Model Order Reduction for Strictly Passive and Causal Distributed Systems

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ABSTRACT

This paper presents a class of algorithms suitable for model reduction of distributed systems. Distributed systems are not suitable for treatment by standard model-reduction algorithms such as PRIMA, PVL, and the Arnoldi schemes because they generate matrices that are dependent on frequency (or other parameters) and cannot be put in a lumped or state-space form. Our algorithms build on well-known projection-based reduction techniques, and so require only matrix-vector product operations and are thus suitable for operation in conjunction with electromagnetic analysis codes that use iterative solution methods and fast-multipole acceleration techniques. Under the condition that the starting systems satisfy system-theoretic properties required of physical systems, the reduced systems can be guaranteed to be passive. For distributed systems, we argue that *causality* of the underlying representation is as important a consideration as *passivity* has become.

Categories & Subject Descriptors: B.7.2 Simulation, B.8.2 Performance Analysis & Design Aids, G.1.1 Interpolation G.1.2, Approximations, I.6 Simulation & Modeling.
General Terms: Algorithms, Performance, Design,
Keywords: Passive reduced order modeling, Distributed systems.

1. INTRODUCTION

Design of modern high-performance electronic systems such as RF circuits, optical transceiver ICs, and global digital signal interconnect requires careful attention to physical modeling so that the intrinsic physical limitations of implementation processes can be accounted for, and undesirable effects such as substrate coupling, signal integrity, and electromagnetic interference (EMI) can be minimized. A high degree of physical fidelity is necessary and can only be achieved by detailed analysis, such as with electromagnetic field solvers. At the same time, effective design of complicated systems requires simple models. Hence, model reduction is now a standard procedure for obtaining simple models of complicated physical systems. Much research has been performed in the model reduction field over the past decade [1, 2, 3], intended to address three primary issues: (a) Model accuracy. (b) Numerically stable and computationally practical generation of models of arbitrary order. (c) Generation of models that are "well-behaved" when embedded into a simulation tool with models of other physical elements.

DAC 2002, June 10-14, 2002, New Orleans, Louisiana, USA.

A reasonable procedure to satisfy the last issue is to require that the models themselves do not possess non-physical properties. For example, components such as interconnect do not generate energy: they are *passive*.

"Lumped" RLC circuits, can be typically represented by matrices that are independent of frequency. For such lumped systems, positive-realness preserving procedures such as those based on congruence transforms [2, 3] are sufficient to guarantee that the reduced models of passive full systems are passive as well. *However, when accounting for high frequency effects, "distributed" systems represented by frequency dependent matrices are typically encountered.* For example, frequency dependent matrices are generated by integral-equation based field solvers that employ full-wave kernels, special Green functions for lossy dielectrics/substrates, or frequency dependent basis functions [4].

There are several approaches to distributed model reduction that essentially convert the model reduction problem to an interpolation or data-fitting problem [5, 6] where it is irrelevant whether the original systems is distributed or lumped. In our experience all of the data-fitting like approaches are limited in some aspect, and to the best of our knowledge, there is no approach that can simultaneously guarantee all the above three requirements. Krylov-based model reduction schemes for lumped systems [1, 3], on the other hand, routinely satisfy all these conditions, so we desire to extend their capabilities to distributed systems.

As input, our algorithm takes a time-invariant state-space-like frequency-domain model whose matrix descriptors may be a function of frequency. As output, it produces a time-invariant state-space model with frequency independent matrix descriptors and whose transfer function is a rational approximant of the original (infinite-order, possibly irrational) transfer function. The algorithm requires only matrix-implicit operations such as matrix-vector products, hence it is suitable for incorporation into modern fast integral equation solvers.

2. PASSIVITY AND CAUSALITY

In this section we will be concerned with properties of an abstract system $\mathcal{H} : \mathcal{X} \to \mathcal{X}$, transforming vector input signals $\mathbf{u}$ into vector output signals $\mathbf{y} = \mathcal{H}\mathbf{u}$ within a space of signals $\mathcal{X}$. For simplicity of exposition we will assume that the system inputs, $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^p$ represent port voltages, and that the outputs $\mathbf{y} : \mathbb{R}^+ \to \mathbb{R}^p$ represent port currents, or the converse (the inputs are currents and the outputs voltages). The Laplace-domain representation of the system $\mathcal{H}$ is then a matrix $\mathbf{H}(s)$, s.t. $\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$, where $\mathbf{u}(s)$ and $\mathbf{y}(s)$ are the Laplace-domain representations of inputs $\mathbf{u}(t)$ and outputs $\mathbf{y}(t)$. Hence, $\mathbf{H}(s)$ is an immittance function: either an admittance matrix $\mathbf{Y}(s)$, or an impedance matrix $\mathbf{Z}(s)$. Let us introduce two inner products in $\mathcal{X}$, the standard inner product $\langle \mathbf{u}, \mathbf{y} \rangle = \int_{-\infty}^{\infty} \mathbf{y}(t)^T \mathbf{u}(t) dt$, and a product which acts on truncated signals $\langle \mathbf{u}, \mathbf{y} \rangle_\tau = \langle \mathbf{u}_\tau, \mathbf{y} \rangle = \langle \mathbf{u}, \mathbf{y}_\tau \rangle = \int_{-\infty}^{\tau} \mathbf{y}(t)^T \mathbf{u}(t) dt$, where $\mathbf{u}_\tau(t) \equiv \{\mathbf{u}(t)$ if $t \leq \tau, 0$ if $t > \tau\}$.

If $\mathbf{u}$ and $\mathbf{y}$ are port current/voltage pairs, $\langle \mathbf{u}, \mathbf{y} \rangle_\tau$ is the total energy dissipated by the system up to time τ. We will generally work in the space of signals $x \in \mathcal{X} = \mathcal{L}_2$ that have finite norm $||x||$ for any τ, where $||x||^2 = \langle x, x \rangle_\tau$.

