Guaranteed Passive Balancing Transformations for Model Order Reduction

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***Abstract*—The major concerns in state-of-the-art model reduction algorithms are: achieving accurate models of sufficiently small size, numerically stable and efficient generation of the models, and preservation of system properties such as passivity. Algorithms, such as PRIMA, generate guaranteed-passive models for systems with special internal structure, using numerically stable and efficient Krylov-subspace iterations. Truncated balanced realization (TBR) algorithms, as used to date in the design automation community, can achieve smaller models with better error control, but do not necessarily preserve passivity. In this paper, we show how to construct TBR-like methods that generate guaranteed passive reduced models and in addition are applicable to state-space systems with arbitrary internal structure.**

***Index Terms*—Passive reduced-order modeling, truncated balanced realization (TBR), Lur'e equations, Lyapunov equations, Krylov subspace.**

I. Introduction

MODEL reduction has been an active research field in design automation over the past decade. In an integrated circuits context, initial interest in model reduction techniques stemmed from efforts to accelerate analysis of circuit interconnect [1]. More recently, model reduction has come to be viewed as a method for generating compact models from all sorts of physical system modeling tools [2]–[11].

Because of the need to obtain accurate high-order models at reasonable computational cost, the Krylov-subspace model reduction methods [12]–[14] have occupied the forefront of research over the past five years. The importance of producing *passive* reduced models has been realized, and several algorithms that preserve passivity of *RLC* circuits have appeared [14]–[18].

Recently, it has become apparent that, while very suitable for analysis for analysis of large-scale systems, Krylov techniques such as PRIMA and PVL do not necessarily generate models as compact as desired (that is, small in order for a given accuracy). [2], [19]. Therefore, another approach, that of truncated balanced realization (TBR), already well-developed in the control literature, has been receiving renewed attention in the electronic design automation community.[20].

Truncated balanced realization algorithms (and their close relatives that generate optimal norm approximants [21]) are of importance in their own right. For small systems—a few hundred states or so—they are superior in accuracy to the Krylov and other parameter-matching techniques, and also provide computable bounds on the reduction error. For large systems, direct application of the techniques used to balance and truncate the systems is computationally infeasible, since the computations required have $O(n^3)$ complexity when performed directly (n being the order of the system to be reduced). Therefore, the TBR methods are of more interest when combined with iterative Krylov-subspace procedures. One formulation of this method is to directly solve the large Lyapunov equations via a Krylov subspace method [22]–[26]. The reduced models are obtained directly from the reduced Lyapunov equation. Another viewpoint is to obtain an initial reduced model via some initial reduction or approximation technique and then further compress it using a TBR method. This second viewpoint is somewhat more general since the initial approximation can be generated by any desired method, for example rational fitting [27], [18] or a now standard Krylov-subspace technique [2], [28].

An issue with the TBR-type methods that has not been addressed in most of the above mentioned works is that they cannot be relied on to preserve passivity. The techniques in [2], [28] use a passivity-preserving initial reduction, but follow this reduction with a standard TBR method. There is no guarantee that the second TBR step will not destroy the passivity of the initial model. More problematic, no means is given in either work to determine if the final model is passive – or not.

Less widely appreciated is another dilemma: Krylov methods such as PRIMA have practical issues that prevent their wide application to systems outside the class of *RLC* circuits. These methods rely on congruence transformations to preserve positive semi-definiteness of the matrices that are internal to the state-space representation. However, whether or not a state-space model represents a passive system is a property of the input-output transfer function, not a property of the internal representation. Many passive systems are not conveniently put into a form for which algorithms such as PRIMA are applicable: they may have system and descriptor matrices that are not positive semi-definite. It may not be possible to perform a change of basis to a convenient form without destroying sparse structure that may be present in the system, meaning that for large-scale systems such an approach is infeasible. Some examples include

Manuscript received June 17, 2002; revised December 22, 2002. This paper was recommended by Associate Editor D. T. Blaauw.

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Digital Object Identifier 10.1109/TCAD.2003.814949

the systems that come from rational approximation of tabular data [29], the magnetic charge formulation of the inductance problem [30], and general linear circuits, in particular those with gyrators, formulated in the sparse tableau form. This issue even appears in *RLC* circuits: the positive definiteness of the matrices in the MNA formulation depends on the choices of signs (the circuit response is of course invariant to this choice).

Further, positive semi-definiteness or even more generally positive-realness are not necessarily the right properties to seek. If the state-space model represents scattering (S) parameters of a passive system, the system is passive if the norm of the S-parameter matrix is bounded by unity, and so even the transfer function has no relation to positive-realness. Such systems cannot be reduced by congruence with any passivity guarantees. On the other hand, while not well known in the circuit simulation and design automation communities, there is a wealth of knowledge in the systems and control literature pertaining to passivity-related concepts (e.g., [31]–[33]). Likewise, there exist reduction algorithms (e.g., [34]) with potentially relevant properties, though again the significance of the connection does not appear to be widely appreciated, as, to the best of our knowledge, no effective truly general-purpose passivity-preserving algorithms are now widely available. While we will not present a truly general, large-scale, passivity-preserving, completely structure-independent algorithm in this paper, by collecting, applying, and extending previously obscure techniques, all in the context of large-scale integrated circuit analysis and Krylov methods, we hope to provide a first step to that goal.

In this paper, we discuss TBR-like model reduction algorithms that can preserve system passivity, have computable error bounds, and, unlike other algorithms such as PRIMA, pose no constraints on the internal structure of the state-space model. We describe variants that preserve both positive-realness (useful for systems that represent Y or Z parameters) and bounded-realness (useful for systems that represent S parameters). These algorithms can be applied directly to a given state-space description [27], or can be used as the second stage of a Krylov-subspace-based procedure [22], [24], [2]. In circuit-related applications, extra care must be taken so that TBR-type methods produce models with accurate steady-state response. We show how to incorporate a particular solution [35] into our overall approach.

The paper is organized as follows. In Section II, we briefly present the relevant concepts and properties of the systems we will be treating, as well as review Krylov-subspace based methods in the context of model order reduction. In Section III, we review balanced realizations, present an algorithm for the procedure as well as some physical interpretation, and recall available error bounds for truncation of the models. We also discuss an important special case in which this technique actually produces passive reduced models. Then, in Section IV, we present a procedure for constructing TBR-like methods that guarantee passive reduced models and in addition are applicable to state-space systems with arbitrary internal structure. Algorithmic procedures are shown, and a physical interpretation is provided, along with error bounds for the algorithms introduced. In Section V we discuss various computational issues and present techniques needed to compute the passivity-preserving balancing transformations. Then in Section VI we show examples that illustrate the relevance of the various algorithms presented in this paper. Finally, in Section VII, conclusions and acknowledgments are presented.

II. BACKGROUND

A. State-Space Models

Given a state-space model in descriptor form

$$E\frac{dx}{dt} = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) + Du(t) \tag{2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times p}$, $y(t)$, $u(t) \in \mathbb{R}^p$, model reduction algorithms seek to produce a similar system

$$\tilde{E}\frac{d\tilde{x}}{dt} = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \tag{3}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \tag{4}$$

where $\tilde{E}, \tilde{A} \in \mathbb{R}^{q\times q}$, $\tilde{B} \in \mathbb{R}^{q\times p}$, $\tilde{C} \in \mathbb{R}^{p\times q}$, $\tilde{D} \in \mathbb{R}^{p\times p}$, of order q much smaller than the original order n, but for which the outputs $y(t)$ and $\tilde{y}(t)$ are approximately equal for inputs $u(t)$ of interest. Often the transfer functions

$$H(s) = D + C(sE - A)^{-1}B \tag{5}$$

$$\tilde{H}(s) = \tilde{D} + \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} \tag{6}$$

are used as a metric for approximation: if $\|H(s) - \tilde{H}(s)\| < \varepsilon$, in some appropriate norm, for some given allowable error ε and allowed domain of the complex frequency variable s, the reduced model is accepted as accurate.

