in this paper is a rational-approximation-based model reduction framework but with the following three distinctions.

1) Instead of solving the model reduction directly, the proposed methodology solves a relaxation of it.
2) The objective function to be minimized is not the $\mathcal{H}_2$ norm but rather, the $\mathcal{H}_\infty$ norm. As it turns out, the resultant optimization problem, as described in Section III, is equivalent to a quasi-convex program, i.e., an optimization of a quasi-convex function (all sublevel sets are convex sets) over a convex set. This property implies the following: 1) there exists a unique optimal solution to the problem and 2) there exist polynomial-time algorithms for its solution. In addition, since the proposed method involves only a single optimization problem, it is near optimal with respect to the objective function used ($\mathcal{H}_\infty$ norm of error).
3) In addition to the aforementioned benefits, it will be demonstrated in this paper that some commonly encountered constraints or additional objectives can be added to the proposed optimization setup *without* significantly increasing the complexity of the problem. Among these features are guaranteeing stability, positive realness (passivity of impedance systems), bounded realness (passivity of scatter-parameter systems), and quality factor-error minimality. In addition, the optimization setup can be modified to generate an optimal parameterized reduced model that is stable for the range of parameters of interest.

The rest of this paper is organized as follows. Section II provides some background. Section III describes the proposed relaxation and explains why it is quasi-convex after a change of decision variables. Section IV gives an overview of the setup of the proposed method and some detail of it. Section V demonstrates how to modify the basic optimization setup to incorporate various desirable constraints. Section VI focuses on

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the extension of the optimization setup to the case of PMOR. In Section VII, more design-oriented modifications will be discussed. As a special case, the RF inductor design algorithm will be given. In Section VIII, the complexity of the proposed algorithm is analyzed. In Section IX, several application examples are shown to evaluate the practical value of the proposed method in terms of accuracy and complexity.

II. Background

A. Tustin Transform and Model Reduction

In order to work with polynomials of transfer functions in a numerically reliable way, the following procedure will be employed throughout this paper. Given a continuous-time (CT) system with transfer matrix $H(s)$, a standard technique borrowed from the control system community for model reduction is employed. First, apply a Tustin transform (see, for example, [23]) $s = \lambda(z-1)/(z+1)$ to construct an equivalent discrete-time (DT) system; then, reduce the DT system; and finally, convert it back to CT by applying the inverse transform $z = (\lambda + s)/(\lambda - s)$. The frequency responses of the CT and DT systems are the frequency axis scaled versions of each other. No aliasing occurs when using the Tustin transform. In addition, since the Tustin transform preserves the order of models, the orders of reduced models remain the same.

The choice of the center frequency λ is somewhat arbitrary. While it is true that extreme choices (e.g., picking the center frequency to be 1 Hz, while the frequency range of interest is at 1 GHz) can be harmful for the proposed model-reduction framework, numerical experiments have shown that a broad choice of center frequencies would allow the proposed framework to work without suffering any CT/DT conversion problem. In addition, in our implementation, we employ an automatic procedure that chooses the center frequency by minimizing the maximum slope of the magnitude of the frequency response, hence, completely avoiding any possibly numerically harmful extreme situations.

B. Optimal $\mathcal{H}_\infty$ Norm Model-Reduction Problem

One of the desirable model reduction problems is the $\mathcal{H}_\infty$ norm optimization. Given a stable transfer function $H(z)$ (possibly of large or even infinite order) and an integer m (indicating the order of the reduced model), construct a stable rational transfer function

$$\hat{H}(z) = \frac{p(z)}{q(z)}$$

such that the order of $\hat{H}(z)$ is less than or equal to m, and the error $\|H(z) - \hat{H}(z)\|_\infty$ is minimized

$$\begin{aligned} \underset{p,q}{\text{minimize}} \quad & \left\| H(z) - \frac{p(z)}{q(z)} \right\|_\infty \\ \text{subject to} \quad & \deg(q) = m, \qquad \deg(p) \le m \\ & q(z) \neq 0 \qquad \forall |z| \ge 1 \quad \text{(stability)}. \end{aligned} \tag{1}$$

Unfortunately, (1) is not convex, and it is not known whether it is NP-hard or not. In other words, existence of an efficient algorithm for solving (1) is still an open question.

C. Relaxation of an Optimization Problem

A relaxation of an optimization problem is a related optimization problem such that an optimal solution to the original problem is a feasible solution to the relaxation. A relaxation can be introduced if it is much easier to solve, and the optimal solution to the relaxation is useful in constructing a reasonably good feasible solution to the original problem. However, note that such feasible solution might not be, in general, an optimal solution to the original problem. Typical ways for obtaining a relaxation include enlarging the feasible set and/or replacing the objective function with another (easier to optimize) function, whose sublevel set contains that of the original. It will be shown later in this paper that the same relaxation ideas are useful in simplifying the proposed model reduction problem.

III. Relaxation-Scheme Setup

This section describes the main theory of the proposed model reduction framework.

A. Relaxation of the $\mathcal{H}_\infty$ Norm Optimization

Motivated by the Hankel optimal model reduction [24], the following relaxation of the optimal $\mathcal{H}_\infty$ norm model reduction was proposed in [25]:

$$\begin{aligned} \underset{p,q,r}{\text{minimize}} \quad & \left\| H(z) - \frac{p(z)}{q(z)} - \frac{r(1/z)}{q(1/z)} \right\|_\infty \\ \text{subject to} \quad & \deg(q) = m, \quad \deg(p) \le m, \quad \deg(r) < m \\ & q(z) \text{ is a Schur polynomial (stability)}. \end{aligned} \tag{2}$$

In (2), an anti-stable rational part $(r(1/z)/q(1/z))$ is added to the setup of (1), and because of these extra decision variables, (2) is a relaxation of (1). Aft er solving (2), a (suboptimal) stable reduced model can simply be obtained as $\hat{H}(z) = (p(z)/q(z))$. The following lemma, which is from [25], gives an error bound of the relaxation.

Lemma 3.1: Let (p^*, q^*, r^*) be the optimal solution of (2) with reduced order m

$$\gamma^* = \left\| H(z) - \frac{p^*(z)}{q^*(z)} - \frac{r^*(1/z)}{q^*(1/z)} \right\|_\infty$$

$$\hat{H}(z) := \frac{p^*(z)}{q^*(z)}$$

be a stable reduced model, then

$$\min_{D \in \mathbb{R}} \left\{ \left\| H(z) - \hat{H}(z) - D \right\|_\infty \right\} \le (m+1)\gamma^*. \tag{3}$$

By definition, γ^* is a lower bound of the error of the optimal $\mathcal{H}_\infty$ norm model reduction problem (1), and Lemma 3.1 states that the suboptimal reduced model provided by the proposed framework has an error upper bound $(m+1)$ times its error lower bound γ^*. In the lemma, $\hat{H}(z) := (p(z)/q(z))$ is the outcome in solving (2), or (9) that is to be discussed in the next section. It should be noted that the scalar D in (3) can be incorporated into the reduced model $\hat{H}$, if $\hat{H}$ is not a strictly proper transfer function. Therefore, the reduced model should really be understood as $\hat{H}(z) + D^*$, where D^* is chosen to be the optimizing D. In Section IV, (10) will be discussed to construct a reduced model that always picks the optimizing D.

B. Change of Decision Variables in the Relaxation Scheme

It is not convenient to directly work with (2) as the set of the coefficients of the polynomials

$$\begin{aligned}\Omega^m_{qpr} := \Big\{ & (\vec{q},\vec{p},\vec{r}) \in \mathbb{R}^m \times \mathbb{R}^{m+1} \times \mathbb{R}^m : \\ & q(z) = z^m + \vec{q}_{m-1} z^{m-1} + \cdots + \vec{q}_1 z + \vec{q}_0 \\ & p(z) = \vec{p}_m z^m + \vec{p}_{m-1} z^{m-1} + \cdots + \vec{p}_1 z + \vec{p}_0 \\ & r(z) = \vec{r}_{m-1} z^{m-1} + \vec{r}_{m-2} z^{m-2} + \cdots + \vec{r}_1 z + \vec{r}_0 \\ & q(z) \neq 0 \qquad \forall\, z \in \mathbb{C} : |z| \geq 1 \Big\} \end{aligned} \tag{4}$$

is not convex if $m > 2$. To overcome the problem, the following set of decision variables is proposed:

$$\begin{aligned} & \Omega^m_{abc} := \\ & \Big\{ (\vec{a},\vec{b},\vec{c}) \in \mathbb{R}^m \times \mathbb{R}^{m+1} \times \mathbb{R}^m : \\ & \ a(z) = (z^m + z^{-m}) + \vec{a}_{m-1}(z^{m-1} + z^{-m+1}) + \cdots + \vec{a}_0 \\ & \ b(z) = \vec{b}_m(z^m + z^{-m}) + \vec{b}_{m-1}(z^{m-1} + z^{-m+1}) + \cdots + \vec{b}_0 \\ & \ c(z) = \frac{1}{j}\left(\vec{c}_m(z^m - z^{-m}) + \cdots + \vec{c}_1(z - z^{-1})\right) \\ & \ \text{satisfying} \quad a(z) > 0 \qquad \forall\, z \in \mathbb{C} : |z| = 1 \Big\}. \end{aligned} \tag{5}$$

The sets Ω^m_{pqr} and Ω^m_{abc} are equivalent in the sense summarized by the following lemma.