Passive Systems. A *passive* system is a system that cannot produce energy. For the systems of interest here we may define:

DEFINITION 1 (PASSIVITY). *A system $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ is passive if $\langle \mathbf{u}, \mathcal{H}\mathbf{u} \rangle_\tau \geq 0$, $\forall\, \tau \in \mathbb{R}^+$, $\forall\, \mathbf{u} \in \mathcal{X}$, $\mathbf{u} : [0, \tau] \to \mathbb{R}^p$.*

In practice, almost all systems of interest for model reduction are non-ideal and contain some loss. That is, they internally consume energy. If a system consumes energy, it is said to be *strictly passive*.

DEFINITION 2 (STRICT PASSIVITY). *A system $\mathcal{H}$ is strictly passive if there is a $\delta \in \mathbb{R}^+$ s.t. $\langle \mathbf{u}, \mathcal{H}\mathbf{u} \rangle_\tau \geq \delta ||u_\tau||^2$, $\forall\, \tau \in \mathbb{R}^+$, $\forall\, \mathbf{u} : [0, \tau] \to \mathbb{R}^p$.*

Positive Real Systems. A related concept in network theory is that of positive realness.

DEFINITION 3 (POSITIVE REALNESS). *A matrix valued function $\mathbf{H}(s)$ is positive-real [7] if*

$$\overline{\mathbf{H}(\overline{s})} = \mathbf{H}(s), \tag{1}$$

$$\mathbf{H}(s) \text{ is analytic in } Re(s) > 0, \tag{2}$$

$$\Pi_{\mathbf{H}}(s) \equiv \mathbf{H}(s) + \mathbf{H}(s)^* \geq 0 \text{ in } Re(s) > 0. \tag{3}$$

DEFINITION 4 (STRICT POSITIVE REALNESS). *A matrix valued function $\mathbf{H}(s)$ is strictly-positive-real [8] if there exists an $\varepsilon \in \mathbb{R}^+$ s.t. $\mathbf{H}(s - \varepsilon)$ is positive-real.*

Positive realness is of interest because of its relation to passivity for lumped networks:

THEOREM 1. *A system $\mathcal{H}$ with rational system transfer function $\mathbf{H}(s)$ is passive and stable if and only if $\mathbf{H}(s)$ is positive-real [7].*

In the context of model reduction, the implication for state-space systems is that if a reduction algorithm for lumped RLC networks produces models with positive-real transfer functions $\mathbf{H}(s)$, then it generates guaranteed passive models. Often the positivity condition (3) can be restricted to the imaginary axis, because of the following result [7]:

THEOREM 2. *A rational $\mathbf{H}(s)$ is positive-real if and only if (1) and (2) hold, and*

$$\Pi_{\mathbf{H}}(i\omega) \geq 0,\ \forall\, \omega \in \mathbb{R} \tag{4}$$

except for simple poles $i\omega_0$ of $\mathbf{H}(s)$, where the residue matrix must be nonnegative definite [7]. $\mathbf{H}(s)$ is strictly positive real if the inequality is strict [7, 8].

THEOREM 3. *If a rational matrix valued function $\mathbf{H}(s)$ is a representation of a passive system, then $\mathbf{H}^{-1}(s)$ is positive-real.*

Causal Systems. A *causal* system is a system whose output depends only on past inputs, not future inputs.

DEFINITION 5 (CAUSALITY). *A system $\mathcal{H}$ is causal if $\mathcal{H}\mathbf{u} = \mathcal{H}\mathbf{u}_\tau$, $\forall\, \tau \in \mathbb{R}^+$, $\forall\, \mathbf{u} : [0, \tau] \to \mathbb{R}^p$.*

All physical systems are causal. Hence, causality is a necessary property of all models intended to be used in any simulator that has a concept of time. However, it is often neglected when modeling distributed systems. When constructing model reduction algorithms for distributed systems, we must keep in mind that the condition in Equation (4) is not sufficient by itself to insure passivity. In particular, it is known that passive systems are required to be causal [9, 10]. If a system is not causal, it cannot be passive.

EXAMPLE 1. *Consider the one port ($p = 1$) network function $\mathbf{Z}_{skin}(i\omega) = \mathbf{R}_0 + \mathbf{R}_{ac}\sqrt{|\omega|}$ that is commonly used as a model for the "resistance" of interconnect in the skin-effect regime. This function satisfies the condition $\Pi_Z(i\omega) > 0$, $\forall\, \omega \in \mathbb{R}$. However, it is* not *a representation of any passive system, because it is not a causal function. In fact, it can be shown that any physical, passive network function that is purely real must be constant with respect to the frequency ω. In Section 6.3 we further illustrate the non-causality of this model.*

Algorithms that generate successively better rational approximations of non-causal systems (for example, $Z_{\text{skin}}(i\omega)$) must in some limit fail to be passive, for example by generating unstable approximants. Therefore we will require that all the systems we manipulate be causal. The algorithms we shall present shortly will produce strictly positive-real (and thus strictly passive) approximations to causal and strictly positive-real (and thus strictly passive) system functions $\mathbf{H}(s)$.