B. Passivity

When modeling passive systems—those that cannot produce energy internally—it is desired that the reduced models also be passive. Otherwise, the reduced models may cause nonphysical behavior when used in later simulations, such as by generating energy at high frequencies that causes erratic or unstable time-domain behavior. For many electrical systems of interest, passivity is implied by positive-realness of the transfer function. The function $H(s)$ is positive-real (PR)[1] if

$$\overline{H(s)} = H(\overline{s}), \tag{7}$$

$$H(s) \text{ is analytic in } \{s : \mathrm{Re}(s) > 0\}. \tag{8}$$

$$H(s) + H(s)^H \geq 0 \text{ in } \{s : \mathrm{Re}(s) > 0\}. \tag{9}$$

In the above, $\overline{H}$ denotes complex conjugate, H^H denotes Hermitian (complex conjugate and transpose), and ≥ 0 in a matrix context denotes semi-definiteness. In particular, if $H(s)$ represents the Y (admittance) or Z (impedance) parameters of a system, positive-realness of $H(s)$ implies that the underlying state-space description is a representation of a passive system [33]. If, however, $H(s)$ represents the S (scattering) parameter matrix, then to represent a passive system, it is necessary that

[1]Actually condition (8) is implied by (9).

$H(s)$ be bounded-real [33]. A function $H(s)$ is bounded-real (BR) if

$$\overline{H(s)} = H(\overline{s}), \tag{10}$$
$$H(s) \text{ is analytic in } \{s : \mathrm{Re}(s) > 0\}. \tag{11}$$
$$I - H(s)^H H(s) \geq 0 \text{ in } \{s : \mathrm{Re}(s) > 0\}. \tag{12}$$

The term "bounded" arises as (12) is equivalent to stating that $||H(s)||_2 \leq 1$ in the open right-half plane.

C. Krylov Methods

Recently developed model reduction methods suitable for application to large systems are based on Krylov-subspace techniques.

Definition 1 (Krylov Subspace): The Krylov subspace $K_m(\mathcal{T}, r)$ generated by a matrix $\mathcal{T}$ and vector r, of order m, is the space spanned by the set of vectors $\{r, \mathcal{T}r, \mathcal{T}^2 r, \ldots, \mathcal{T}^{m-1} r\}$. Mathematically, the reduced models are obtained via a projection operation [36]

$$\tilde{E} = W^T E V \quad \tilde{A} = W^T A V \quad \tilde{B} = W^T B \quad \tilde{C} = CV. \tag{13}$$

and usually V and W are constructed so that their columns span a Krylov subspace. For example, a typical implementation (PRIMA [37]) is to construct $V = W$ by using the Arnoldi algorithm, thereby spanning a Krylov subspace with $\mathcal{T} = A^{-1}E$, $r = A^{-1}B$. Because of the moment-matching properties of Krylov-subspaces, the reduced transfer function $\tilde{H}(s)$ will agree with the original $H(s)$ up to the first q derivatives on an expansion around some chosen point in the complex plane (usually $s = 0$). PVL [12] uses the Lanczos algorithm to construct two Krylov spaces for formation of V and W. Multipoint approximation algorithms use unions of multiple Krylov spaces to match the frequency response about several points in the complex plane [36], [38].

The Krylov-subspace methods are very effective if operations with $\mathcal{T}$ can be obtained cheaply (via efficient matrix solves or matrix-vector products), the dimension p of the input space is not too large, and sufficient accuracy can be obtained with a reasonable model order q. These conditions usually hold in practical applications.

Algorithm 1: Reduction via Congruence
1) Compute V (e.g., via the Arnoldi algorithm)
2) Compute realization of reduced model as $\tilde{E} = V^T E V$, $\tilde{A} = V^T A V$, $\tilde{B} = V^T B$, $\tilde{C} = CV$, $\tilde{D} = D$

The PRIMA algorithm has another interesting property. Given a starting passive model, if the original state-space model can be formulated with positive semi-definite A and E and $B = C^T$, then the transfer function of the final reduced model will be positive-real, meaning the reduced system is also passive. This is essentially because the projection operation in (13) becomes a congruence transform for $W = V$, and since congruence transforms preserve positive semi-definiteness, the reduced $\tilde{E}, \tilde{A}$ inherit the numerical range properties of their parents, implying that the reduced function $\tilde{H}(s)$ is positive-real. A typical algorithm for reduction using congruence transformations is shown as Algorithm 1. Note however that it is entirely possible to have systems with positive-real $H(s)$, and thus underlying passive models, for which the conditions necessary for using PRIMA do *not* hold. Such systems cannot be reduced in a guaranteed positive-real manner via congruence transformations. Likewise, congruence-based techniques cannot guarantee bounded-real reduced models from bounded-real starting systems.

A simple example serves to illustrate. Consider the state-space model described by the matrices

$$E = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1.2 & 12 \\ 0 & 0 & -1 \end{bmatrix} \tag{14}$$
$$B = C^T = \begin{bmatrix} 1 \\ 0.2 \\ -0.2 \end{bmatrix}, \quad D = 0.02. \tag{15}$$

By use of the positive-real lemma [33], it can be shown that the transfer function $H(s) = D + C(sI - A)^{-1}B$ is positive-real. However, since the A matrix is indefinite (i.e., $A + A^T$ has eigenvalues of both signs), not all $\tilde{A}$ obtained by orthogonal projection are definite, so the reduced transfer function is not necessarily positive-real. In particular, choose $Q = (sI - A)^{-1}B$ with $s = 1$, and let the projectors be $W = V = Q/||Q||_2$. Then, the one-state reduced model has $\tilde{A} = +0.358$, and $\tilde{H}(0) = -0.301$. The reduced model is not only not positive-real, it is not even stable.

III. Truncated Balanced Realizations

Complementary model reduction techniques are based on truncated balanced realization. We are mostly interested in applying TBR procedures as the second stage of a composite model reduction procedure [2], the first stage being reduction by a Krylov-based projection method. Note that most of the algorithms in [22], [24], are essentially equivalent to a first-stage Krylov projection followed by a second-stage TBR procedure. We first discuss the most commonly used approach before presenting passivity-preserving variants. Primarily for clarity, in this section we will assume $E = I$, as this assumption simplifies the computational procedure, and also facilitates comparisons with the literature related to truncated balanced realization procedures. In addition, most of the cases of interest for this paper can be easily manipulated to the $E = I$ form. When E is nonsingular, the mapping $E \rightarrow I$, $A \rightarrow E^{-1}A$, $B \rightarrow E^{-1}B$ will put a descriptor system into standard form for a system of ordinary differential equations. Even though it is common in electrical engineering applications to have situations where E is in fact singular and cannot be inverted, in the situations of interest to us, where an initial projection step has taken place, usually E is nonsingular. The reason for this is that most Krylov-subspace algorithms in the literature build projection spaces from powers of the matrix $A^{-1}E$ acting on a seed space given by B. To obtain a singular E-matrix after a projection step, the space used for projection would need to include the nullspace of E, but the nullspace is "filtered out" by the Krylov procedure, and so will not enter the projection procedure except for special choices of the seed vectors, or unless deliberately included by special means. See [39] for a reduction procedure that can result in singular E. We will defer treatment of cases with singular E to Section V-B, where it is shown how to manipulate these cases into the form discussed here. We do emphasize that it *is* possible to formulate the computational procedure to work with E directly,[2] but we have chosen not to do this as a matter of convenience.

A. Standard Approach

The TBR procedure as first presented in [20] is centered around information obtained from the controllability Grammian W_c, which can be obtained from solving the Lyapunov equation

$$AW_c + W_cA^T = -BB^T \tag{16}$$

for W_c, and the observability Grammian W_o, which can be obtained from the dual Lyapunov equation

$$A^TW_o + W_oA = -C^TC \tag{17}$$

for W_o.

Under a similarity transformation of the state-space model

$$A \to T^{-1}AT, \quad B \to T^{-1}B, \quad C \to CT \tag{18}$$

the input-output properties of state-space model, such as the transfer function, are invariant (only the internal variables are changed). The grammians, however, vary under the rules

$$W_c \to T^{-1}W_cT^{-T}, \quad W_o \to T^TW_oT \tag{19}$$

and so are not invariant. The TBR procedure is based on two observations about W_o and W_c. First, the eigenvalues of the product W_cW_o are invariant. These eigenvalues, the Hankel singular values, contain useful information about the input–output behavior of the system. In particular, "small" eigenvalues of W_cW_o correspond to internal sub-systems that have a weak effect on the input-output behavior of the system and are, therefore, close to nonobservable or noncontrollable or both. Second, since the Grammians transform under congruence, and as any two symmetric matrices can be simultaneously diagonalized by an appropriate congruence transformation [40], it is possible to find a similarity transformation T that leaves the state-space system dynamics unchanged, but makes the (transformed) $\hat{W}_o$ and $\hat{W}_c$ equal and diagonal[3]. In these coordinates, with $\hat{W}_c = \hat{W}_o = \Sigma$, we may partition Σ into

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{20}$$

where Σ_1 describes the "strong" sub-systems to be retained and Σ_2 the "weak" sub-systems to be deleted. Conformally partitioning the transformed matrices as

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{C} = [\hat{C}_1 \quad \hat{C}_2] \tag{21}$$

[2]For example, (16), becomes $AW_cE^T + EW_cA^T = -EBB^TE^T$.