Lemma 3.2: There exists a one-to-one map $\tau_m : \Omega^m_{qpr} \mapsto \Omega^m_{abc}$, which is defined as follows.

- Given $(\vec{q},\vec{p},\vec{r}) \in \Omega^m_{qpr}$, $(\vec{a},\vec{b},\vec{c}) = \tau_m(\vec{q},\vec{p},\vec{r}) \in \Omega^m_{abc}$ is defined by

$$\begin{aligned} a(z) &= |q(z)|^2 \\ b(z) &= \mathrm{Re}\left\{p(z)q(z^{-1})\right\} \\ c(z) &= \mathrm{Im}\left\{p(z)q(z^{-1})\right\}. \end{aligned}$$

- Given $(\vec{a},\vec{b},\vec{c}) \in \Omega^m_{abc}$, $(\vec{q},\vec{p},\vec{r}) = \tau^{-1}(\vec{a},\vec{b},\vec{c}) \in \Omega^m_{qpr}$ is defined as follows: Let $k \in \{1, 2, \ldots, 2m\}$ and z_k be the (maybe repeated) roots of

$$z^m a(z) = 0$$

  then, $\vec{q}$ is found by

$$q(z) = \prod_{k:|z_k|<1} (z - z_k) \tag{6}$$

  and $\vec{p}$ and $\vec{r}$ are uniquely found by the relation

$$p(z)q(z^{-1}) + q(z)r(z^{-1}) = b(z) + jc(z). \tag{7}$$

Then, the map τ satisfies the following property:

$$H(e^{j\omega}) = \frac{p(e^{j\omega})}{q(e^{j\omega})} + \frac{r(e^{-j\omega})}{q(e^{-j\omega})} = \frac{b(e^{j\omega}) + jc(e^{j\omega})}{a(e^{j\omega})} \qquad \forall \omega. \tag{8}$$

Proof: See the Appendix. ■

Lemma 3.2 implies that the stability constraint $q(z) \neq 0, \forall\ z \in \mathbb{C} : |z| \geq 1$ in (4), which makes the feasible set of (1) nonconvex, can be replaced by the easier to handle (to be shown) positivity constraint $a(z) > 0, \forall\ z \in \mathbb{C} : |z| = 1$, and this paves the way to the discovery of efficient algorithms for solving the relaxation problem. With the change of variables given by the previous lemma and by applying the identity $e^{j\omega} = \cos(\omega) + j\sin(\omega)$, (2) can equivalently be formulated as

$$\begin{aligned} \underset{\tilde{a},\tilde{b},\tilde{c},\gamma}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \left| H(e^{j\omega})\tilde{a}(\omega) - \tilde{b}(\omega) - j\tilde{c}(\omega) \right| < \gamma \tilde{a}(\omega) \\ & \tilde{a}(\omega) > 0 \qquad \forall\, \omega \in [0, \pi] \\ & \deg(\tilde{a}) = m, \deg(\tilde{b}) \leq m, \deg(\tilde{c}) \leq m \end{aligned} \tag{9}$$

with

$$\begin{aligned} \tilde{a}(\omega) &= 1 + \tilde{a}_1\cos(\omega) + \cdots + \tilde{a}_m\cos(m\omega) \\ \tilde{b}(\omega) &= \tilde{b}_0 + \tilde{b}_1\cos(\omega) + \cdots + \tilde{b}_m\cos(m\omega) \\ \tilde{c}(\omega) &= \tilde{c}_1\sin(\omega) + \cdots + \tilde{c}_m\sin(m\omega). \end{aligned}$$

Lemma 3.3: Program (9) is quasi-convex (i.e., optimizing a quasi-convex function over a convex set).

Proof: See the Appendix. ■

IV. Model-Reduction Setup

This section deals with the solution procedure of the proposed model reduction framework. A summary of the procedure is given as follows.

Algorithm 1: MOR
Input: $H(z)$
Output: $\hat{H}(z)$

i. Solve program (9) using a cutting plane algorithm (details in Section IV-A) to obtain the relaxation solution $(\tilde{a}, \tilde{b}, \tilde{c})$.
ii. Compute the denominator $q(z)$ using spectral factorization (6). This is explained in Section IV-B.
iii. Solve a convex optimization problem to obtain the numerator $p(z)$ (see Section IV-B).
iv. Synthesize a state space realization of the reduced model $\hat{H}(z) = p(z)/q(z)$ (see [23] for details).

Steps i. and iii. in the earlier algorithm description are more involved, and they are explained as follows.

A. Solving the Relaxation

Program (9) is a large-scale (with infinitely many constraints) quasi-convex program, and it can be solved by a localization/cutting plane strategy. Note that the localization method is a standard optimization technique, and it is given here for completeness. Suppose the objective of the program is to find a point in a target set X (e.g., the suboptimal level set of a minimization problem) or to verify that X is empty. The basic algorithm of the localization scheme is as follows.

a. At each step k, maintain a localization set $\mathcal{P}_k : X \subset \mathcal{P}_k$.
b. Compute a query point, namely, the vector of the current trial of the decision variables $x_k \in \mathcal{P}_k$, and check if $x_k \in X$.
c. If $x_k \in X$, then terminate the algorithm and successfully return x_k. Otherwise, return a "cut" (e.g., a hyperplane)

such that all points in X must be in one side of the hyperplane (i.e., a halfspace). Denote the corresponding halfspace as $\mathcal{H}$.

d. Update the localization set to $\mathcal{P}_{k+1} : \mathcal{P}_k \cap \mathcal{H} \subset \mathcal{P}_{k+1}$.

e. If volume $(\mathcal{P}_{k+1}) < \varepsilon$, for some small ε (which, for instance, is determined by the desired suboptimality level), then assert that X is empty and terminate the algorithm. Otherwise, go back to step a.

The choice of the localization set $\mathcal{P}_k$ and the query point x_k distinguishes one method from another. Reasonable choice of localization set/query point can be as follows: 1) a covering ellipsoid/center of the ellipsoid or 2) covering polytope/analytic center of the polytope. The former choice results in the ellipsoid algorithm (see [26] or [27] for detailed reference), while the latter choice results in the analytic center cutting plane method (see [28] for reference).

Finding $\mathcal{P}_1 : X \subset \mathcal{P}_1$ is problem-dependent and it will be discussed in Section V in the context of MOR.

The particular implementation in step c. is problem-dependent, and the subroutine used in such step is commonly referred to as an "oracle." Different oracles pertaining different model reduction constraints will be discussed in detail in Section V.

B. Constructing the Reduced Model

The denominator $q(z)$ and the numerator $p(z)$ of the reduced model could be found by applying (6) and (7) in Lemma 3.2. However, the following more practical procedure yields a reduced model whose approximation quality is no worse than that obtained with (7). Once $q(z)$ is found, calculate $p(z)$ as the optimal solution to the following:

$$\begin{aligned} &\underset{p,\gamma}{\text{minimize}} \quad \gamma \\ &\text{subject to} \quad \left| H(e^{j\omega}) - \frac{p(e^{j\omega})}{q(e^{j\omega})} \right| < \gamma, \qquad \forall\, \omega \in [0, \pi], \\ &\qquad\qquad\quad \deg(p) \leq m. \end{aligned} \tag{10}$$

Note that (10) is convex and can be solved by a localization method. In addition, note that, since the degree of the numerator p can be m, the transfer function is not strictly proper, and the optimal constant term D in (3) is automatically chosen when (10) is solved.

C. Obtaining Models of Increasing Orders

In the proposed model reduction framework, the information from an order m model reduction can relatively cheaply be reused to find the reduced models of order $m+k$ (with $k>0$). We describe here the update procedure for order $m+1$ reduced model (the procedure for higher order reduced models is the same). Suppose $(\tilde{a}^*_m, \tilde{b}^*_m, \tilde{c}^*_m)$ is the optimal trigonometric polynomials for order m reduction and assume the corresponding error is γ^*_m, then

$$\frac{\tilde{b}^*_m(\omega) + 0 \cdot \cos((m+1)\omega) + j\,(\tilde{c}^*_m(\omega) + 0 \cdot \sin((m+1)\omega))}{\tilde{a}^*_m(\omega) + 0 \cdot \cos((m+1)\omega)}$$

is automatically a valid (stable, passive, etc.) candidate for the order $m+1$ reduction problem. Therefore, it can be used as the initial center of the localization set (e.g., covering ellipsoid) for the $m+1$-order problem. The localization set for the $m+1$-order problem can also be inherited from that of the order m problem by appending the previous localization set in the following way. Let x_m be the vector of decision variables of the order m problem, x^*_m be coefficients of the optimal trigonometric polynomials $(\tilde{a}^*_m, \tilde{b}^*_m, \tilde{c}^*_m)$ of order m, and P^*_m be the symmetric positive semidefinite matrix that defines the ellipsoid of the order m localization set, then

$$(x_m - x^*_m)'\, P^*_m\, (x_m - x^*_m) \leq 1.$$

Now, let x^{m+1}_a, x^{m+1}_b, and x^{m+1}_c be the coefficients of the $m+1$-degree terms in the $m+1$-degree trigonometric polynomials of the $m+1$-order reduction problem. If there exists some $M>0$ s.t. $|x^{m+1}_a| < M$, $|x^{m+1}_b| < M$, $|x^{m+1}_c| < M$, then

$$(x_m - x^*_m)' P^*_m (x_m - x^*_m) + |x^{m+1}_a|^2 + |x^{m+1}_b|^2 + |x^{m+1}_c|^2 \leq 1 + 3M^2$$

can be used as the initial ellipsoid (i.e., localization set) for the $m+1$ model reduction problem. The order m optimal objective value γ^*_m can be used as the initial objective value when the $m+1$-order procedure starts. Using these initial iterates for the $m+1$-order problem, relatively few cuts will be required to obtain the $m+1$-order optimal trigonometric polynomials.