3. TECHNICAL BACKGROUND

Distributed Systems in Descriptor Form. Let us assume the original distributed system (e.g. an interconnect network) has been described, for instance by the discretization step of an integral equation method, in terms of a frequency dependent matrix $\mathbf{Z}(s)$. $\mathbf{Z}(s)$ describes the couplings between all the discretization basis functions and may be very large in the applications of interest. Many integral equation methods, when applied to distributed systems, produce $\mathbf{Z}(s)$ as a linear combination of matrices. One example is $\mathbf{Z}(s) = \mathbf{R}(s) + s\mathbf{L}(s)$, where $\mathbf{R}(s)$ and $\mathbf{L}(s)$ can still be in general frequency dependent. Incidentally this particular form for $\mathbf{Z}(s)$ may be advantageous for our approach, although in general not necessary. We assume input and output information is of interest at some "ports" of the network for which the model is to be generated. The frequency-domain description of the system can be written as

$$[\mathbf{R}(s) + s\mathbf{L}(s)]\mathbf{i}_m(s) = \mathbf{B}\mathbf{v}_p(s), \quad \mathbf{i}_p(s) = \mathbf{B}^T\mathbf{i}_m(s), \tag{5}$$

where $\mathbf{v}_p(s) \in \mathbb{C}^p$ and $\mathbf{i}_p(s) \in \mathbb{C}^p$ are Laplace-domain representations of voltages and currents at the p defined ports of interest, $\mathbf{i}_m(s) \in \mathbb{C}^n$ are the internal currents, and $\mathbf{B} \in \mathbb{R}^{n \times p}$ is a matrix relating ports to internal currents. In this case $\mathbf{u} = \mathbf{v}_p$ represents the system inputs (voltages), $\mathbf{y} = \mathbf{i}_p$ the system outputs (currents), and $\mathbf{x} = \mathbf{i}_m$ the internal states (also in this case currents). The transfer function from inputs to outputs is $\mathbf{H}(s) = \mathbf{B}^T[\mathbf{R}(s) + s\mathbf{L}(s)]^{-1}\mathbf{B}$, $\mathbf{i}_p(s) = \mathbf{H}(s)\mathbf{v}_p(s)$, and one view of model reduction is that it seeks an approximation to the transfer function $\mathbf{H}(s)$.

Projection Methods. The Krylov-subspace based model reduction algorithms [1, 3, 2] are projection methods. One of the key insights in reducing distributed systems is that **projection methods, and their approximation properties, apply to lumped systems *as well as to distributed ones.***

Given a matrix $\mathbf{Q} \in \mathbb{R}^{n \times q}$, an orthogonal projection method obtains a rational approximant (of dimension q), or, equivalently, a reduced model, via the congruence transformations

$$\hat{\mathbf{R}}(s) = \mathbf{Q}^*\mathbf{R}(s)\mathbf{Q},\ \hat{\mathbf{L}}(s) = \mathbf{Q}^*\mathbf{L}(s)\mathbf{Q},\ \hat{\mathbf{B}} = \mathbf{Q}^*\mathbf{B}. \tag{6}$$

The post-multiplication by matrix $\mathbf{Q}$ corresponds to projecting the original state vector of variables $\mathbf{i}_m$ into the reduced space of basis functions represented by the q columns of matrix $\mathbf{Q}$: $\hat{\mathbf{i}}_m = \mathbf{Q}\mathbf{i}_m$. The pre-multiplication by matrix $\mathbf{Q}^*$ in (6) reduces the number of equations to the new size of the state q.

After the congruence transformation step the linear system is

$$[\hat{\mathbf{R}}(s)+s\hat{\mathbf{L}}(s)]\hat{\mathbf{i}}_m=\hat{\mathbf{B}}\mathbf{v}_p(s),\quad \mathbf{i}_p(s)=\hat{\mathbf{B}}^*\hat{\mathbf{i}}_m(s) \tag{7}$$

where $\hat{\mathbf{R}},\hat{\mathbf{L}}\in\mathbb{C}^{q\times q}$; $\hat{\mathbf{i}}_m(s)\in\mathbb{C}^q$, $\hat{\mathbf{B}}\in\mathbb{R}^{q\times p}$. At this point, the projection operation is conceptual; the system of (7) contains a frequency-parameter and is thus infinite-dimensional. We will discuss finite-dimensional realizations later in the paper.

For lumped systems, the most popular approaches to model reduction generate the columns of the matrix $\mathbf{Q}$ by matching moments of the frequency domain response $\mathbf{H}(s)$, or, equivalently, derivatives of the time domain response. An exhaustive discussion on this step is not among the purposes of this work. However, we mention here that one possible way to obtain a very simple projection matrix $\mathbf{Q}$ for the congruence transformation in (6) is to construct each of the q columns of $\mathbf{Q}$ by solving the original full system (5) at q different frequency points [2]. When the underlying system matrices $\mathbf{R}(s)$ and $\mathbf{L}(s)$ are defined implicitly, as in large-scale electromagnetic codes, this is not an inconvenience, and is in fact preferred as derivative information can be awkward to obtain.

The importance of congruence transformations in generating well-behaved reduced models stems from the following theorem.

THEOREM 4. *If $\mathbf{Z}(s)$ is positive-real, then $\hat{Z}(s)=\mathbf{Q}^*\mathbf{Z}(s)\mathbf{Q}$ is also positive-real.*

PROOF. Conditions (1) and (2) in Definition 3 are easily verified. Condition (3) follows from observing that $\Pi_{\hat{\mathbf{Z}}}(s)=\mathbf{Q}^*\Pi_{\mathbf{Z}}(s)\mathbf{Q}$, so range$(\Pi_{\hat{\mathbf{Z}}})\subset$ range$(\Pi_{\mathbf{Z}})$. □

The theorem implies that if the given infinite-dimensional system to be reduced is passive and can be described by a positive-real system matrix $\mathbf{Z}(s)$, then the model with smaller descriptor matrices obtained from the projection operation will represent a *passive* system as well, although still infinite-dimensional.

LEMMA 1. *Given a system of the form in (5), if $\hat{\mathbf{Z}}(s)=\hat{\mathbf{R}}(s)+s\hat{\mathbf{L}}(s)$ is [strictly] positive-real, $\hat{\mathbf{H}}(s)$ is [strictly] positive-real [3].*

The proof follows from Theorems 3 and 4. From this Lemma, one can observe that one way to obtain a positive-real $\hat{\mathbf{H}}(s)$ is to find a positive-real $\hat{\mathbf{Z}}(s)$. Note particularly that positive-realness of $\hat{\mathbf{Z}}(s)$ implies that all the poles and zeros of $\hat{\mathbf{H}}(s)$ are in the left half-plane.