[3]To see this it may help to note that W_c^{-1} and W_o transform according to the same congruence operation; but if W_c^{-1} is diagonalized, so is W_c.

and truncating the model, retaining $\tilde{A} = \hat{A}_{11}, \tilde{B} = \hat{B}_1, \tilde{C} = \hat{C}_1$ as the reduced system, therefore has the effect of deleting the "weak" internal subsystems. A complete TBR algorithm [41] is shown as Algorithm 2. An approach with improved numerical properties may be found in [42].

Algorithm 2: Truncated Balanced Realization (TBR)

1) Solve $AW_c + W_cA^T = -BB^T$ for W_c
2) Solve $A^TW_o + W_oA = -C^TC$ for W_o
3) Compute Cholesky factors $W_c = L_cL_c^T$, $W_o = L_oL_o^T$,
4) Compute SVD of Cholesky product $U\Sigma V = L_o^TL_c$ where Σ is diagonal positive and U, V have orthonormal columns
5) Compute the balancing transformations $T = L_cV\Sigma^{-1/2}$, $T^{-1} = \Sigma^{-1/2}U^TL_o^T$
6) Form the balanced realization as $\hat{A} = T^{-1}AT$, $\hat{B} = T^{-1}B$, $\hat{C} = CT$
7) Select reduced model order and partition $\hat{A}$, $\hat{B}$, $\hat{C}$ conformally
8) Truncate $\hat{A}$, $\hat{B}$, $\hat{C}$ to form the reduced realization $\tilde{A}$, $\tilde{B}$, $\tilde{C}$

B. Error Bounds

One of the attractive aspects of TBR methods is that computable error bounds are available. If the i-th diagonal entry of the matrix Σ in Algorithm 2 is given by σ_i, and the σ_i ordered $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$, then the error in the transfer function of the order-q reduced model is bounded by [21]

$$\|H(s) - \hat{H}_q(s)\|_\infty \leq 2 \sum_{k=q+1}^{N} \sigma_k. \tag{22}$$

C. Physical Interpretation of the TBR Procedure

In order to later contrast the physical significance of passivity-preserving TBR methods, here we review the physical interpretation of the method in Algorithm 2.

The observability Grammian W_o is related to the L_2 norm of the output produced in free evolution ($u(t) = 0, \forall t \geq 0$) from an initial state x_0 at time 0

$$x_0^TW_ox_0 = \int_0^\infty y(t)^Ty(t)dt, \quad x(0) = x_0,\ u(t) = 0 \forall t \geq 0. \tag{23}$$

The controllability Grammian W_c, on the other hand, is related to the minimum L_2 norm of the input over all possible input that can control the system to the state x_0 at time 0.

$$x_0^TW_c^{-1}x_0 = \inf\left\{\int_{-\infty}^0 u(t)^Tu(t)dt,\ u(t) \text{ controlling to } x(0) = x_0\right\}. \tag{24}$$

Noting that $\int_0^\infty y(t)^Ty(t)dt$ and $\int_{-\infty}^0 u(t)^Tu(t)dt$ are the L_2 norms of the system output (restricted to $t \geq 0$) and the system input (on $t \leq 0$) respectively, it is seen that small eigenvalues of the observability Grammian W_o are associated with "normal modes" [20] (state eigenvectors) that produce small free evolution L_2 output norms. These modes are, therefore, relatively unimportant for the system response. Similarly, small eigenvalues of the controllability Grammian W_c are associated with state eigenvectors that we can control only with an input with large L_2 norm (regardless of what trajectory we follow to reach them). Hence the system is not very likely to be driven into those states and they are not likely to be important for the system response. It can be noticed that some modes, although difficult to be controlled by the input, could produce large outputs. Vice versa, there can be some modes that, although producing small output norms, are controlled with small input norm. This is the reason for the balancing procedure that transforms to coordinates that "balance" the importance of past inputs and future outputs, the weighting being revealed by the eigenvalues of the product of the observability and controllability Grammians. The algorithm will keep in the final reduced model only modes that are:

- both easily controllable, meaning they do not require a large input L_2 norm to reach, and
- easily observable, meaning that they produce free evolution outputs with large L_2 norms.

We now turn to the question of when TBR procedures produce passive reduced models.

D. Passivity Preservation in Symmetrizable Systems

It turns out that there is a special system case, of relevance to integrated circuits applications, for which the standard TBR procedure (Algorithm 2) always produces positive-real reduced models. Suppose that the state-space model is *symmetric*, that is $A = A^T, B = C^T$, and furthermore A is negative semi-definite. Since $\mathrm{Re}\{j\omega I - A\} = -1/2(A + A^T) = -A \geq 0$, the system is positive-real. From (16) and (17) it follows that $W_o = W_c$. From inspecting step 5 in Algorithm 2, we find that $T^{-1} = T^T$. *Thus the similarity transformation is a congruence transformation.* The reduced $\tilde{A}$ must be negative semi-definite, and we will likewise have $\tilde{B} = \tilde{C}^T$. Therefore, the reduced system is positive-real. We, therefore, have the following broader result:

Theorem 1: Suppose a state-space system is linearly transformable to a system of the form in (1), with $E = I, B = C^T$, $A = A^T, A \leq 0, D \geq 0$. A reduced model generated via Algorithm 2 is positive-real.

Proof: If the system is already in the special stated form, it is positive-real [14]. That the TBR procedure applies to systems transformable to this form follows because the balancing transformation is essentially unique as explained in [20]. In particular, given a state-space model in balanced coordinates, the matrix $\hat{A}$ can differ from any other $\hat{A}'$ also in balanced coordinates by at most a similarity transformation by a matrix P that is diagonal with diagonal entries ± 1. Therefore, there exists some P and some T such that

$$\hat{A}' = P^{-1}\hat{A}P = P^T T^T A T P = (PT)^T A (PT) = (T')^T A T'. \tag{25}$$

Likewise, for any $\hat{B}$, $\hat{C}$ in balanced coordinates, there is some T' such that $\hat{B} = (T')^T B$, $\hat{C} = CT'$. ∎

Theorem 1 would seem to state that the TBR procedure has passivity-preserving properties similar to PRIMA, but it is actually more general in one sense, and more restrictive in another. It is more general in the sense that the starting system need only be *transformable* to a symmetric, internally positive-real system. The passivity-preserving reduction property is independent of the system coordinates. This is a very desirable property. In contrast, the positive semi-definiteness preserving properties of congruence transformations depend on the coordinate system used and are *not* preserved under similarity transformations.

Theorem 1 is undesirably restrictive in the sense that it only applies to systems that fall into the symmetrizable class, such as *RL* and *RC* circuits in MNA form, and reductions of such forms via congruence. Not all systems, however, fall into this class, and more powerful techniques are needed to preserve passivity in TBR methods. These techniques will be coordinate-independent, and completely general.

IV. Passivity-Preserving Truncated Balanced Realizations

A. Positive Real Conditions

We will show in Section VI, by means of a simple example, that the TBR procedure of Algorithm 2 does not necessarily produce passive models. In making assessments about passivity, we require a tool that can assess the positive-realness of a state-space model in a global manner. One such tool is the positive-real lemma [33], which states that $H(s)$ is positive-real if and only if there exist matrices $X_c = X_c^T$, J_c, K_c such that the Lur'e equations

$$\begin{aligned} AX_c + X_c A^T &= -K_c K_c^T && (26)\\ X_c C^T - B &= -K_c J_c^T && (27)\\ J_c J_c^T &= D + D^T && (28)\end{aligned}$$

are satisfied, and $X_c \geq 0$ (X_c is positive semi-definite). X_c is analogous to the controllability Grammian. In fact, it is the controllability Grammian for a system with the input-to-state mapping given by the matrix K_c. It should not be surprising that there is a dual set of Lur'e equations for $X_o = X_o^T > 0$, J_o, K_o that are obtained from (26)–(28) by the substitutions $A \to A^T$, $B \to C^T$, $C^T \to B$. The dual equations

$$\begin{aligned} A^T X_o + X_o A &= -K_o^T K_o && (29)\\ X_o B - C^T &= -K_o^T J_o && (30)\\ J_o^T J_o &= D + D^T. && (31)\end{aligned}$$

have a corresponding observability quantity $X_o \geq 0$ for a positive-real $H(s)$. It is easy to verify that X_c, X_o transform under similarity transformation just as W_c, W_o (19), that their eigenvalues are invariant, and in fact in most respects they behave as the Grammians W_c, W_o.