V. Constructing Oracles

Nontrivial oracles will be discussed in this section.

A. Stability: Positivity Constraint

From Lemma 3.2, it can be seen that the positivity constraint $\tilde{a}(\omega) > 0$ in (9) is equivalent to the stability constraint in (2) requiring $q(z)$ to be a Schur polynomial. Therefore, the positivity constraint must be strictly imposed for all ω ranging from zero to π, and therefore, the common engineering practice of enforcing such constraint on only a finite set of points in that interval will not suffice. In order to address this issue, consider the positivity constraint

$$\tilde{a}(\omega) = 1 + a_1 \cos(\omega) + \cdots + a_m \cos(m\omega) > 0 \qquad \forall\, \omega \in [0, \pi]. \tag{11}$$

It is sufficient to check whether

$$\min_{\omega \in [0,\pi]} \tilde{a}(\omega) > 0.$$

Since $\tilde{a}(\omega)$ is continuous over $[0, \pi]$, the minimum is attained, and it can only be at the roots of

$$\frac{d\tilde{a}(\omega)}{d\omega} = -\tilde{a}_1 \sin(\omega) - \cdots - m\tilde{a}_m \sin(m\omega) = 0 \tag{12}$$

as the boundary points are included with

$$\frac{d\tilde{a}(0)}{d\omega} = \frac{d\tilde{a}(\pi)}{d\omega} = 0.$$

If there exists ω_0 among the roots of (12) s.t. $\tilde{a}(\omega_0) \leq 0$, then $\tilde{a}(\omega_0) > 0$ defines a cut; otherwise, the positivity constraint is met.

In order to find the roots of (12), the identity $e^{j\omega} = \cos(\omega) + j\sin(\omega)$ can be applied to (12)

$$\begin{aligned}\frac{d\tilde{a}(\omega)}{d\omega} &= -\frac{1}{2j}\left(a_1(z - z^{-1}) + \cdots + ma_m(z^m - z^{-m})\right)\\ &:= \partial\tilde{a}(z)\\ &= 0\end{aligned}$$

with $z = e^{j\omega}$. Note that $z^m\partial\tilde{a}(z)$ is an ordinary polynomial of degree $2m$ and $e^{j\omega} \neq 0, \forall\, \omega \in \mathbb{R}$. Therefore, any ω_0 is a root of (12) if and only if it is a root of $\partial\tilde{a}(e^{j\omega})$, and the root finding task can be performed by finding (unit circle) roots of an ordinary polynomial $z^m\partial\tilde{a}(z)$ of degree $2m$.

B. Passivity for Impedance Systems: Positive Real Constraint

For some applications, it is desirable that the reduced model transfer function has positive real part. In order to impose this constraint, it suffices to note that the real part of the transfer-function approximation in (9) is $\tilde{b}(\omega)/\tilde{a}(\omega)$. Therefore, the only modification to (9) is to add the constraint

$$\tilde{b}(\omega) > 0, \qquad \forall\, \omega \in [0, \pi]$$

and the treatment of this oracle is similar to that of the positivity constraint discussed in Section V-A.

However, it should be noted that (10) should accordingly be modified to guarantee the positive realness of the final reduced model. That is, the following constraint should be added.

$$p(e^{j\omega})q(e^{-j\omega}) + p(e^{-j\omega})q(e^{j\omega}) > 0, \qquad \forall\, \omega \in [0, \pi]. \quad (13)$$

It is important to realize that (13) is linear with respect to the decision variable $p(z)$, and the left side defines a trigonometric polynomial.

C. Passivity for S-Parameter Systems: Bounded Real Constraint

For S-parameter models, the notion of dissipative system is given by the bounded real condition (i.e., $|H(z)| < 1, \forall\, z \in \mathbb{C}$, $|z| = 1$). To model this property, (9) can be modified by adding the constraint

$$\tilde{a}(\omega) > |\tilde{b}(\omega) + j\tilde{c}(\omega)|, \qquad \forall\, \omega \in [0, \pi].$$

To construct the oracle, first check the positivity of the trigonometric polynomial

$$\tilde{a}(\omega)^2 - \tilde{b}(\omega)^2 - \tilde{c}(\omega)^2 > 0, \qquad \forall\, \omega \in [0, \pi].$$

If this condition is met, then bounded realness is satisfied at the current query point; otherwise, if the constraint is violated for some $\omega_0 \in [0, \pi]$, then the constraint

$$\tilde{a}(\omega_0) > \left|\tilde{b}(\omega_0) + j\tilde{c}(\omega_0)\right|$$

defines a cut. It is noted that (10) should be modified similarly to preserve the passivity of the final reduced model.

D. Multiport Positive Real Passivity

For a multiport transfer matrix $H(z) \in \mathbb{C}^{n\times n}$ with real coefficients, positive real passivity means

$$H(e^{j\omega}) + H(e^{j\omega})' > 0, \qquad \forall\, \omega \in [0, \pi], \qquad (14)$$

with $'$ denoting complex conjugate transpose. The following is a procedure to construct the multiport positive real passivity oracle.

Let

$$\begin{aligned}x[k+1] &= Ax[k] + Bu[k]\\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

be a state space realization of $H(z)$ and define the 2×2 block matrix

$$\Sigma := \begin{bmatrix} 0 & C' \\ C & D + D' \end{bmatrix} := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}. \qquad (15)$$

Consider the generalized eigenvalue problem

$$z\begin{bmatrix} -\Sigma_{11} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & A' + \Sigma_{12}\Sigma_{22}^{-1}B' \\ -I & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ -A + B\Sigma_{22}^{-1}\Sigma_{21} & -B\Sigma_{22}^{-1}B' \end{bmatrix} = 0. \quad (16)$$

The following Lemma presents the oracle construction procedure.

Lemma 5.1: If the generalized eigenvalue problem [(16)] does not have any eigenvalue on the unit circle, then (14) is satisfied. Otherwise, there exists $\omega_0 \in [0, \pi]$, such that $e^{j\omega_0}$ is an eigenvalue of (16) and $H(e^{j\omega_0}) + H(e^{j\omega_0})' \not> 0$. In this case, if $v_0 \in \mathbb{C}^n$ is an eigenvector associated with a nonpositive eigenvalue of $H(e^{j\omega_0}) + H(e^{j\omega_0})'$, then

$$v_0'\left(H(e^{j\omega_0}) + H(e^{j\omega_0})'\right)v_0 > 0 \qquad (17)$$

defines a (real coefficient) linear cut with respect to the coefficients of the numerator of H.

Proof: See the Appendix. ■

E. Objective Oracle

In the case where the transfer function H of the original system is explicitly fully specified (in terms of system matrices, numerator/denominator, or pole/zero/gain) and the exact $\mathcal{H}_\infty$ norm is to be minimized, one can use the following oracle. Given the current iterates $(\tilde{a}, \tilde{b}, \tilde{c})$ and the desired level of optimality γ, an unstable transfer function

$$\hat{H}(e^{j\omega}) := \frac{\tilde{b}(\omega) + j\tilde{c}(\omega)}{\tilde{a}(\omega)}$$

can be realized. Then, the difference system $H - \hat{H}$ can be formed to check (e.g., using MATLAB function `norm`) if its

$\mathcal{L}_\infty$ norm (similar to $\mathcal{H}_\infty$ norm but not restricted to stable systems) is less than γ. If the corresponding $\mathcal{L}_\infty$ norm is not smaller than γ, then a violating frequency ω_0 can be identified, and the cut

$$\left|\tilde{b}(\omega_0) + j\tilde{c}(\omega_0) - \tilde{a}(\omega_0)H(\omega_0)\right| < \gamma\tilde{a}(\omega_0)$$

can be enforced.

In the case where the transfer function H of the original system is specified as sample data $(\omega_i, H(\omega_i))$, where $i = 1, 2, \ldots, N$, the $\mathcal{L}_\infty$ norm check of the difference $H - \hat{H}$ can be simplified to checking N inequalities.

Finally, if the original transfer function H is again explicitly given (e.g., system matrices) but the $\mathcal{L}_\infty$ norm oracle aforementioned is deemed too expensive to compute, the frequency response of H can be sampled and the proposed algorithm still applies (although the $\mathcal{H}_\infty$ norm error is no longer guaranteed). Uniform sampling of the DT frequency axis over the range of interest is generally a good choice for the proposed algorithm.

VI. Extension to PMOR

This section discusses how the setup in (9) can be extended to solve the problem of PMOR (i.e., to construct a parameter-dependent model, stable for all parameter values of interest). In particular, the positivity (stability) oracle will be discussed. The main idea of the oracle construction is to relate the following three mathematical objects: positive multivariate trigonometric polynomial, finite sum of squares (SOS) of multivariate trigonometric polynomials, and (convex) SOS programming.