4. THE OPTIMAL GLOBAL INTERPOLATION APPROACH

To illustrate some of the problems encountered in model reduction for distributed systems, consider the algorithm presented in [11]. The central approach of the algorithm is a Taylor expansion of the system matrix descriptor $\mathbf{Z}(s)\approx\sum_{k=0}^{N}\mathbf{Z}_k s^k$, using polynomials as interpolants. A standard Krylov method is then applied to a system constructed from the Taylor expansion. This approach does not generate well-behaved models because the Taylor approximation is not *globally* well-behaved. In fact, all polynomials diverge in the $s\to\infty$ limit. Hence, although good accuracy can be achieve in a given frequency band of interest, *global* properties such as positive-realness cannot be guaranteed. Empirically, the resulting reduced models are often found to have unstable poles, the models are not passive, and thus the algorithm is of little practical value.

In our approach, we will seek to combine approximation of the $\mathbf{Z}(s)$ internal matrix descriptors with a Krylov method as in [11]. But our method differs in the following fundamental aspect.

4.1 The key idea

Almost all systems for which one would wish to extract reduced models are non-ideal (non-ideality is why they must be modeled in detail) and so contain a small amount of loss. These systems are strictly passive and can typically be described by strictly positive real system matrices.

The key idea of our algorithm is based on the observation that **if a system descriptor is strictly positive-real to begin with, a *globally* and *uniformly* convergent interpolant will eventually (for a large enough order of the interpolant) be positive-real as well.** (This will be seen in the proof of Theorem 5.) Furthermore, a well-chosen global interpolant will be positive-real for low enough orders to be practical. Local approximations based on Taylor and Padé typically do not have these properties.

There is one more point, subtle yet of great importance, that we wish to underline before proceeding with the main algorithm. In our algorithm, as in others [3], we require that, not only the transfer function $\mathbf{H}(s)$ of the given large system be strictly positive-real, but also that its internal system matrix descriptor $\mathbf{Z}(s)$ be strictly positive real (i.e. that the state-space description be internally positive-real). However, as discussed in Section 2 in a physical system, $\mathbf{H}(s)$ must also be causal. Hence, as for the the positive realness property, we shall require that not only the transfer function $\mathbf{H}(s)$ of the given large system be strictly positive-real *and causal* but also that its internal system matrix descriptor $\mathbf{Z}(s)$ be strictly positive real *and causal* (i.e. that the state-space description be internally positive-real *and causal*). In this case we can restrict our search for approximations of $\mathbf{Z}(s)$ to the set of stable, positive-real interpolants. For non-causal $\mathbf{Z}(s)$, either accuracy or stability/passivity would have to be eventually sacrificed.

4.2 Proposed Algorithm

We propose an eight step procedure, which we term **G**lobal **R**ational **I**nterpolation, **P**assive (GRIP):

1. Obtain/estimate/given a set of q points at which the transfer function at the network ports $\mathbf{H}(s)\in\mathbb{C}^{p\times p}$ is to be matched,
2. Compute the basis $Q\in\mathbb{C}^{n\times q}$ for the projection operation (see Section 3).
3. Project the internal system matrices $\mathbf{R}(s),\mathbf{L}(s)\in\mathbb{C}^{n\times n}$ to obtain smaller $\hat{\mathbf{R}}(s),\hat{\mathbf{L}}(s)\in\mathbb{C}^{q\times q}$ as in (6). Note that this is a conceptual operation; the reduced matrices are still frequency-dependent, so the system is still of potentially infinite order.
4. Perform a *global* and *uniform* interpolation of the (projected) internal system matrices $\hat{\mathbf{R}}(s),\hat{\mathbf{L}}(s)$ (see Section 5).
5. Check the passivity (see [12]), and accuracy of the matrix interpolants. If not passive, or if matrix interpolants are not accurate, go to Step 4 and increase the order N of the global interpolant.
6. Check the accuracy of the reduced model transfer function $\hat{\mathbf{H}}(s)$. If not accurate, go to Step 1 and add additional matching points q.
7. Realize as state-space system.
8. Perform a second-stage guaranteed-passive optimal reduction step, if desired [13].

Steps 1,2,3, and 6 are standard in lumped-system model reduction. Various approaches are possible, and many are described in the literature. As they are not the main focus here, they will not be discussed further. Step 5 can be performed solving the Lur'e equation in the Positive-Real Lemma [12], for which computational procedures are available in the literature [12]. Step 7 is dependent on how Step 4 is performed, but is always possible if Steps 4-6 are feasible.

It can be noticed that our algorithm is posed in such a way that if it terminates, accuracy, stability, and passivity are guaranteed. However, we have not yet shown that it is possible to construct specific instantiations that *will* terminate. Such task is equivalent to finding for the key Step 4 a suitable interpolant that is guaranteed to converge globally and uniformly. To this purpose, in the next Section, one possible choice will be described.

5. A LAGUERRE-BASIS IMPLEMENTATION

5.1 Choosing the global uniform interpolant

Several approaches are possible to the global interpolation problem. One possibility is to use algorithms developed for general-purpose interpolation or data-fitting [6] that guarantee passivity by construction. These algorithms are very computationally demanding. For many applications of interest, a simpler alternative is available. First, it is advantageous (although not necessary) to find some decomposition of matrix $\mathbf{Z}(s)$ into for instance some matrices $\mathbf{R}(s)$ and $\mathbf{L}(s)$ for which the individual matrix entries do not have sharply discontinuous behavior in the frequency parameter s. Many integral-equation-based electromagnetic field solvers for distributed systems already produce such a decomposition. In order to use our method, particular attention will need to be dedicated to making sure that such solvers generate strictly positive-real and causal system matrices.