B. Guaranteed Passive Balanced Truncations

A passivity-preserving reduction procedure follows by noting that the Lur'e equations can be solved for the quantities X_c, X_o which may then be used as the basis for a TBR procedure. We may find a coordinate system in which $\hat{X}_c = \hat{X}_o = \Sigma$, with Σ again diagonal. In this coordinate system, the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ may be partitioned and truncated, just as for the standard TBR procedure, to give the reduced model defined by ($\tilde{A}$, $\tilde{B}$, $\tilde{C}, D$). We present this as Algorithm 3 and call it PR-TBR, as it preserves positive-realness of the transfer function. Several approaches that turn out to give essentially similar results have appeared previously in different contexts [34], [43], [44].

Algorithm 3: Positive-Real TBR (PR-TBR)
1) Solve **(26)–(28)** for X_c and **(29)–(31)** for X_o.
2) Proceed with steps 3–8 in Algorithm 2, substituting X_c for W_c and X_o for W_o.

Theorem 2: Algorithm 3 applied to systems with positive-real transfer functions produces reduced models with positive-real transfer functions.

Proof: From the form of the partitioning, (20) and (21), likewise partitioning either $\hat{K}_c$ or $\hat{K}_o$, it is clear that the reduced system, in the PR-balanced coordinates, satisfies

$$\hat{A}_{11}\Sigma_1 + \Sigma_1 \hat{A}_{11}^T = -\hat{K}_1 \hat{K}_1^T \tag{32}$$
$$\Sigma_1 \hat{C}_1^T - \hat{B}_1 = -\hat{K}_1 \hat{J}_c^T, \tag{33}$$
$$\hat{J}_c \hat{J}_c^T = \hat{D} + \hat{D}^T. \tag{34}$$

Therefore, the reduced system satisfies the Lur'e equations with positive semi-definite Σ_1 ($\Sigma_1 \geq 0$ as $\Sigma \geq 0$). By the positive-real lemma, the reduced system is positive-real. ■

We emphasize that Theorem 2 holds regardless of the internal form of the state-space system. Again, this is not true for congruence-based procedures.

C. Bounded-Real Conditions

To obtain equivalent TBR procedures that guarantee a final transfer function that is bounded-real, useful when working with transfer functions representing S-parameters, we need the bounded real equations

$$AY_c + Y_c A^T = -BB^T - K_c K_c^T \tag{35}$$
$$Y_c C^T + BD = -K_c^T J_c \tag{36}$$
$$J_c J_c^T = I - D^T D \tag{37}$$

and the corresponding dual equations

$$A^T Y_o + Y_o A = -C^T C - K_o^T K_o \tag{38}$$
$$Y_o B + C^T D = -K_o J_o^T, \tag{39}$$
$$J_o^T J_o = I - D^T D \tag{40}$$

that are satisfied with $Y_c \geq 0$, $Y_o \geq 0$ if the system transfer function is bounded-real. Algorithm 4 performs truncated balanced realization while guaranteeing the boundedness of the reduced transfer function[4].

Algorithm 4: Bounded-Real TBR (BR-TBR)
1) Solve **(35)–(37)** for Y_c and **(38)–(40)** for Y_o.
2) Proceed with steps 3–8 in Algorithm 2, substituting Y_c for W_c and Y_o for W_o.

D. A Hybrid Approach

In many cases, while not guaranteed by construction, it is often the case that the TBR approximants produced by Algorithm 2 turn out to be positive-real. Therefore, we propose Algorithm 5, which performs the TBR procedure, solves the positive-real (or bounded-real) equations for the reduced model in order to check its passivity, and if it turns out not to be passive, discards it and proceeds to Algorithm 3 (or Algorithm 4). There is an advantage in this procedure as often the TBR approximates are more accurate for a given order than PR-TBR. Because of the cubic scaling of cost, it is relatively cheap, compared to the cost of the TBR reduction, to check a reduced model for passivity since the reduced system is presumably of lower order. As often the TBR models are passive, the net effect of the composite algorithm is to approximately double the cost in the worst case, versus usually getting better models at smaller cost (PR-TBR "costs" more than TBR) in the more-common average case.

Algorithm 5, which appropriately combines all of the previously presented algorithms, can be used as generic flow for generating accurate guaranteed passive reduced-order models of systems with arbitrary structure.

Algorithm 5: Hybrid TBR
1) Perform Algorithm 2.
2) Using the reduced model matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, solve **(26)–(28)** for $\tilde{X}_c$ (or **(35)–(37)** for Y_c).
3) **if (26)–(28)** (or **(35)–(37)**) are solvable and $\tilde{X}_c \geq 0$ (or $\tilde{Y}_c \geq 0$),
then terminate and return $\tilde{A}$, $\tilde{B}$, $\tilde{C}$.
else discard TBR-reduced model and proceed with Algorithm 3 (or 4).

E. Error Bounds

As mentioned in Section III-B, one of the attractive aspects of TBR-like methods is that computable error bounds are available. Fortunately that is also the case for the positive-real (bounded-real) algorithms introduced in this paper. For bounded-real systems, as discussed in Section IV-C, if the ith diagonal entry of the matrix Σ obtained from Algorithm 4 is given by ξ_i, and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_N$, then the error in the transfer function of the order-q reduced model is bounded by

$$\|H(s) - \tilde{H}(s)\|_\infty \leq 2 \sum_{k=q+1}^{N} \xi_k \tag{41}$$

an expression strikingly similar to (22) (see [43]). For the positive-real case unfortunately no such simplified bound exists. Under the same definitions, the best available error bound is given by

$$\|H(s) - \tilde{H}(s)\|_\infty \leq \lambda_{\max}(D + D^T) \sum_{k=q+1}^{N} \frac{2\xi_k}{(1-\xi_k)^2} \left(1 + \sum_{j=1}^{k} \frac{2\xi_j}{1-\xi_j}\right)^2 \tag{42}$$

(see [44] where a similar technique was proposed and the error bound was derived). Discussion and derivation of these bounds is beyond the scope of this paper. Note however, that for the specific procedure discussed here, the ξ_i will all be less that unity if D is nonsingular. If D is singular, and has rank r, then $p - r$ singular values will be identically unity. Modes corresponding

[4]The bound does not have to be unity; it can be any positive constant.

to these singular values cannot be truncated. In the case where $r < p$, the bound of (42) becomes nonsensical. In this case, we are not aware of a procedure to compute a bound.

F. Physical Interpretation of the PR/BR-TBR Procedures

In Section III-C we discussed how the TBR procedure, and eigenvalues of associated Grammians, could be interpreted in terms of the relative importance of system modes to the system input and output norms. It turns out that the PR/BR-TBR techniques have a similar interpretation, but one that is more closely tied to a circuit-theoretic notion of energy. To make this connection, we draw upon concepts from the theory of dissipative dynamical systems, discussion of which can be found in [31], [32], [43].