Enabling PMOR capability of (9) means allowing the coefficients of the trigonometric polynomials to be parameter-dependent. That is

$$a(z,\mathbf{p}) = a_0(\mathbf{p})+a_1(\mathbf{p})(z+z^{-1})+\cdots+a_m(\mathbf{p})(z^m+z^{-m})$$

$$b(z,\mathbf{p}) = b_0(\mathbf{p})+b_1(\mathbf{p})(z+z^{-1})+\cdots+b_m(\mathbf{p})(z^m+z^{-m})$$

$$c(z,\mathbf{p}) = \frac{1}{j}\left(c_1(\mathbf{p})(z-z^{-1})+\cdots+c_m(\mathbf{p})(z^m-z^{-m})\right)$$

where $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^{n_\mathbf{p}}$. That is, $\mathbf{p}$ denotes the vector of design parameters whose values vary in the set $\mathcal{P}$. The parameterized version of (9) is

$$\begin{aligned} \underset{\tilde{a},\tilde{b},\tilde{c},\gamma}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \left|H(e^{j\omega},\mathbf{p})\tilde{a}(\omega,\mathbf{p})-\tilde{b}(\omega,\mathbf{p})-j\tilde{c}(\omega,\mathbf{p})\right| \\ & < \gamma\tilde{a}(\omega,\mathbf{p}) \qquad \forall\,\omega \in [0,\pi] \qquad \forall\,\mathbf{p} \in \mathcal{P} \\ & \tilde{a}(\omega,\mathbf{p}) > 0 \qquad \forall\,\omega \in [0,\pi], \mathbf{p} \in \mathcal{P} \\ & \deg(\tilde{a}) = m, \quad \deg(\tilde{b}) \le m, \quad \deg(\tilde{c}) \le m. \end{aligned} \tag{18}$$

The key problem with the application of the localization methods to (18) is that the positivity oracle is different from the nonparameterized case if $\mathcal{P}$ is not a finite set. In general, it would be very difficult to satisfy the parameterized positivity constraint if the parameter dependence of $\tilde{a}$ on $\mathbf{p}$ is arbitrary. However, if the dependence is polynomial

$$\tilde{a}(\omega,\mathbf{p}) = \sum_k \left(\sum_i c_{i_1,i_2,\ldots,i_{n_\mathbf{p}}}\mathbf{p}_1^{i_1}\mathbf{p}_2^{i_2},\ldots,\mathbf{p}_{n_\mathbf{p}}^{i_{n_\mathbf{p}}}\right)\cos(k\omega) \tag{19}$$

then it can be shown that a sufficient condition for positivity can be found using a SOS argument. The resulting oracle will involve the solving of a semidefinite program, which can be performed in polynomial time. The underlying reason is as follows. If $\mathcal{P}$ is bounded (with $\underline{\mathbf{p}}_k$ and $\overline{\mathbf{p}}_k$ being the lower and upper bounds for the kth parameter, respectively), then $\mathbf{p}_k = (\overline{\mathbf{p}}_k + \underline{\mathbf{p}}_k + (\overline{\mathbf{p}}_k - \underline{\mathbf{p}}_k)\cos(\omega_k))/2$, and therefore, $\tilde{a}(\omega,\mathbf{p})$ is a multivariate trigonometric polynomial. The following theorem, from [29], provides a sufficient condition for the positivity of $\tilde{a}(\omega,\mathbf{p})$.

Theorem 6.1: A trigonometric polynomial is positive if and only if it is a finite SOS of trigonometric polynomials.

Although the statement in the earlier theorem is sufficient and necessary, care must be taken to apply the necessary direction of the theorem. Specifically, to apply this direction, one must consider the set of all finite SOS, which is impossible to characterize. Nevertheless, the theorem does provide a guideline to efficiently construct an oracle, since for every finite SOS of trigonometric polynomials, there exists a quadratic form $\theta' Q\theta$, where θ is a vector of appropriately chosen monomials and $Q = Q' \ge 0$ (see [30] and [31]).

Searching for $P = P' \ge 0 : \theta' P\theta = \tilde{a}(\omega,\mathbf{p})$ is a semidefinite programming feasibility problem. The positivity oracle that uses this search is summarized as follows.

Algorithm 2: PMOR POSITIVITY ORACLE
Input: query point $\tilde{a}$
Output: declaration of constraint met, or a cut $(\alpha,\beta) : \alpha' x_a > \beta$, for all vector of coefficients x_a of positive trigonometric polynomials

i. Given trigonometric polynomial $\tilde{a}(\omega, p)$, pick a vector of monomials $\theta \in \mathbb{C}^k$, for instance, as guided by [30] and [31].
ii. Solve the semidefinite program

$$\begin{aligned} \underset{y\in,\mathbb{R},P}{\text{minimize}} \quad & y \\ \text{subject to} \quad & \theta' P\theta = \tilde{a}(\omega,\mathbf{p}) + y \quad \forall\,\omega \in [0,\pi] \quad \forall\,\mathbf{p} \in \mathcal{P} \\ & P = P' \ge 0. \end{aligned} \tag{20}$$

iii. **if** (20) is feasible and optimal $y^* < 0$,
  return Positivity constraint is met
 else
  return Cut (α,β) constructed using the information of the solution to (20).

The following lemma certifies the correctness of the oracle and gives a constructive proof of the existence of (α,β).

Lemma 6.1: If (20) is feasible and the optimal value $y^* < 0$, then $\tilde{a}(\omega,\mathbf{p}) > 0, \forall\omega \in [0,\pi], \mathbf{p} \in \mathcal{P}$. Otherwise, a cut $(\alpha,\beta) \in \mathbb{R}^{n_a} \times \mathbb{R}$ can be returned. The cut has the following property: $\alpha' x_a > \beta$ for all $x_a \in \mathbb{R}^{n_a}$ such that the optimal objective value of (20) is negative.

Proof: See the Appendix. ∎

It must be noted that the cut returned by (20) is restrictive in the sense that it eliminates all the candidates x_a that do not result in $y^* < 0$, while some of which can still produce positive trigonometric polynomials. Nevertheless, it is generally true that this is not too conservative if the vector of monomials is properly chosen.

While the specific construction of the positivity constraint oracle in Lemma 6.1 requires the dependence of $\tilde{a}$ on the design parameter to be polynomial, there is no restriction in the dependence of $\tilde{b}$ and $\tilde{c}$, and they can be chosen to best fit the problem at hand. Finally, it is noted that (20) can be solved using free semidefinite programming solvers such as SeDuMi [32].

The details of how to construct (20) and (34), used in the proof of Lemma 6.1 in the Appendix, will be shown in the Appendix through the special case in which two design parameters are allowed.

VII. Additional Modifications Based on Designers' Need

We show here that the proposed Algorithm 1 (MOR), which is given in Section IV, and Algorithm 2 (PMOR), which is given in Section VI, are quite flexible, and they can serve as a basic framework which can easily be modified to account for several additional desirable constraints devised, for instance, from a designer's knowledge about the specific system to be modeled.

A. Explicit Approximation of Quality Factor

When the transfer function H is, for instance, the impedance of an RF inductor, the accurate representation of the quality factor

$$Q(\omega) := \frac{\mathrm{Im}\left(H(e^{j\omega})\right)}{\mathrm{Re}\left(H(e^{j\omega})\right)}, \qquad \omega \in [0, \pi]$$

is of critical importance for the designers in order to evaluate the system performance. In this case, the basic problem in (9) can be modified to guarantee a very good quality factor accuracy

$$\begin{aligned}
&\underset{\tilde{a},\tilde{b},\tilde{c},\gamma}{\text{minimize}} \quad \gamma \\
&\text{subject to} \quad \left| H(e^{j\omega})\tilde{a}(\omega) - \tilde{b}(\omega) - j\tilde{c}(\omega) \right| < \gamma \tilde{a}(\omega) \\
&\qquad \left| \frac{\mathrm{Im}\left(H(e^{j\omega})\right)}{\mathrm{Re}\left(H(e^{j\omega})\right)} \tilde{b}(\omega) - \tilde{c}(\omega) \right| < \rho\gamma\tilde{b}(\omega) \\
&\qquad \tilde{a}(\omega) > 0 \quad \tilde{b}(\omega) > 0 \qquad \forall\, \omega \in [0, \pi] \\
&\qquad \deg(\tilde{a}) = m,\ \deg(\tilde{b}) \le m,\ \deg(\tilde{c}) \le m.
\end{aligned} \tag{21}$$

ρ in the second set of the constraint is a tuning parameter of the relative accuracy between match on frequency response and on quality factor. The oracles for (21) are similar to those for (9). The positive real part constraint and the reduced model should be constructed using

$$\begin{aligned}
&\underset{p,\gamma}{\text{minimize}}\ \gamma \\
&\text{subject to} \left| H(e^{j\omega}) - \frac{p(e^{j\omega})}{q(e^{j\omega})} \right| < \gamma \qquad \forall\, \omega \in [0, \pi] \\
&\left| \frac{\mathrm{Im}\big(H(e^{j\omega})\big)}{\mathrm{Re}\big(H(e^{j\omega})\big)} - \frac{p(e^{j\omega})q(e^{-j\omega}) - p(e^{-j\omega})q(e^{j\omega})}{p(e^{j\omega})q(e^{-j\omega}) + p(e^{-j\omega})q(e^{j\omega})} \right| < \rho\gamma \\
&p(e^{j\omega})q(e^{-j\omega}) + p(e^{-j\omega})q(e^{j\omega}) > 0 \qquad \forall\, \omega \\
&\deg(p) \le m.
\end{aligned} \tag{22}$$

Again, this program is quasi-convex, and the oracle procedure with (13) can also be applied here.