Second, the frequency dependency of projected matrices $\hat{\mathbf{R}}(s)$ and $\hat{\mathbf{L}}(s)$ in (6) can then be captured for instance using the set of basis functions [14], $E_k(s) = \left(\frac{\lambda - s}{\lambda + s}\right)^k$, where λ is a positive real number. In this way we can write:

$$\hat{\mathbf{R}}(s) = \sum_{k=0}^{\infty} \hat{\mathbf{R}}_k E_k(s), \quad \hat{\mathbf{L}}(s) = \sum_{k=0}^{\infty} \hat{\mathbf{L}}_k E_k(s). \tag{8}$$

The basis created by the functions $E_k(s)$, sometimes called the Laguerre basis, is a member of a larger family [15] of bases, all of which consist of sets of stable rational functions orthonormal over the imaginary axis $s = i\omega$. An interesting contrast with the Taylor series approach is that the $E_k(s)$ are, in a sense, band-limited. For $|\omega| > \lambda$, the $E_k(s)$ have monotonic magnitude, and for $|\omega| < \lambda$, they are nearly equi-ripple, much like Chebyshev polynomials. This implies that with suitable choice of λ, the approximations to $\mathbf{R}(s), \mathbf{L}(s)$ will be well behaved outside the approximation interval, and convergence will be fast within it.

The Laguerre basis is particularly interesting because, under the bilinear transformation, $s = \lambda(1-z)/(1+z)$, the series expansion in terms of the basis functions $E_k(s)$ is mapped to a Fourier series of complex exponentials, since $E_k(s) = z^k$, where $z = e^{i\phi}, \phi \in [0, 2\pi)$. The problem of rationally approximating the matrix functions $\hat{\mathbf{R}}(s), \hat{\mathbf{L}}(s)$ is reduced to the problem of approximating a function on the circle via a Fourier series, or equivalently computing a Discrete Fourier Transform (DFT), as the entries of $\hat{\mathbf{R}}(s), \hat{\mathbf{L}}(s)$ may be approximated term-wise.

5.2 Global Uniform Convergence

If the entries of $\hat{\mathbf{R}}(s), \hat{\mathbf{L}}(s)$ are smooth when mapped to the circle, then the partial sums $\hat{\mathbf{R}}^{(N)}(s), \hat{\mathbf{L}}^{(N)}(s)$ converge uniformly to $\hat{\mathbf{R}}(s), \hat{\mathbf{L}}(s)$

$$\hat{\mathbf{R}}^{(N)}(s) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\mathbf{R}}_k z^k, \quad \hat{\mathbf{L}}^{(N)}(s) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\mathbf{L}}_k z^k. \tag{9}$$

If $\hat{\mathbf{R}}(s), \hat{\mathbf{L}}(s)$ are not smooth, but are continuous, it is still possible to obtain uniformly convergent approximates by summing the Fourier series in the sense of Cesaro [16]. Practically speaking, this means replacing the summations (9) by

$$\tilde{\mathbf{R}}^{(N,C)}(s) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\mathbf{R}}_k \left[1 - \frac{k}{N}\right] z^k, \quad \tilde{\mathbf{L}}^{(N,C)}(s) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\mathbf{L}}_k \left[1 - \frac{k}{N}\right] z^k. \tag{10}$$

Equivalently, we may say that as successive approximates, we take the arithmetic means of the partial sums $\hat{\mathbf{R}}^{(N)}(s), \hat{\mathbf{L}}^{(N)}(s)$, rather than the partial sums themselves. Summation in this manner has the property of suppressing the Gibbs effect, and also insuring uniform convergence on a broader class of functions.

LEMMA 2. *If the entries of $\hat{\mathbf{R}}^{(N)}(s), \hat{\mathbf{L}}^{(N)}(s)$ are continuous when mapped to the circle, $\tilde{\mathbf{R}}^{(N,C)}(s), \tilde{\mathbf{L}}^{(N,C)}(s)$ converge uniformly to $\hat{\mathbf{R}}^{(N)}(s), \hat{\mathbf{L}}^{(N)}(s)$ as $N \to \infty$ [16].*

Shortly we will need the following definitions:

DEFINITION 6 (STRONG-η CONDITION). *Satisfied for $\hat{\mathbf{Z}}(s)$ if $\Pi_{\hat{\mathbf{Z}}}(s) - \eta I \geq 0$ for any $\eta > 0$.*

where $\Pi_{\hat{\mathbf{Z}}}(s)$ was defined in (3).

DEFINITION 7 (WEAK-η CONDITION). *Satisfied for $\hat{\mathbf{Z}}(s)$ if for any $\varepsilon \geq 0$, there is an $\eta > 0, \eta < \varepsilon$ s.t. $\Pi_{\hat{\mathbf{Z}}}(s) + \eta I > 0$.*

From Lemma 2 we obtain a major result of the paper:

THEOREM 5. *Given a system description $\hat{\mathbf{Z}}(s) = \hat{\mathbf{R}}(s) + s\hat{\mathbf{L}}(s)$ where matrices $\hat{\mathbf{R}}(s)$ and $\hat{\mathbf{L}}(s)$ are causal, strictly positive real, and continuous on the imaginary axis, there exists an integer N and coefficients $\tilde{\mathbf{R}}_k^{(N,C)}, \tilde{\mathbf{L}}_k^{(N,C)}$ for the partial sums in (10) such that the matrix rational function $\tilde{\mathbf{Z}}(s) = \tilde{\mathbf{R}}^{(N,C)}(s) + s\tilde{\mathbf{L}}^{(N,C)}(s)$ is a positive-real rational interpolant of $\hat{\mathbf{Z}}(s)$ whose error can be bounded from above by any chosen positive constant.*