In order to provide a physical interpretation for the PR/BR-TBR algorithm let us introduce the concept of a supply function $s[u(t), y(t)]$. A supply function describes the rate at which power is supplied by the environment into the system, and typically is defined such that $s[u(t), y(t)] > 0$ implies a positive amount of energy input, while $s[u(t), y(t)] < 0$ means energy is extracted from the system back to the environment. When the system inputs and outputs are currents or voltages, i.e., when the system transfer function represents impedance or admittance matrices, we may use the supply function $s[u(t), y(t)] = u(t)^T y(t)$. When the system transfer function represents scattering parameters, we may use the supply function $s[u(t), y(t)] = u(t)^T u(t) - y(t)^T y(t)$. Regardless of the particular form of supply function we can further define the following two quantities:

$$V_{\mathrm{av}}(x_0) = \sup\left\{-\int_0^\infty s[u(t), y(t)]dt, x(0) = x_0\right\} \tag{43}$$

$$V_{\mathrm{req}}(x_0) = \inf\left\{\int_{-\infty}^0 s[u(t), y(t)]dt, u(t) \text{ controlling to } x(0) = x_0\right\} \tag{44}$$

where $V_{\mathrm{av}}(x_0)$ is the *available storage energy*, or maximum energy that can be extracted from the system over any possible trajectory of the state from an initial state x_0 at time 0. $V_{\mathrm{req}}(x_0)$ can be interpreted as the *required supply*, or the minimum amount of energy that must be provided by the environment to the system in order to control the system to state x_0 at time 0 over any possible trajectory. It can be shown ([31], [32]) that for dissipative and controllable systems, $V_{\mathrm{av}}(x_0)$ is always a positive number not larger than $V_{\mathrm{req}}(x_0)$

$$0 \le V_{\mathrm{av}}(x_0) \le V_{\mathrm{req}}(x_0). \tag{45}$$

Furthermore, it can be shown ([31], [32]) that the solutions X_o and X_c to the positive real Lur'e (29)–(31) and their dual (26)–(28), respectively, obtained from the procedure in Section V have a physical interpretation for passive immittance systems in terms of the energy quantities $V_{\mathrm{av}}(x_0)$ and $V_{\mathrm{req}}(x_0)$

$$x_0^T\, X_o\, x_0 = V_{\mathrm{av}}(x_0) \tag{46}$$

$$x_0^T\, X_c^{-1}\, x_0 = V_{\mathrm{req}}(x_0). \tag{47}$$

Using a similar argument to the classical TBR interpretation, small eigenvalues of X_o are associated with modes for which the maximum energy we can extract, $V_{\mathrm{av}}(x_0)$, is small. They are, therefore, not likely to be important "energy-wise" for the system response. Small eigenvalues of X_c are associated with modes for which the minimum amount of energy $V_{\mathrm{req}}(x_0)$ we have to supply in order to reach them is large. Hence it is relatively difficult to drive the system into those states and they are not likely to be important "energy-wise" for the system response.

As in the classical TBR, it can be noticed that some modes, although energy-wise hardly accessible, are energy-wise important and we can extract back from them large amounts of energy. Vice-versa, there can be some modes for which, although we cannot extract large amounts of energy, they require a small amount of energy to reach. Thus, in a similar way as classical TBR, PR-TBR balances the importance of past energy inputs and future energy outputs by transforming to a coordinate system in which X_o and X_c are equal and diagonal, and in which the invariant quantities that are the eigenvalues of the product of X_o and X_c are easily calculated. The algorithm will keep in the final reduced model only modes that are:

- both **"energy-wise"** easily "controllable", that is they do not need much energy input to be reached;
- and **"energy-wise"** easily "observable", that is, it is possible to extract a lot of energy from them.

It is also interesting to note ([34], [31], [43]) that the solutions X_o and X_c of the positive real Lur'e (29)–(31) and their dual (26)–(28), are related and not unique. Specifically, there exists a minimal solution $X_{o,\min}$ and a maximal solution $X_{o,\max}$ for (29)–(31), a minimal solution $X_{c,\min}$ and a maximal solution $X_{c,\max}$ for (26)–(28), such that

$$0 \le X_{o,\min} = X_{c,\max}^{-1} \le X_o \tag{48}$$

$$= X_c^{-1} \le X_{o,\max} = X_{c,\min}^{-1}. \tag{49}$$

The procedure in Section V produces the minimal solutions used in (46)–(47) respectively

$$X_o = X_{o,\min} = X_{c,\max}^{-1} \tag{50}$$

$$X_c = X_{c,\min} = X_{o,\max}^{-1}. \tag{51}$$

The same physical interpretation presented above for positive real systems representing impedance or admittance can be given to bounded real systems representing scattering parameters by defining as in [43]

$$x_0^T\, Y_o\, x_0 = V_{\mathrm{av}}(x_0) \tag{52}$$

$$x_0^T\, Y_c^{-1}\, x_0 = V_{\mathrm{req}}(x_0). \tag{53}$$

where Y_o and Y_c are the minimal solutions of the bounded real Lur'e (38)–(40) and their dual (35)–(37), respectively, obtained from the procedure in Section V.

V. Computational Considerations

In this section, we discuss the computational techniques needed to compute the passivity-preserving balancing transformations. The complexity of the algorithms presented is cubic in the number of state variables, due to the use of direct, dense linear algebra for eigenvector computations and matrix–matrix products. Thus, standard TBR and the passive-TBR variants cannot be directly applied to extremely large systems such as large collections of interconnect because of the high cubic computational complexity. However, this cost is acceptable if the algorithms are being applied to systems that are moderate in size, as is usually the case with systems that result from a prior reduction step. Therefore, we wish to reiterate that in the case of large systems, one would use the TBR algorithms as a *"second step"* of a "two-step" reduction procedure. During the first step, whenever possible (see Section II), one would use the less computationally demanding (but less efficient) Krylov subspace guaranteed passive reduction techniques such as PRIMA to reduce the originally very large system to order around few hundreds. At such point one can easily use without much computational effort passive-TBR to reduce the system to order around 10 to 20. This "two-step" procedure produces a much better compression (i.e., better accuracy for the same final order) than using PRIMA to reduce in one single step the original very large system to the final order around 10 to 20.

Focusing now exclusively on the second-step reduction, first we show how the basic computations may be performed for systems in the standard form of (1)–(2). Next, we show how more general models described by differential-algebraic systems can be put into this form for purposes of reduction. Finally, we discuss techniques needed to handle a special case (singular or zero D-matrix) that often occurs in integrated circuit (IC) applications.

A. Solving the Lur'e Equations

Solution of the Lur'e equations and solution of algebraic Riccati equations (ARE's) are closely related. An overview of basic numerically robust computational procedures is given in [45]. We summarize this procedure below. See [46] and references therein for more recent coverage of related computational problems.

Solution of the Lur'e equations, and related AREs, can be done by computing the invariant subspace of a matrix pencil $\lambda\mathcal{E} - \mathcal{A}$, i.e., solving a generalized eigenvalue problem. Of course, as is usual, for reasons of numerical stability, it is desirable to avoid eigenvector computations, and instead work with Schur (or generalized Schur) forms that can expose invariant subspaces in a numerically stable manner. In the positive-real case, the pencil we need is $\mathcal{E} - \mathcal{A}$,

$$\mathcal{E} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D + D^T \end{bmatrix}. \tag{54}$$

Suppose that via some means we have computed an invariant subspace $Z \in \mathbb{R}^{(2n+p)\times n}$ that satisfies $\mathcal{E}Z\Lambda = \mathcal{A}Z$, $\Lambda \in \mathbb{R}^{n\times n}$, of the special form

$$Z = \begin{bmatrix} I \\ X \\ \tilde{X} \end{bmatrix} \tag{55}$$

where $I, X \in \mathbb{R}^{n\times n}$, $\tilde{X} \in \mathbb{R}^{p\times n}$. Then from the invariance condition $\mathcal{E}Z\Lambda = \mathcal{A}Z$, it can be verified that X is indeed the solution to (26)–(28). To compute an invariant subspace of such a special form, the rank-p singularity of the pencil is first compressed [45] using a QR factorization of $\mathcal{A}$ to reduce the dimension of the pencil to $2n$. Then, we find an invariant subspace $Z_r \in \mathbb{R}^{2n\times n}$ (for example, via the QZ method [47]) of the form

$$Z_r = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{56}$$

X can be computed as $X = X_2 X_1^{-1}$. To obtain the extremal solutions discussed in Section IV-F, which incidentally correspond to minimum-phase spectral factors [43], we take the subspace that corresponds to stable eigenvalues of the pencil. In the regular case, if a stable subspace (in particular, if the pencil has pure imaginary eigenvalues) cannot be constructed, the system is not positive-real.