B. Weighted Frequency Response Setup

In some applications, the desired approximation accuracy is different in different frequency ranges. For those applications, the objective function of (9) can be replaced by

$$\left\| W(z)\left(H(z) - \hat{H}(z)\right) \right\|_\infty$$

where $W(z)$ are weights that can be chosen to be larger for the "more important" frequency range.

C. Matching of Frequency Samples

Program (9) can be modified so that the reduced transfer function matches exactly the original transfer function at some particular frequencies ω_k between zero and π. In order to do this, equality constraints such as

$$H(e^{j\omega_k})\tilde{a}(\omega_k) - \tilde{b}(\omega_k) - j\tilde{c}(\omega_k) = 0 \qquad \forall\, k$$

can be imposed. Similarly, (10) can be modified to make sure that the final reduced model matches the full model at those frequencies. Besides the intended use of exact sample matching, this modification has the practical meaning of reducing the number of optimization decision variables in (9) and (10), hence, reducing the runtime significantly.

D. System With Obvious Dominant Poles

Algorithm 3 implements a PMOR procedure, and it is specialized in the case where the full model has a pair of "dominant poles." It is given because it can take advantage of the problem-specific insight common, for instance, in RF inductor design. Note that the reduced model $\hat{H}(z, \mathbf{p})$ is stable, because, as described in Algorithm 3, $|\hat{z}^*(\mathbf{p})| < 1$, and $\hat{H}(z, \mathbf{p})$ is stable $\forall \mathbf{p} \in \mathcal{P}$.

Algorithm 3: PMOR: RF INDUCTOR DESIGN
Input: $H(z, \mathbf{p})$
Output: $\hat{H}(z, \mathbf{p})$

i. Construct reduced models $\tilde{H}_{\mathbf{p}}(z)$ for each $\mathbf{p} \in \mathcal{P}_1 \subset \mathcal{P}$, where $\mathcal{P}_1$ is a finite (training) set

ii. Identify the dominant poles $z^*_{\mathbf{p}}$ of models $\tilde{H}_{\mathbf{p}}(z)$

iii. Construct proper "nondominant" systems $H^1_{\mathbf{p}}(z)$ s.t.

$$\tilde{H}_{\mathbf{p}}(z) = \frac{K_{\mathbf{p}}}{\left(z - z^*_{\mathbf{p}}\right)\left(z - \bar{z}^*_{\mathbf{p}}\right)} \left(z^2 + A_{\mathbf{p}} z + H^1_{\mathbf{p}}(z)\right) \tag{23}$$

where $K_{\mathbf{p}} \in \mathbb{R}$ and $A_{\mathbf{p}} \in \mathbb{R}$

iv. Construct global interpolation model $\hat{K}(\mathbf{p})$, $\hat{A}(\mathbf{p})$, and $\hat{z}^*(\mathbf{p})$. Special attention should be paid to the model $\hat{z}^*(\mathbf{p})$ to make sure that $|\hat{z}^*(\mathbf{p})| < 1, \forall \mathbf{p} \in \mathcal{P}$

v. Solve (18) to find a parameterized model $\hat{H}^1(z, \mathbf{p})$ with nondominant systems $H^1_{\mathbf{p}}(z)$ as inputs.

vi. Construct reduced model of the original system using (23). That is

$$\hat{H}(z, \mathbf{p}) = \frac{\hat{K}(\mathbf{p})}{\left(z - \hat{z}^*(\mathbf{p})\right)\left(z - \overline{\hat{z}^*(\mathbf{p})}\right)} \left(z^2 + \hat{A}(\mathbf{p})z + \hat{H}^1(z, \mathbf{p})\right).$$

Note that, in order to make sure the final model $\hat{H}(z, \mathbf{p})$ is passive, pole and zero information of the "dominant" system can be taken into account to form the numerator of the overall system when parameterized "nondominant" system $\hat{H}^1(z, \mathbf{p})$ is being computed.

VIII. Computational Complexity

There are two sources that contribute to the complexity. The first part is the computation of the frequency samples, which, when using accelerated solvers [33]–[35], is $O(n \log(n))$ for each frequency point, with n being the order of the full model. The examples in Section IX usually required from 20 to 200 frequency samples. The second part is the cost of running the optimization algorithm. The complexity analysis here is based on the specific method of ellipsoid algorithm (which is implemented as a test code). If q and n_{v} are the order of the reduced model and the number of decision variables in the optimization, respectively, then $n_{\mathrm{v}} = O(q)$. Based on the fact that the volume of the bounding ellipsoid is reduced by at least a factor of $1 - (1/n_{\mathrm{v}})$, it can be concluded that it takes $O(n_{\mathrm{v}}^2) = O(q^2)$ iterations to terminate the algorithm. At each iteration of the ellipsoid algorithm, the cost is $O(q^2)$ (matrix vector product performed when updating the bounding ellipsoid). Therefore, the cost of the second part is $O(q^4)$. The overall complexity of the algorithm is summarized as

$$O\left(n \log(n) n_{\mathrm{s}}\right) + O(q^4)$$

with n_{s} being the number of frequency samples computed. Similarly, for the parameterized case, $n_{\mathrm{v}} = O(q\Pi q_{pk})$, where q_{pk} is the degree of the polynomial with each parameter p_k, as in (19), and the complexity is

$$O\left(n \log(n) n_{\mathrm{s}}\right) + O(q\Pi q_{pk})^4. \tag{24}$$

Based on our experience in running the examples in Section IX, the bottleneck for nonparameterized model reduction is represented by the computation of the frequency response samples, i.e., the first term in (24), unless the samples are available as measured data. For parameterized applications, on the contrary, the bottleneck is solving the relaxation, as there are many more decision variables. Therefore, the second term of (24) becomes the dominating factor.

TABLE I
Reduction of RF Inductor From Field Solver Data Using QCO and PRIMA

	QCO	QCO	PRIMA
order	14	20	20
cost (# of solves)	20	40	20
error (%) : H	6.9×10^{-3}	7.1×10^{-4}	1.8×10^{-3}

IX. Applications and Examples

In this section, several application examples are shown to illustrate how the proposed optimization based model reduction algorithm works and performs in practice. All the examples in this section were implemented in MATLAB and run on a Pentium IV laptop with 1-GHz clock, 1 GB of RAM, and running Windows XP.

A. ROM: Comparison With PRIMA

In this section, the proposed algorithm is compared with the commonly used model reduction method of moment matching. The first two examples are nonparameterized comparison. The last example is a parameterized modeling problem for a two-turn RF inductor, as described in [9].

RF Inductor Example: The first example is a comparison between multipoint moment matching (PRIMA) [36] and the proposed algorithm for reducing a seven-turn spiral RF inductor model generated by an electro-magneto-quasi-static (EMQS) mixed-potential integral equation (MPIE) solver [35]. The original model has an order of 1576. PRIMA is set to match two moments at DC, six moments at each of the following frequencies: 4, 8, and 12 GHz. The resulting model has an order of 20. On the other hand, two models are constructed using the proposed method. One has an order of 14 using 20 frequency samples (same computational cost as PRIMA), and the other has an order of 20 using 40 frequency samples (same order as PRIMA). When using the proposed method, both stability and positive real passivity oracles are checked in this example. The following error metric is computed: $\max(|H(f) - \hat{H}(f)|/|H(f)|, f \in [0, 14\,\mathrm{GHz}]$. Comparison results are shown in Table I, with QCO being the shorthand for the proposed quasi-convex optimization method.

RLC-Line Example: This is a cooked-up example in which the full model is not quite reducible. The example is presented here in order to examine how PRIMA and the proposed method perform in a poorly defined setup. In this example, we reduce an RLC line segmented into ten sections (full model order 20) with an open circuit termination. The transfer function is the admittance. The model is obtained as follows: inductor currents and capacitor voltages are the state variables. Kirchhoff's Current Law is imposed at each capacitor node, and the branch equation is used between adjacent nodes. The reduced models of both methods have an order of ten, and PRIMA is set to match four moments at 10^4 rad/s, four moments at 5×10^4 rad/s, and two

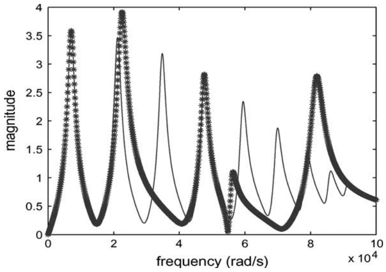

Fig. 1. Magnitude of admittance of an RLC line. (Solid line) Full model. (Solid with stars) PRIMA 10th order ROM.

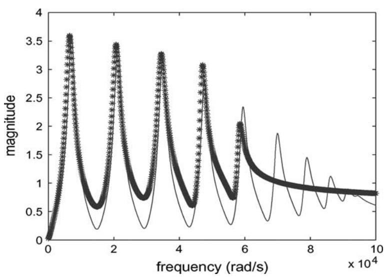

Fig. 2. Magnitude of admittance of an RLC line. (Solid line) Full model. (Solid with stars) QCO 10th order ROM.