PROOF. Property (1) follows by construction as the E_k satisfy (1). Property (2) also follows by construction, since by inspection the E_k have poles only in the left half-plane. Due to Theorem 2, proof of (4) is now sufficient to complete the proof. Case 1: $\Pi_{\mathbf{Z}}$ is Strongly-η. From Lemma 2, if $\hat{\mathbf{R}}(s)$ and $\hat{\mathbf{L}}(s)$ are continuous when mapped to the circle, $\tilde{\mathbf{R}}^{(N,C)}(s), \tilde{\mathbf{L}}^{(N,C)}(s)$ converge uniformly and so does $\tilde{\mathbf{Z}}(s) = \tilde{\mathbf{R}}^{(N,C)}(s) + s\tilde{\mathbf{L}}^{(N,C)}(s)$. Thus $\forall \eta > 0$, $\exists N$ s.t. $||\tilde{\mathbf{Z}}(i\omega) - \hat{\mathbf{Z}}(i\omega)||_2 < \eta/4$, $\forall \omega \in \mathbb{R}$. Hence $||\Pi_{\tilde{\mathbf{Z}}}(i\omega) - \Pi_{\hat{\mathbf{Z}}}(i\omega)||_2 \leq 2||\tilde{\mathbf{Z}}(i\omega) - \hat{\mathbf{Z}}(i\omega)||_2 \leq \eta/2$, $\forall \omega \in \mathbb{R}$. $\Pi_{\tilde{\mathbf{Z}}}(i\omega) > \Pi_{\hat{\mathbf{Z}}}(i\omega) - ||\Pi_{\tilde{\mathbf{Z}}}(i\omega) - \Pi_{\hat{\mathbf{Z}}}(i\omega)||_2 > \eta - \eta/2$, so[1] $\Pi_{\tilde{\mathbf{Z}}} > \eta/2$ and $\Pi_{\tilde{\mathbf{Z}}}$ is Strongly-η, which implies (4) (see [8]) and that $\tilde{\mathbf{Z}}$ is strictly positive-real. Case 2: $\Pi_{\hat{\mathbf{Z}}}$ is [strictly] positive-real but not Strongly-η. Choose any $\eta > 0$ and Map $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}} + \eta I$. $\hat{\mathbf{Z}}$ is now Strongly-η. Go to Case 1. [2] □

Theorem 5 proves that an order of interpolation N large enough *does exist* and therefore that the algorithm in Section (4.2 terminates. A practical algorithm would require a small N. The order of the interpolant is related to the smoothness of the function being approximated. Hence, although we could use this algorithm to approximate $\mathbf{H}(s)$ or $\mathbf{Z}(s)$ directly, that would require evaluation of an awful lot of matching points around resonances, and most likely a very large order of the interpolant. A small N is instead needed when the algorithm is used on some internal decomposed matrices $\mathbf{R}(s)$ and $\mathbf{L}(s)$ which are almost always continuous within a given band of interest. Out-of-band smoothness (for example, for delay functions, which create essential singularities at ∞) can be insured by filtering operations which must be designed to preserve passivity and causality of the original matrices.

5.3 Computing the DFT Coefficients

The DFT coefficients in the sums (9) can be efficiently calculated for instance using a Fast Fourier Transform (FFT) algorithm.

[1] In this we needed to use the result that perturbations of a Hermitian matrix ($\Pi_{\hat{\mathbf{Z}}}$) result in perturbations of the eigenvalues bounded by the 2-norm of the perturbation.

[2] The upshot of all this is that the Strongly-η condition is slightly stronger than strict positive-realness and may not be satisfied for all strictly passive systems. But by introducing an additional error of $O(\eta)$ (i.e., roughly doubling the interpolation error bound) through the η-shifting procedure we may guarantee strict positive-realness of the final model. An alternative is to not perform the η-shifting, in which case we may prove that $\tilde{\mathbf{Z}}$ is Weakly-η, which allows $\tilde{\mathbf{Z}}$ to have an excess energy gain of $O(\eta)$. Since we may drive $\eta \to 0$, neither deviation is of practical consequence in systems with loss modeled over a finite bandwidth.

That is equivalent to evaluating the DFT coefficients via numerical integration using the trapezoidal rule. For functions on a circle, the trapezoidal rule is near-optimal, and achieves rapid (spectral) convergence for smooth functions. Hence the steps for one possible global approximation procedure are

1. For a desired interpolation order N, choose the size M of the FFT as some power of two: $M = 2^n > N$.
2. Calculate the frequency points s_k on the imaginary axis corresponding to the M equally spaced FFT points $z_k = exp(j2\pi k/M)$, $k = 1,\ldots,M$ on the unit circle using the bilinear transform: $s_k = \lambda(1-z_k)(1+z_k)$, where $\lambda = 2\pi f_0$ is a parameter to be chosen around the center of the frequency band of interest for the system response.
3. Use (6) to evaluate each individual projected matrix $\hat{\mathbf{R}}(s_k)$ and $\hat{\mathbf{L}}(s_k)$ at the selected frequency points s_k, $k = 1,\ldots,M$.
4. Use an FFT algorithm to calculate the M coefficients $\hat{\mathbf{R}}_k$ and $\hat{\mathbf{L}}_k$ in (9) from the sequences $\hat{\mathbf{R}}(s_k)$ and $\hat{\mathbf{L}}(s_k)$, $k = 1,\ldots,M$.
5. Apply to each of the M FFT coefficients the Cesaro's transformation in (10) and obtain the coefficients $\tilde{\mathbf{R}}_k$ and $\tilde{\mathbf{L}}_k$.

Note that, since the $\mathbf{R}(s)$ and $\mathbf{L}(s)$ matrices usually satisfy conjugate symmetry relations, $\mathbf{R}(s), \mathbf{L}(s)$ need to be evaluated at only half the points on the circle. Also, once the M Cesaro's FFT coefficients are available one can construct at no additional cost several interpolants of increasing order $N < M/2$ simply truncating the sums in (10) to the first N coefficients.