B. Additive Decomposition of Nonstandard Systems

In general, state-space models must be written in the form

$$E\frac{dx}{dt} = Ax(t) + Bu(t) \tag{57}$$

$$y(t) = Cx(t) + Du(t) \tag{58}$$

where the matrix E is singular. Extensions of the positive-real lemma are available for models in the descriptor form where E is singular such that the transfer function cannot be put into the standard form [46]. A full coverage of all special cases, and the details of manipulating E directly, is beyond the scope of this paper. Instead we propose a simple procedure that will allow us to use the method in Section V-A for solving the positive real equations. We propose performing an additive decomposition of the transfer function $H(s) = D + C(sE - A)^{-1}B$ into the form

$$H(s) = H_\infty + H_p(s) \tag{59}$$

where H_p is strictly proper, i.e., it is a purely rational function, $H_p(s) \to 0$ as $s \to \infty$. H_∞ will contain the portion of the transfer function that is nonzero as $s \to \infty$.

For a bounded real function, $H(s)$ can approach a constant, but can have no higher order terms in s, so

$$H_\infty(s) = D_\infty \tag{60}$$

for some D_∞ such that $\|D_\infty\| \le 1$. For a positive-real function, $H(s)$ may have a pole at infinity, but if so it must be simple with a Hermitian, nonnegative definite, residue matrix [33]. Therefore, in the positive-real case

$$H_\infty = D_\infty + sK_\infty \tag{61}$$

for some D_∞ such that $D_\infty + D_\infty^H \ge 0$, $K_\infty = K_\infty^H \ge 0$.

To perform such a decomposition, by using the procedure from [48] we first transform the system (A, B, C, D, E) to an equivalent system (A_d, B_d, C_d, D, E_d) where the matrix pencil $sE_d - A_d$ is block-diagonal. In this form the system may be written as

$$E_d = \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix} \; A_d = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \tag{62}$$

$$B_d = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \; C_d = [\, C_1 \quad C_2 \,]. \tag{63}$$

The block partitioning is chosen such that the sub-pencil $sE_{22} - A_{22}$ contains all the 'infinite' structure of the system. E_{22} will have zeros on the diagonal, and A_{22} will be nonsingular. (E_{11}, A_{11}) will contain all the finite structure, and E_{11} will be nonsingular by construction. Next, by defining

$$A_p = E_{11}^{-1} A_{11},\; B_p = E_{11}^{-1} B_1,\; C_p = C_1 \tag{64}$$
$$N = A_{22}^{-1} E_{22},\; B_\infty = E_{11}^{-1} B_2,\; C_\infty = C_2 \tag{65}$$

we may transform to an equivalent system $(A_w, B_w, C_w, D_w, E_w)$ that has a structure resembling the Weierstrass form [49],

$$E_w = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \; A_w = \begin{bmatrix} A_p & 0 \\ 0 & I \end{bmatrix}, \tag{66}$$
$$B_w = \begin{bmatrix} B_p \\ B_\infty \end{bmatrix} \; C_w = [\, C_p \quad C_\infty \,]. \tag{67}$$

Here, A_p will be nonsingular and N will be a nilpotent matrix, that is, a matrix such that $N^k = 0$ for some $k > 0$. For the systems of interest in this paper, it turns out that generally $k = 1$ or $k = 2$.

With the system in this form, the transfer function can be decomposed as

$$H(s) = C_p(sI - A_p)^{-1} B_p - C_\infty (I - sN)^{-1} B_\infty \tag{68}$$

so it is clear that the strictly proper term $H_p(s)$ is given by

$$H_p(s) = C_p(sI - A_p)^{-1} B_p. \tag{69}$$

Likewise, it turns out that the matrices N, C_∞, B_∞ contain all information about the "infinite" behavior of the transfer function,

$$H_\infty = -C_\infty (I - sN)^{-1} B_\infty. \tag{70}$$

To see that indeed H_∞ is composed only of terms with powers of s^k for $k \geq 0$, i.e., the terms $D_\infty + sK_\infty$ in our particular case, we inspect the form of $(I - sN)^{-1}$. Because N is nilpotent, $(I - sN)^{-1}$ can be written as a finite series in k terms of powers of sN.

$$\begin{aligned}(I - sN)^{-1} \\ = [I + sN - (sN)^2 + (sN)^3 + \cdots + (-1)^k (sN)^{k-1}].\end{aligned} \tag{71}$$

For example, consider $k = 2$. In this case, noting that $N^2 = 0$

$$(I - sN)(I + sN) = I - sN + sN - (sN)^2 = I - (sN)^2 = I \tag{72}$$

we conclude that $(I - sN)^{-1} = I + sN$. The first two terms in the expansion are enough to calculate the information we need. The first term leads to

$$D_\infty = D - C_\infty B_\infty \tag{73}$$

and the second to

$$K_\infty = -C_\infty N B_\infty. \tag{74}$$

Note that, for a positive-real system, it *must* be the case that either $k = 2$, or $C_\infty N^l B_\infty = 0$ for all $l > 1$. In any event these terms may be deleted from the model without harm since they do not appear in the input/output relation. For bounded real systems, all terms N^l for $l > 0$ may be removed. Note additionally that k is also known as the *index* of the DAE in (57). DAE's with index greater than 1 are not accurately solved by the algorithms typically used in circuit simulators [50].

Finally, consider the state-space model in standard form

$$\frac{dx}{dt} = A_p x(t) + B_p u(t) \tag{75}$$
$$y(t) = C_p x(t) + D_\infty u(t) \tag{76}$$

with transfer function

$$H_a(s) = D_\infty + C_p(sI - A_p)^{-1} B_p = H(s) - sK_\infty. \tag{77}$$

If $H(s)$ is bounded-real, then $K_\infty = 0$ and $H(s) = H_a(s)$. $H_a(s)$ is positive-real if $H(s)$ is positive-real as well, because $H_a(s)$ is analytic in the open right half-plane, and

$$\begin{aligned} H_a(j\omega) + H_a^H(j\omega) &= H(j\omega) + H^H(j\omega) - j\omega(K_\infty - K_\infty^H) \\ &= H(j\omega) + H^H(j\omega) \geq 0. \end{aligned}$$

Therefore, to obtain a passive reduced model of the original system of (57) and (58), we may apply the passivity-preserving model reduction methods to the system in (75) and (76).

As an aside, note that this procedure also provides a means to perform a passivity check on descriptor systems. In particular, the descriptor system of (1) and (2) is positive-real if and only if $K_\infty = K_\infty^T \geq 0$, and the matrices A_p, B_p, C_p, D_∞ satisfy the positive-real conditions (26)–(28) or (29)–(31). The system is bounded-real if and only if $K_\infty = 0$ and the matrices A_p, B_p, C_p, D_∞ satisfy the bounded-real conditions (35)–(37) or (38)–(40).

C. *Infinite Zeroes*

It is known that the simple procedure in Section V-A breaks down when the transfer function $H(s)$ has a zero exactly on the imaginary axis (including a zero at infinity). The positive-real equations can still be solved in this case [34], but more sophisticated computational procedures are necessary. Of particular interest in IC applications is the case $D = 0$. In this case, we have used the method for solving the positive-real equations given in [51].

Note that physical systems, modeled to a high level of fidelity, are usually strictly positive real, so do not have troublesome zeros. The additive decomposition procedure of Section V-B may be necessary to reveal this. Even if D is singular, D_∞ may be nonsingular.

D. *Achieving DC Accuracy*

It is known that the TBR technique in Algorithm 2 often leads to reduced models that exhibit a mismatch of the dc gain when compared to the original unreduced model. It was noticed previously that the technique tends to give good approximations of the impulse response, but the approximation may have a large steady-state error for the step response. In general, in fact, the algorithm tends to perform better at high frequencies than at dc, showing a high-frequency error that tends to zero. While this might be acceptable in certain applications of the technique, it is a significant problem when considered in the context of circuit level model order reduction. Therefore, such property has always been regarded as a significant drawback of the technique.

In order to address this problem, a useful computational tool is transformation to the *reciprocal system*. This approach has previously been presented in the context of what is called the singular perturbation approximation, first when applied to balanced system [35], [52] and also in the context of positive- and bounded-real systems [53]. This approximation relies typically on the fact that it is possible to partition the state vector into two sets of variables. One of these sets is composed of variables which are termed "fast," i.e., its states have very fast transient dynamics and decay rapidly to certain steady values in the neighborhood of a given frequency (for instance $s = 0$ for the dc case that we are interested here, but the technique is easily generalized for any frequency s_0).

Given an initial system with transfer function $H(s)$, the reciprocal system is a state-space system with transfer function $H(s^{-1})$. In some problems, there is an advantage to performing balanced truncation on the reciprocal system because of the frequency properties of the error. The reciprocal transformation maps $s = 0$ into $s = \infty$, so balancing the reciprocal system tends to produce better approximations at low frequencies.