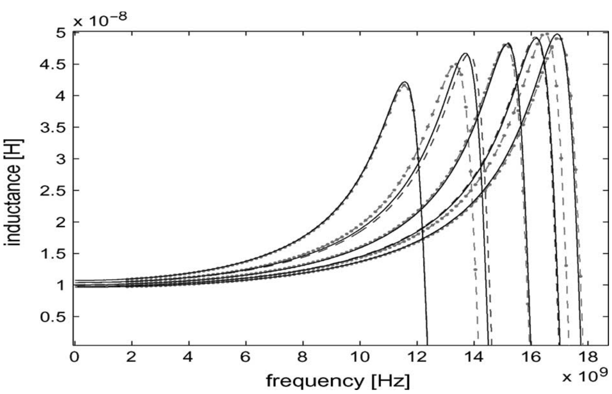

Fig. 3. Inductance of RF inductor for different wire separations. (Dashed) Full model. (Dashed–dotted) Moment matching 12th order. (Solid) QCO 8th order.

moments at 10^5 rad/s, respectively. Figs. 1 and 2 compare the magnitudes of the admittance of the full model and the reduced models by PRIMA and by the proposed method, respectively. The difficulties encountered when modeling this example with PRIMA are discussed in [37]. As expected, in this example, PRIMA performs better locally, but the proposed method does better for the whole frequency range of interest.

PMOR of Two Turn RF Inductor: In this example, the two turn RF inductor in [9] is analyzed. In [9], A 12th order parameterized reduced model was constructed using a moment matching method. On the other hand, we have constructed an 8th order PROM using the proposed method. Fig. 3 show the comparison results in [9] for the case of wire width $D = 1\ \mu$m and wire separation $W = 1, \ldots, 5\ \mu$m, with the additional result of the proposed method superimposed.

B. ROM: Comparison With a Rational Fit Algorithm

In the third example, we compare the proposed method with an existing optimization-based rational fit [17], [18], [21] by constructing a reduced model from measured frequency response of a fabricated spiral RF inductor [38]. In this example, the order of the reduced model is ten, and the positive real part constraint is imposed. Frequency weights (preferring samples of up to 3 GHz) are used, and the quality factor is explicitly minimized. In particular, (21) is solved with tuning parameter $\rho = 10^{-4}$. Runtime for the proposed method was 60 s. On the other hand, rational fit [18], vector fitting [17], and passivity enforcement [21] were used in combination to construct another passive model for comparison. The runtime in running the mentioned algorithms was 30 s.

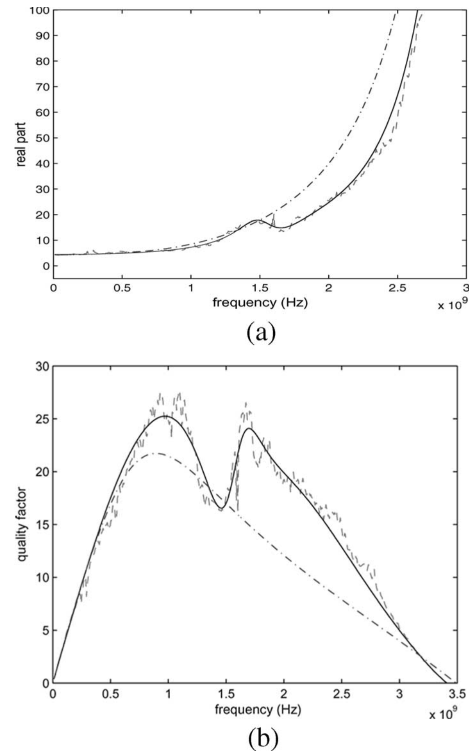

Fig. 4. Identification of RF inductor. (Dashed line) Measurement. (Solid line) QCO-reduced model. (Dashed-dotted line) Methods from [17], [18], and [21]. (a) Real part of impedance. (b) Quality factor.

Fig. 4(a) and (b) shows the real part of the impedance and the quality factor of the model produced by the proposed approach comparing it to measured data and to a model of the same order [(10)] generated using the optimization-based approaches in combination.

C. ROM: Comparison to Measured S-Parameters From an Industry-Provided Example

In the fourth example, we identify a reduced model from measured multiport S-parameter data. On a commercial graphic card, 390 frequency-response samples have been measured. The internal architecture and implementation details are not available. Although the original data is multiple-input–multiple-output, data from only one port are used to construct the reduced model. Fig. 5 shows the comparison result for the corresponding ports. The reduced model is order 20. The model was identified in 30 s.

D. ROM: Frequency-Dependent Matrices Example

In the fifth example, we apply the proposed method to reduce a model of an RF inductor generated by a full-wave MPIE

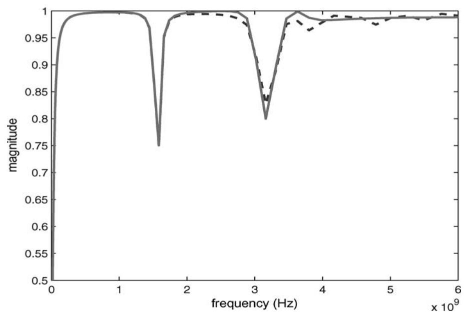

Fig. 5. Magnitude of one of the port S-parameters for an industry-provided example. (Solid line) Reduced model (order 20). (Dashed line) Measured data (almost overlapping).

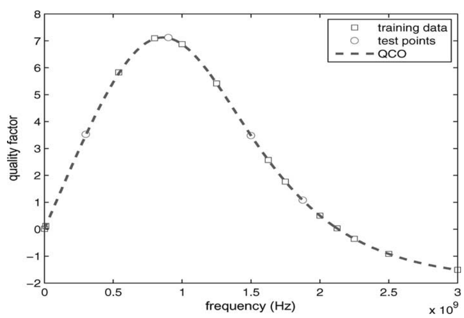

Fig. 6. Quality factor of an RF inductor with substrate captured by layered Green's function. Full model is infinite order and QCO reduced model order is six.

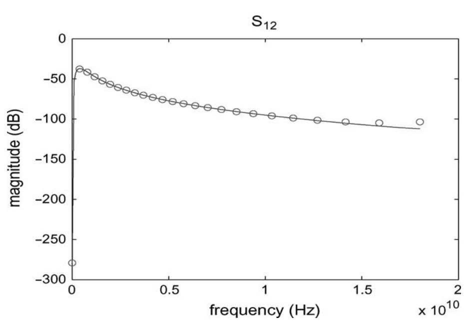

Fig. 7. Magnitude of S12 of the coupled inductors. (Circle) Full model. (Solid line) QCO reduced model.

solver accounting for the substrate effect using layered Green's functions [35], [39]. Since the system matrices are frequency-dependent, the order of the full model is infinite. The order of the reduced model is six, and the positive real part constraint is imposed. Computation time was 2 s. Fig. 6 shows the result of the quality factor.

E. ROM: Two Coupled RF Inductors

A 10th order passive reduced model of two coupled four-turn RF inductors (identical, side by side) was constructed. It took about 120 s to build the reduced model. Fig. 7 shows the result for the magnitude of S12, and Fig. 8 shows the phase of S12.

F. PMOR of Full-Wave RF Inductor With Substrate

In this example, an 8th order passive parameterized reduced model is constructed for an RF inductor with substrate. The

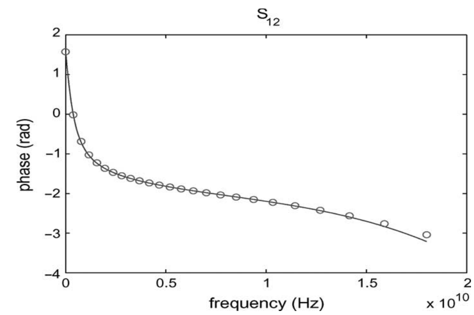

Fig. 8. Phase of S12 of the coupled inductors. (Circle) Full model. (Solid line) QCO reduced model.

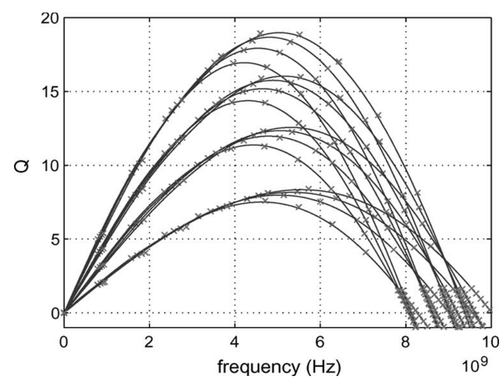

Fig. 9. Quality factor of parameterized RF inductor with substrate. (Cross) Full model from field solver. (Solid line) QCO reduced model.

design parameters are wire width (W) and wire separation (D). The parameter space is a square from (1, 1) to (5, 5) μm. In constructing the reduced model, 25 (W, D) pairs forming a grid of $(1:5) \times (1:5)$ were used as training data. The reduced model is validated with simulation results from field solver on a $((1.5:1:4.5) \times (1.5:1:4.5))$ grid, and Fig. 9 shows the result. Construction of the reduced model took overnight.

For the inductor application in this section, the initial model has more than 2000 states (quasi-static), while the reduced model has an order of eight. To construct the parameterized reduced model in this example, 25 individual models were used as training data.

G. PMOR of a Large Power-Distribution Grid

In this example, a passive parameterized reduced model of a power-distribution grid is built using the techniques in Section VII-C and those similar to Algorithm 3. The design parameters are die size $D \in [7, 9]$ mm and wire width $W \in [2, 20]$ μm. Distributed uniformly in the design space, 25 full models are used as training points for the reduced model of order 32. To test the parameterized reduced model, comparison of full model and reduced model is done at parameters $D \in \{8.25, 8.75\}$ mm and $W \in \{4, 8, 12, 14, 18\}$ μm. Figs. 10 and 11 show the result at $D = 8.25$ mm and $D = 8.75$ mm, respectively.