5.4 Realization

In this section we describe how to perform Step 7 in the general algorithm 4.2, realization as a state-space model. Having performed the global rational approximation on the projected matrix functions $\hat{\mathbf{R}}(s)$ and $\hat{\mathbf{L}}(s)$, the system (5) is now:

$$\left[\frac{1}{M}\sum_{k=0}^{N-1}\tilde{\mathbf{R}}_k z^k + \lambda\left(\frac{1-z}{1+z}\right)\frac{1}{M}\sum_{k=0}^{N-1}\tilde{\mathbf{L}}_k z^k\right]\hat{\mathbf{i}}_m = \hat{\mathbf{B}}\mathbf{v}_p \qquad (11)$$

where $\tilde{\mathbf{R}}_k$ and $\tilde{\mathbf{L}}_k$ contain already the Cesaro's correction (10). We then collect the terms corresponding to the same powers of z, define an augment state

$$\mathbf{x} = \left[\; z^{N-1}\mathbf{i}_m^T \quad z^{N-2}\mathbf{i}_m^T \quad \ldots \quad z^2\mathbf{i}_m^T \quad z\mathbf{i}_m^T \quad \mathbf{i}_m^T \;\right]^T. \qquad (12)$$

and produce a finite dimension discrete linear system, Substituting $z = (\lambda - s)/(\lambda + s)$ we obtain the continuous and final system realization

$$\begin{aligned} s\mathbf{E}_c\mathbf{x}_c &= \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{v}_p \qquad (13)\\ \mathbf{i}_p &= \mathbf{C}_c\mathbf{x}_c \end{aligned}$$

where

$$\mathbf{E}_c = \begin{bmatrix} I & I & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & I \\ 2\mathbf{F}_N & 2(\mathbf{F}_N+\mathbf{F}_{N-1}) & \ldots & 2(\mathbf{F}_N+\ldots+\mathbf{F}_2) & \mathbf{F}_N+\ldots+\mathbf{F}_1-\mathbf{F}_0 \end{bmatrix}$$

$$\mathbf{A}_c = \lambda\begin{bmatrix} I & -I & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & -I \\ 0 & 0 & \ldots & 0 & -(\mathbf{F}_N+\ldots+\mathbf{F}_1+\mathbf{F}_0) \end{bmatrix},$$

$$\mathbf{B}_c = -2\lambda\left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \hat{\mathbf{B}} \;\right]^T,$$

$$\mathbf{C}_c = \hat{\mathbf{B}}^*\left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; I \;\right],$$

$$\mathbf{F}_k = \begin{cases} \frac{1}{M}\left[\tilde{\mathbf{R}}_0 + \lambda\tilde{\mathbf{L}}_0\right] & k = 0, \\ \frac{1}{M}\left[\tilde{\mathbf{R}}_k + \tilde{\mathbf{R}}_{k-1} + \lambda\tilde{\mathbf{L}}_k - \lambda\tilde{\mathbf{L}}_{k-1}\right] & 1 \le k \le N-1, \\ \frac{1}{M}\left[\tilde{\mathbf{R}}_{N-1} - \lambda\tilde{\mathbf{L}}_{N-1}\right] & k = N. \end{cases}$$

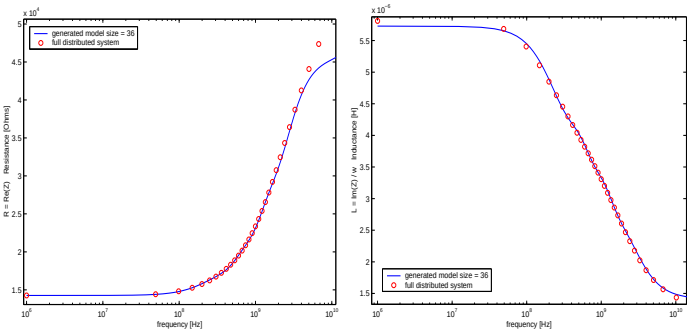


Figure 1: Real part and imaginary part divided by ω of the frequency response for the lossy substrate example.

6. EXAMPLES

6.1 Effect of Lossy Substrate on Line Impedance

The geometry in this example consists of two wires over a lossy substrate. The two wire volumes are discretized into short and thin filaments using a set of piece-wise constant basis functions. A standard Galerkin technique is used to calculate the resistance and partial inductance matrices $\mathbf{R}(s)$ and $\mathbf{L}(s)$. *A frequency dependent Green function is used in the kernel of the Galerkin integration to account for the effects of the lossy substrate. Hence, the resulting matrices are frequency dependent.* The system, before model reduction, appears as in (5). The descriptor matrices have been projected to a reduced space of size $q = 4$ obtained by solving the original full system at frequencies $f = 0,\ 0.4GHz,\ 1GHz,\ 2.4GHz$. which correspond to the points on the unit circle: $z = 1,\ exp(-j\pi/4),\ -j,\ exp(-j3\pi/4)$. As center frequency for our band of interest we have chosen $\lambda = 2\pi 10^9$. In this example we have chosen $M = 64$ points for the FFT size. Fig. 4.b shows the real part of the DFT coefficients for the inductance matrices. Coefficients $32,\ldots,63$ are very small indicating the original system matrix is causal. We have then truncated the DFT series to N=8 coefficients, producing a final model of order 36. Fig. 1 compares the frequency response of the original full system with the frequency response of the final realized linear state space model. All the poles of the realized model are in the half-plane $Re(s) < 0$, hence stable. Using the Positive-Real Lemma [12], we confirmed that the generated system is passive.