In [35] it was shown that the model obtained with the singular perturbation approximation to an internally balanced system, i.e., the model obtained by truncating the reciprocal system of an internally balanced system, enjoys the same error bounds as the truncated balanced approximation. Furthermore, since it is easy to prove that the reciprocal of a (strictly) positive-real (bounded-real) system is also (strictly) positive-real (bounded-real) (see [53]) it is then clear that one can combine the reciprocal system technique with Algorithm 3. The resulting reduced-order model will, therefore, be guaranteed positive-real (bounded-real). We, therefore, propose Algorithm 6 in order to generate reduced-order models that do not suffer from dc accuracy problems. In fact, the models generated by Algorithm 6 will match the transfer function exactly at dc ($s = 0$).

Algorithm 6: DC-Accurate TBR

1) Perform the reciprocal transformation **[53]** $A \rightarrow A^{-1}$, $B \rightarrow -A^{-1}B$, $C \rightarrow CA^{-1}$, $D \rightarrow D - CA^{-1}B$
2) Perform Algorithm 2, Algorithm 3, or Algorithm 4.
3) Perform the inverse reciprocal transformation $\tilde{A} \rightarrow \tilde{A}^{-1}$, $\tilde{B} \rightarrow -\tilde{A}^{-1}\tilde{B}$, $\tilde{C} \rightarrow \tilde{C}\tilde{A}^{-1}$, $\tilde{D} \rightarrow \tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B}$

It is interesting to note that this behavior of the TBR-like algorithms is reminiscent of the situation in moment-matching Krylov methods: Krylov spaces based on A tend to match well at infinity, while those based on A^{-1} tend to match well at dc. The contrast is much less strong in the TBR case because of its near-optimality properties. On transfer functions with large norm away from either dc or infinity, TBR will achieve good relative error near the large-norm areas preferentially, as must be done to achieve an absolute error bound.

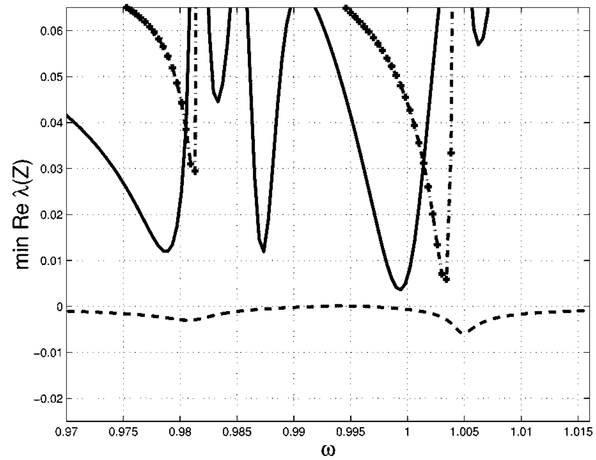


Fig. 1. Minimum eigenvalue of transfer functions used to illustrate non-positive-real reduced model generated by standard TBR procedure. Solid line: original (positive-real) order-26 transfer function. Dashed line: TBR result of order 7. Dashed–dotted line: PR-TBR result of order 7. Note negative sign of TBR results indicating non-positive-realness.

VI. Results

In this section, we show examples that illustrate the relevance and applicability of the various algorithms presented in this paper.

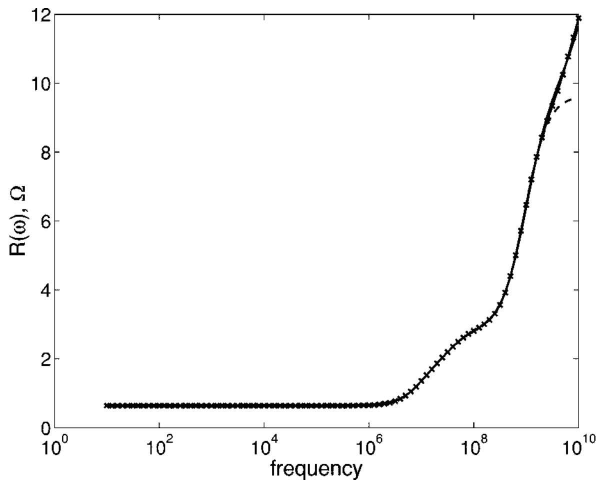
A. A Non-Passive ROM Generated by TBR

First, we demonstrate empirically that standard TBR (Algorithm 2) can generate models that are not passive by examining a 26-state lumped circuit model of a crystal filter. We configured the circuit to model the two-port impedance parameters from the input to a differential output. This created a 26-state state-space model that is positive-real. We then generated all the possible TBR models of orders 1–26, and used the positive real lemma to inspect them for positive-realness (equivalent to passivity in this case). Several of the models were found to be nonpassive (see Fig. 1). We then generated all the possible PR-TBR models. All were found to be positive-real as expected.

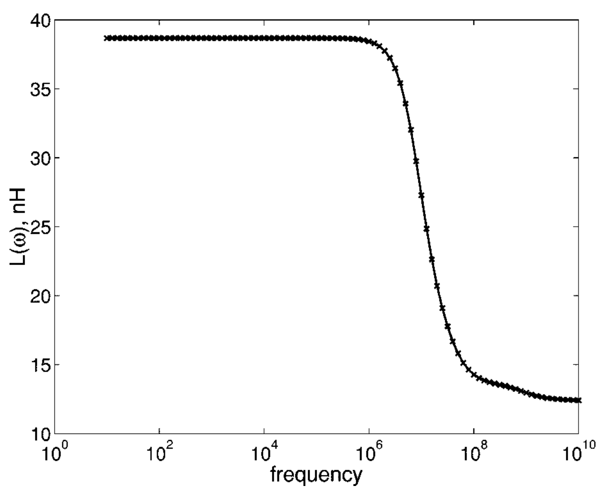
Note that the A, E matrices for this test case were obtained via MNA analysis, and in particular, satisfy the conditions necessary to apply PRIMA (they are of the form given by (2) in [37] with $C = E$ and $G = A$, C, and G nonnegative definite). Since this example could form a subsystem of a larger passive system, or be the result of some previous reduction procedure, its existence proves that examples exist for which PRIMA is applicable and produces passive models, but for which TBR generates nonpassive approximations in a second-stage procedure.

B. A Symmetrizable System

Our next example (Fig. 2) is a spiral inductor modeled with the magnetoquasistatic electromagnetic tool FASTHENRY [54]. This example first appeared in [2]. In general, as we mentioned already in many other parts of this paper, model order reduction is a "two-step" procedure. During the first step one would use "nonoptimal" but computationally tractable Krylov subspace moment matching techniques such as PRIMA, since the original system is typically too large to be handle by the more optimal TBR techniques. For this example the



(a)



(b)

Fig. 2. (a) Real part of reduced models of impedance (resistance) of spiral inductor. (b) Imaginary part of impedance divided by $2\pi j\omega$ (inductance) of spiral inductor. Solid line: 60–state PRIMA model which is the result of the first step of a typical reduction procedure. As a second step one could apply either TBR or PR-TBR. Dashed line: model of order 5 generated via TBR. Dash–cross line: Model of order 5 generated using PR-TBR on inverse system. All models in this example are positive-real.

initial system of around 1500 states is reduced to a 60-state positive-real model using PRIMA. This model is nearly exact in the frequency range shown.