X. Conclusion

In this paper, a relaxation framework for the optimal $\mathcal{H}_\infty$ norm MOR problem is proposed. The framework has been

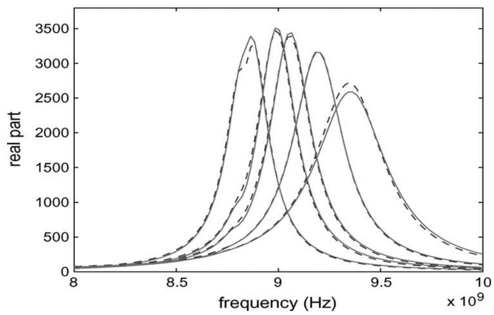

Fig. 10. Real part of power-distribution grid at $D = 8.25$ mm and $W = 4, 8, 12, 14, 18$ μm. (Dashed) Full model. (Solid) QCO reduced model.

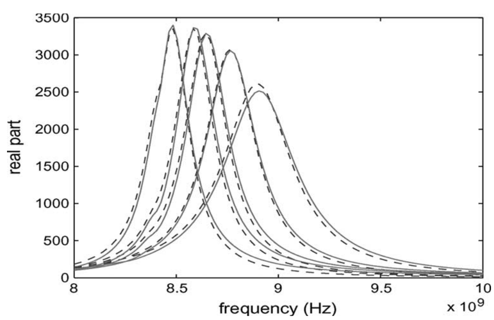

Fig. 11. Real part of power-distribution grid at $D = 8.75$ mm and $W = 4, 8, 12, 14, 18$ μm. (Dashed) Full model. (Solid) QCO reduced model.

demonstrated to perform approximately as well as PRIMA when reducing large systems and better than PRIMA for examples that require a more global accuracy in frequency response. Unlike PRIMA, the proposed method has a guaranteed error bound, and it can reduce models with frequency-dependent system matrices; hence, it can capture, for instance, substrate and full-wave effects. Unlike other optimization-based methods, the proposed method has been shown to be very flexible in preserving stability and passivity. Finally, the proposed optimization setup has also been extended to solve parameterized MOR problems. Several examples have been presented validating both the MOR and PMOR approaches against field solvers and measured data on large RF inductors, IC power-distribution grids, and industrial-provided package examples.

APPENDIX

A. *Proof of Lemma 3.2*

Given $(\vec{q}, \vec{p}, \vec{r}) \in \Omega_{qpr}^m$, it is shown that in order to satisfy (8), $(\vec{a}, \vec{b}, \vec{c})$ must be defined as $\tau_m(\vec{q}, \vec{p}, \vec{r})$. In addition, condition $a(z) > 0$, $|z| = 1$ is satisfied, because $a(z) = |q(z)|^2$ and $q(z) \neq 0$ for $|z| \geq 1$.

Now, consider the case when $(\vec{a}, \vec{b}, \vec{c}) \in \Omega_{abc}^m$ is given. Condition (8) implies that $a(z) = p(z)p(1/z)$, $|z| = 1$, and therefore $a(z) = p(z)p(1/z), \forall z \in \mathbb{C}$, and hence, $z^m a(z) = z^m p(z)p(1/z)$. Since $z^m a(z) \neq 0$ on the unit circle, $p(z)p(1/z) \neq 0$ also on the unit circle. In addition, if z_0 is a zero of $q(z)q(1/z)$, then $1/z_0$ is also a zero of $q(z)q(1/z)$ by symmetry. Therefore, $z^m p(z)p(1/z)$ has exactly m stable zeros and m unstable zeros, and therefore, the Schur polynomial $q(z)$ as given in (6) is unique. Now, show that there exists a unique $(\vec{p}, \vec{r}) : (\vec{q}, \vec{p}, \vec{r}) \in \Omega_{qpr}^m$. Similar to the argument for $a(z) = p(z)p(1/z)$, (8) implies that (7) holds $\forall z \in \mathbb{C}$. Again, this implies that

$$z^m \left(p(z)q(1/z) + q(z)r(1/z)\right) = z^m \left(b(z) + jc(z)\right) \quad \forall\, z \in \mathbb{C} \tag{25}$$

and this means that the equality between two ordinary polynomials of degree no larger than $2m$. That is, (25) can be represented as

$$c_{2m}z^{2m} + c_{2m-1}z^{2m-1} + \cdots + c_0 \equiv z^m \left(b(z) + jc(z)\right)$$

where $c_k, \forall k \in \{0, \ldots, 2m\}$ are linear functions of coefficients of $(\vec{p}, \vec{r})$. Now, consider the linear map $M : \mathbb{R}^{2m+1} \mapsto \mathbb{R}^{2m+1} : c = M(\vec{p}, \vec{r})$, where c is the vector of all c_k. That is, the map from the $2m + 1$ coefficients of $p(z)$ and $r(z)$ ($\deg(p) \leq m$ and $\deg(r) < m$) to the coefficients c_k. If $\exists (\vec{p}, \vec{r}) \in \mathbb{R}^{2m+1} : M(\vec{p}, \vec{r}) = 0$, then from (25), it holds that

$$p(z)q(1/z) \equiv -q(z)r(1/z). \tag{26}$$

Since $q(z)$ is a Schur polynomial, (26) implies that $r(z)$ has at least m zeros, which means that $r(z) \equiv 0$ as $\deg(r) < m$, and by (26) again, $p(z) \equiv 0$. Therefore, $\mathrm{Null}(M) = 0$ and (25) [or (7)] uniquely defines $(\vec{p}, \vec{r}) : (\vec{q}, \vec{p}, \vec{r}) \in \Omega_{qpr}^m$. ■

B. *Proof of Lemma 3.3*

First, note that $\tilde{a}(\omega) > 0, \forall \omega \in [0, \pi]$ defines the intersection of infinitely many halfspaces (each defined by a particular $\omega \in [0, \pi]$), and therefore, the feasible set is convex. Second, consider a sublevel set of the objective function (for any fixed γ). Since $|z| = \max_{|\theta|=1} \mathrm{Re}(\theta z) \forall\, z \in \mathbb{C}$ condition $|H(e^{j\omega})\tilde{a}(\omega) - \tilde{b}(\omega) - j\tilde{c}(\omega)| < \gamma \tilde{a}(\omega) \forall \omega \in [0, \pi]$ is equivalent to

$$\mathrm{Re}\left(\theta\left(H(e^{j\omega})\tilde{a}(\omega) - \tilde{b}(\omega) - j\tilde{c}(\omega)\right)\right) < \gamma \tilde{a}(\omega) \tag{27}$$

$\forall \omega \in [0, \pi]$, $|\theta| \leq 1$, which is the intersection of halfspaces parameterized by θ and ω. Therefore, the sublevel sets of the objective function of (9) are convex, and the quasi-convexity of the program is established. ■

C. *Proof of Lemma 5.1*

Note that (14) is the same as

$$u' H(e^{j\omega})u + u' H(e^{j\omega})' u > 0 \quad \forall\, u \in \mathbb{C}^n, u \neq 0, \omega \in [0, \pi] \tag{28}$$

and it is equivalent to [with Σ as defined in (15)]

$$\begin{bmatrix} x \\ u \end{bmatrix} \Sigma \begin{bmatrix} x \\ u \end{bmatrix}' > 0 \tag{29}$$

subject to "system constraints"

$$zx = Ax + Bu \tag{30}$$

$Hu = Cx + Du$ for $z \in \mathbb{C}$. According to KYP lemma [40], frequency-dependent inequality [(29)] subject to "system constraint" [(30)] holds if and only if the system (with unknowns x, u, and ψ)

$$\begin{aligned} zx &= Ax + Bu \\ \frac{\psi}{z} &= A'\psi - \Sigma_{11}x - \Sigma_{12}u \\ B'\psi &= \Sigma_{21}x + \Sigma_{22}u \end{aligned} \tag{31}$$

does not have nonzero solutions for $|z| = 1$. Assuming that Σ_{22} is invertible and solving for u from the last equation of (31), the earlier condition is equivalent to the condition that the generalized eigenvalue problem (the DT counterpart of what is known as Hamiltonian in CT)

$$z\begin{bmatrix} -\Sigma_{11} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & A' + \Sigma_{12}\Sigma_{22}^{-1}B' \\ -I & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ -A + B\Sigma_{22}^{-1}\Sigma_{21} & -B\Sigma_{22}^{-1}B' \end{bmatrix} = 0 \tag{32}$$

does not have any eigenvalue on the unit circle, which is exactly the condition given in Lemma 5.1.

If the earlier condition is satisfied, then (14) is met. Otherwise, let $e^{j\omega_0}$ be an eigenvalue of (32), and it needs to be shown that

$$H(e^{j\omega_0}) + H(e^{j\omega_0})' \not> 0. \tag{33}$$

Indeed, $e^{j\omega}$ being an eigenvalue of (32) implies that (31) is satisfied with $e^{j\omega_0}$ and the corresponding x, u, and ψ; then, the quadratic form in (29) becomes

$$\begin{aligned} &x'\Sigma_{11}x + x'\Sigma_{12}u + u'\Sigma_{21}x + u'\Sigma_{22}u \\ &= x'(A'\psi - e^{-j\omega_0}) + u'B'\psi \\ &= 0 \end{aligned}$$

and the result is (33). The fact that (17) defines a linear cut should be obvious. ■

D. Proof of Lemma 6.1

First, consider the case when (20) is feasible. Since $\theta' P\theta > -\infty$ and $|\tilde{a}(\omega, \mathbf{p})| < \infty$, an optimal solution exists. Let it be y^*. If $y^* < 0$, then $\tilde{a}(\omega, \mathbf{p}) = \theta' P\theta - y^* > \theta' P\theta > 0$.