6.2 Full-Wave PEEC kernel

In this example we consider two parallel wires 4um wide, 1um thick and 750 um long. The two wires are separated by 3cm. The wire volumes are discretized into short and thin filaments using a set of piece-wise constant basis functions. A Galerkin technique is used to calculate the resistance and partial inductance matrices $\mathbf{R}(s)$ and $\mathbf{L}(s)$. *Since the separation between the two wires is not small compared to the minimum wavelength of interest, a frequency dependent full-wave Green Function needs to be employed in the kernel of the Galerkin integration. Hence, the resistance and inductance matrices are frequency dependent.* The system, before model reduction, appears as in (5). The descriptor matrices have been projected to a reduced space of size $q = 4$ obtained by solving the full system at frequencies $f = 0,\ 1.3GHz,\ 3.3GHz,\ 8GHz$. which correspond to the points on the unit circle: $z = 1,\ exp(-j\pi/4),\ -j,\ exp(-j3\pi/4)$. As center frequency we have chosen $\lambda = 2\pi \times 3.3GHz$. In this example we have chosen $M = 128$ points for the FFT size. We have then truncated the DFT series to N=23 coefficients, producing a final model of order 96. Fig. 2 compares the frequency response of the original full system with the frequency response of the final realized linear state space model. Fig. 3 shows that all the poles of the realized model are in the half-plane $Re(s) < 0$, hence the system is stable. Using the Positive-Real Lemma [7], we confirmed that the generated system is passive.

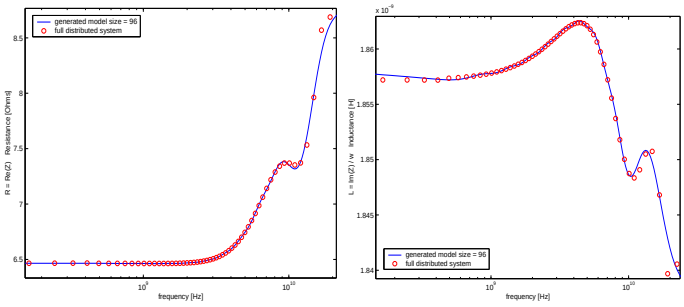


Figure 2: Real part and imaginary part divided by ω of the frequency response for the full-wave example.

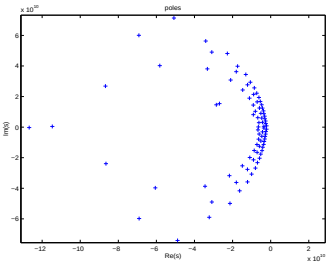


Figure 3: All poles of the reduced fullwave model are stable.

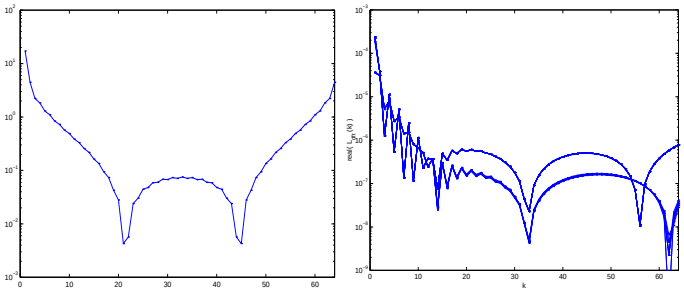
6.3 A non-causal example

Since $\mathbf{R}(s)$ and $\mathbf{L}(s)$ are frequency domain representations, when we calculate their FFT interpolants we obtain something related to their time domain impulse response (actually the impulse response of a discrete-time system obtained by sampling the continuous-time system at rate λ). Since the FFT produces M coefficients that repeat periodically, the $k = M/2, \ldots, M-1$ coefficients are related to negative-time part of the time domain impulse response of $\mathbf{R}(s)$ and $\mathbf{L}(s)$. In Fig. 4.a we show the FFT coefficients of a common *non-causal* example mentioned in Section 2: $\mathbf{Z}_{\text{skin}}(i\omega) = \mathbf{R}_0 + \mathbf{R}_{ac}\sqrt{|\omega|}$ used as a model for the "resistance" of interconnect in the skin-effect regime. We can easily notice in such Figure that the coefficients $k = M/2, \ldots, M-1$ related to non-causal coefficients of the time domain response are non-zero. If a model order reduction is attempted on such an originally non-causal system, one will obtain non-stable models. Alternatively one could deliberately ignore the non-causal coefficients and set them to zero before beginning the reduction. However in this case stable but highly inaccurate models will be produced.

In our approach we require therefore that the original system descriptor matrices $\mathbf{R}(s)$ and $\mathbf{L}(s)$ be causal. This means checking that the non-causal $k = M/2, \ldots, M-1$ coefficients of the FFT be zero except for some aliasing phenomena. Fig. 4.b shows the real part of the DFT coefficients for the $\mathbf{L}(s)$ in example 6.1. One can easily verify that such original system matrix descriptor is actually a causal one.

7. CONCLUSIONS

In this paper we have presented a class of algorithms for guaranteed passive model order reduction of strictly passive and causal linear systems with frequency dependent matrices (distributed systems). Our approach is based on the key idea that if a system is strictly positive-real to begin with, a *globally* and *uniformly* convergent interpolant will eventually (for a large enough order of the interpolant) be positive-real as well.Laguerre basis are a set of well-behaved uniformly convergent interpolation functions whose coefficients can be conveniently calculated using the FFT algorithm. An implementation using a Laguerre basis as interpolant is given and examples are presented. While the Laguerre basis reduces the



a) b)

Figure 4: a) FFT coefficients for the non-causal example $\mathbf{Z}_{\text{skin}}(i\omega) = \mathbf{R}_0 + \mathbf{R}_{ac}\sqrt{|\omega|}$ used as a model for the skin-effect "resistance". b) FFT coefficients of the inductance matrix for the lossy substrate example. Note that non-causal coefficients $32-63$ are in a) very large and in b) very small.

infinite order of the original distributed system to a finite order, a standard Krylov subspace congruence transformation can still be employed to reduce the size of the matrices. The algorithm is also perfectly compatible with fast matrix-vector product algorithms.

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