Next, consider the case when (20) is feasible but $y^* \leq 0$. First, note that two polynomials are equal if and only if the respective coefficients are equal. Second, for symmetric matrices A and B, $\mathrm{Tr}(A, B) = \sum_{i,j} A(i,j)B(i,j)$, and therefore, the sum of coefficients of $\theta' P\theta$ associated with the same monomial can be written as $\mathrm{Tr}(T, P)$ for some symmetric T, whose entries are either one or zero. Finally, note that coefficients of monomials in $\tilde{a}(\omega, \mathbf{p})$ is linear with respect to coefficient vector x_a. Then, (20) can be written as

$$\begin{aligned} &\underset{y,P}{\text{minimize}} && y \\ &\text{subject to} && \mathrm{Tr}(P) = b_0 + y \\ & && \mathrm{Tr}(T_i P) = b_i \qquad \forall\, i \in \mathcal{I} \\ & && P = P' \geq 0 \end{aligned} \tag{34}$$

where $T_i = T_i'$ is the matrix collecting the coefficients of the ith monomial, $b_i = b_i(x_a)$ is a linear function of x_a being equal to the coefficient of the i^{th} monomial, and $\mathcal{I}$ is the index set of all possible monomials in the expression $\tilde{a}(z, z_{\mathbf{p}})$. Now, consider the Lagrangian of (34)

$$\begin{aligned} L(\lambda) &= \underset{y,P=P'>0}{\text{minimize}} \left\{ y + \lambda_0(\mathrm{Tr}(P) - y - b_0) + \sum_{i=1}^{|\mathcal{I}|} \lambda_i \left(\mathrm{Tr}(T_i P) - b_i\right) \right\} \\ &= \underset{y,P=P'>0}{\text{minimize}} \left\{ y(1-\lambda_0) + \mathrm{Tr}\left(P\left(\sum_{i=0}^{|\mathcal{I}|}\lambda_i T_i\right)\right) - \sum_{i=0}^{|\mathcal{I}|}\lambda_i b_i \right\} \end{aligned}$$

with T_0 being the identity matrix. It is true that

$$L(\lambda) = \begin{cases} -\sum_{i=0}^{|\mathcal{I}|}\lambda_i b_i, & \text{if } \lambda_0 = 1, \quad \sum_{i=0}^{|\mathcal{I}|}\lambda_i T_i \geq 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

At the optimum, the optimal primal/dual pair (y^*, λ^*) has the following property:

$$-\sum_{i=0}^{|\mathcal{I}|}\lambda_i^* b_i = y^*.$$

Define $b = [b_1 \quad \cdots \quad b_{|\mathcal{I}|}]'$ and recall that b is a linear function of x_a. That is $b = Mx_a$ for some matrix M of appropriate dimension. Then, under condition $y^* \geq 0$ (hence, a cut is needed), it holds that $-\lambda^{*\prime} Mx_a \geq 0$. Therefore, all solution [of (18)] x_a that pass the positivity check of the oracle (i.e., $y^* < 0$) should satisfy

$$-\lambda^{*\prime} Mx_a < 0 \tag{35}$$

and therefore $(M'\lambda^*, 0)$ is the desired cut.

Finally, consider the case when (20) is infeasible. By argument of the statements of alternatives, infeasibility of (20) implies the existence of feasible dual solution $\lambda \in \mathbb{R}^{|\mathcal{I}|+1}$

$$\lambda_0 = 1 \qquad \sum_{i=0}^{|\mathcal{I}|}\lambda_i T_i \geq 0 \qquad \sum_{i=0}^{|\mathcal{I}|}\lambda_i b_i \leq 0.$$

Therefore $\sum_{i=0}^{|\mathcal{I}|}\lambda_i b_i > 0$ will lead to the same type of cut as in (35). ■

E. PMOR Stability Oracle With Two Design Parameters

Consider the case in which only two design parameters are allowed. Denote the parameters as D and W (i.e., wire separation and wire width for RF inductor design). Let m be reduced order, M and N be the highest degrees of D, and W in the coefficients of "denominator" $\tilde{a}(\omega, \mathbf{p})$. That is

$$\tilde{a}(\omega, D, W) = \sum_{k=0}^{m}\sum_{i=0}^{M}\sum_{j=0}^{N}\tilde{a}_{ijk}D^i W^j \cos(k\omega) \tag{36}$$

$$b_{ijk} := \sum_{p=|i|}^{M} \sum_{q=|j|}^{N} \left(\sum_{s=0}^{\left\lfloor \frac{p-|I|}{2} \right\rfloor} \binom{p}{|i|+2s} D_0^{p-|i|-2s} D_1^{|i|+2s} \binom{|i|+2s}{s} \right) \left(\sum_{t=0}^{\left\lfloor \frac{q-|j|}{2} \right\rfloor} \binom{q}{|j|+2t} W_0^{q-|j|-2t} W_1^{|j|+2t} \binom{|j|+2t}{t} \right) a_{p,q,|k|}$$

$$\binom{p}{q} := \frac{p!}{(p-q)!q!} \tag{40}$$

where index i and j are associated with design parameter D and W and index k is with the frequency variable ω. $\tilde{a}_{ijk} \in \mathbb{R}$ are optimization decision variables in (20).

Now, consider the transition from the right-hand side of (20) to (34). To this end, it is necessary to introduce further change of variables. Let $z := e^{\sqrt{-1}\omega}$ and with a charge of variables $(a_{ijk} = (1/2)\tilde{a}_{ijk})$, (36) becomes

$$\tilde{a}(\omega, D, W) = \sum_{k=0}^{m} \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ijk} D^i W^j (z^k + z^{-k}). \tag{37}$$

Furthermore, assume that $D \in [\underline{D}, \bar{D}]$ and $W \in [\underline{W}, \bar{W}]$, then

$$\begin{aligned} D &= D_0 + D_1 \left(z_D + z_{D^{-1}}\right) \\ W &= W_0 + W_1 \left(z_W + z_{W^{-1}}\right) \end{aligned} \tag{38}$$

where $D_0 = 0.5(\underline{D} + \overline{D})$, $W_0 = 0.5(\underline{W} + \overline{W})$, $D_1 = 0.25(\overline{D} - \underline{D})$, $W_1 = 0.25(\overline{W} - \underline{W})$ and $z_D \in \mathbb{C}$, $|z_D| = 1$, $z_W \in \mathbb{C}$, $|z_W| = 1$. Using the parameterization of D and W in (38), (37) becomes

$$\begin{aligned} &\sum_{k=0}^{m} \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ijk} \left(D_0 + D_1 \left(z_D + z_D^{-1}\right)\right)^i \\ &\cdot \left(W_0 + W_1 \left(z_W + z_W^{-1}\right)\right)^j \left(z^k + z^{-k}\right) \\ &:= \sum_{k=-m}^{m} \sum_{i=-M}^{M} \sum_{j=-N}^{N} b_{ijk} z_D^i z_W^j z^k \end{aligned} \tag{39}$$

where b_{ijk} are linear functions of coefficients a_{ijk} in (37).

Now, consider the transition from the left-hand side of (20) to (34). Define $n := (M+1)(N+1)(m+1)$ and consider the vector of monomials

$$\theta = [\, z_D^0 z_W^0 z^0 \quad \cdots \quad z_D^i z_W^j z^k \quad \cdots \quad z_D^M z_W^N z^m \,] \in \mathbb{C}^n$$

where the indexes change from j to k to i, $|z_D| = |z_W| = |z| = 1$. The quadratic form with $P \in \mathbb{S}_+^{n \times n}$ in the left-hand side of constraints in (20) is

$$\theta' P \theta = \sum_{i=-M}^{M} \sum_{j=-N}^{N} \sum_{k=-m}^{m} \mathrm{Tr}(T_{ijk} P) z_D^i z_W^j z^k \tag{41}$$

where $T_{ijk} \in \mathbb{S}^{n \times n}$

$$\mathrm{Tr}(T_{ijk} P) = \sum_{(s,t):\theta(s)'\theta(t) = z_D^i z_W^j z^k} P(s,t).$$

The constraint $\theta' P \theta = \tilde{a}(\omega, p) + y$ in (34) can be imposed by enforcing the coefficients of $z_D^i z_W^j z^k$ to be equal in right-hand side of (40), which is shown at the top of the page and left-hand side (41) for all necessary triples (i, j, k). Finally, note that the extra scaler y can be incorporated into the comparison with the (0, 0, 0) term.

With the optimization problem in the form of (34) setup, it can be solved by a standard semi-definite programming solver, such as SeDuMi, and the rest is exactly the same as those in the proof of Lemma 6.1.

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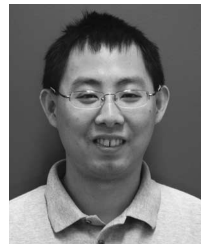

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