Since this order is still considered excessive, and since Krylov subspace model reduction techniques are in general not optimal, when comparing accuracy for a given final order it is standard procedure to further reduce the PRIMA model using TBR. The frequency responses of the PRIMA model, TBR model, and PR-TBR model are shown in Fig. 2, and the time-domain responses are shown in Fig. 3. In [2], it was commented that the reduced models after the TBR procedure appeared to be passive, but no explanation was given. Here we have rigorously checked, using the positive-real lemma, that the models were indeed passive, and gave a proof as to why, for symmetrizable systems such as this, that should be the case. Note, however, that the results shown in Fig. 2 from TBR and PR-TBR are slightly different. This is not surprising as, while both TBR and PR-TBR guarantee passivity in this case, they are different computational techniques with different physical interpretations and different

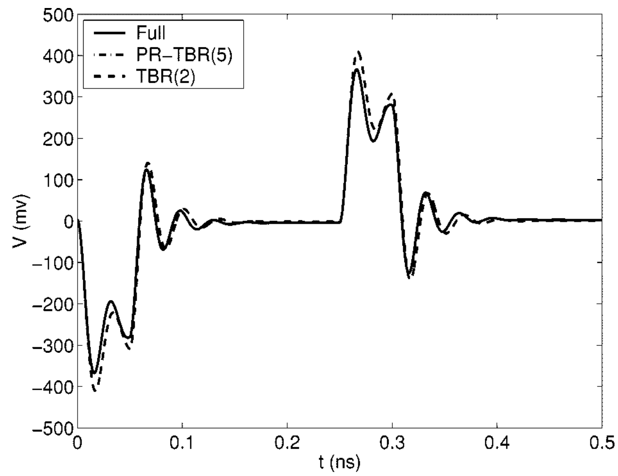


Fig. 3. Voltage response across spiral inductor, in parallel with small capacitor, driven by short current pulse. Response computed from original model (solid line), TBR approximation of second order (dash line) and PR-TBR approximation of fifth order (dash–dotted line, nearly exact) are shown.

error properties. Given that PR-TBR is more computationally involved than TBR, for the special case of symmetrizable systems it makes sense to use TBR exclusively.

C. A Bounded-Real Example From Rational Function Fitting

In the next example we consider the bounded-real variant of the TBR procedure (BR-TBR). First, a rational fitting method was used to fit a high-order model to tabulated two-port S-parameter data originating from a full-wave EM field solver. The fitting algorithm, which has provision for automatic estimation of model order, was tuned to a conservative setting, and generated an order-42 initial model that was nearly an exact fit to the data in the given frequency range. The resulting 42-state model was much larger than desired for final simulation, so the BR-TBR procedure was used to reduce the model to six states. The results are shown in Fig. 4. The reduced model had norm bounded by unity, indicating that it represented a passive element. Several models of orders six to eight were also generated by both TBR and congruence transform strategies, but all had H_∞ norms ranging from 1.05 to 1.9, i.e., they were not passive. Such techniques are, therefore, unusable for this type of systems. We likewise re-iterate that PRIMA is not suitable for these systems either [29].

D. A PEEC Connector

This example features a connector structure from Teradyne Inc. composed of 18 pins with a ground shield around and between the conductors. This structure or portions of it were previously used [19] to illustrate a PEEC formulation based on PRIMA that generates passive reduced-order models. While the resulting model was indeed provably passive, disappointing reductions were reported, due mostly to the inability of the PRIMA algorithm to zero-in on the relevant modes of the system. In fact volume discretization of the interior of the conductors in order to properly model skin-effect leads to the appearance of various internal subsystems that have negligible effect in the structure impedance but which can fool the PRIMA

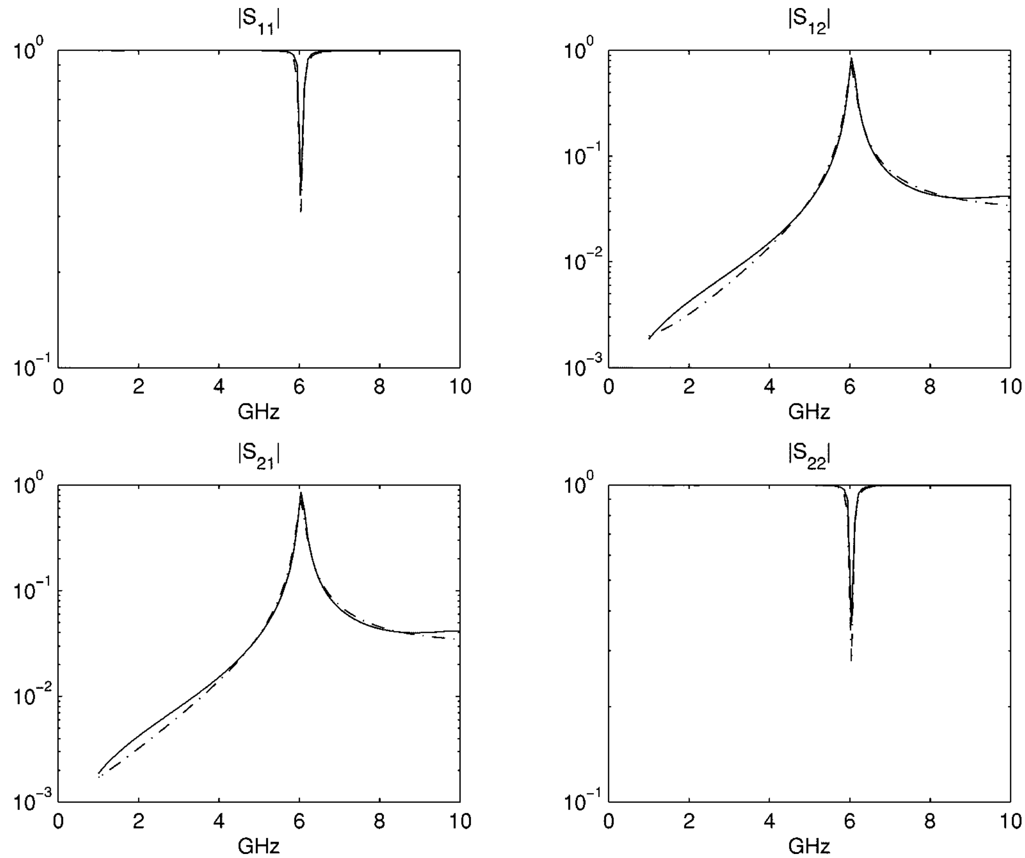


Fig. 4. Magnitude of rational function fit and reduced model for S-parameters tabulated by full-wave field solver. Solid line shows initial data and oder-42 rational fit (complete overlap). Dash-dotted line shows order-6 reduced model obtained via BR-TBR.

algorithm. In order to address such issue in [28] the same example was used to illustrate a two-step algorithm for *RLC* order reduction based on PRIMA followed by TBR, in an attempt to solve the above problem. Significant order reductions were reported after the second step of reduction as TBR is able to determine that those modes are not observable nor controllable. While this clearly shows that further reduction after the PRIMA stage is possible and indeed desirable, passivity was no longer guaranteed in the final, smaller models.

Here we have used the same example and checked the passivity of reduced-order models of various orders. We believe that the modes that are being discarded by TBR are related to the internal subsystems resulting from skin-effect modeling. As such the character of the problem after the initial PRIMA reduction is predominantly *RL*, a type of system for which we know that TBR is passive (see Section III-D). Once more we generated all the possible TBR models for the system obtained after the PRIMA reduction and used the positive real lemma to inspect them for positive-realness (again equivalent to passivity in this case). Due to the *almost* symmetric nature of the systems, almost all the models we obtained were found to be passive. However, models of order 19 and 29 were found to be nonpassive, a problem if the model is to be used in time-domain simulations. This example shows once more that TBR can indeed lead to large reductions in model-order but can produce nonphysical models. The example also presents a strong case for using the hybrid algorithm presented earlier (see Section IV-D and Algorithm 5). Since the majority of the TBR-produced models are likely to be passive it is advantageous to obtain such a model, check it for passivity, and only proceed to the PR-TBR algorithm if the passivity check fails. Of course, it would be possible, using TBR, to compute and check another model of slightly different order, and this is fairly easy to do since TBR essentially produces models of all orders simultaneously. However in general changing the order is not guaranteed to always produce a passive model. In fact, as the next example shows, there are systems where TBR almost never produces a passive model. This is a particular problem when one of the requirements specified by the user is having a model of a particular size or no larger than a certain size. Furthermore, since there is a cost associated with the passivity check, it is not practical to check "too many" alternative models before proceeding to PR-TBR.

E. An RLC Line

For our next example we use a 40-segment uniform *RLC* line that is L-dominated. The values of the line were chosen to be $R = 25$, $C = L = 0.398\,94$. For the purpose of comparison we computed 25th-order models using both TBR and PR-TBR. Fig. 5(a) shows the low-frequency behavior of the exact line impedance as well as that obtained using the two models. For this particular case it turns out that PR-TBR performs much better than regular TBR in terms of the model error. More important, however is the result shown in Fig. 5(b) where we plot the minimal eigenvalue of the symmetric part of the transfer function as a function of frequency. As can be seen from the plot, the minimal eigenvalue for the TBR model can go below zero

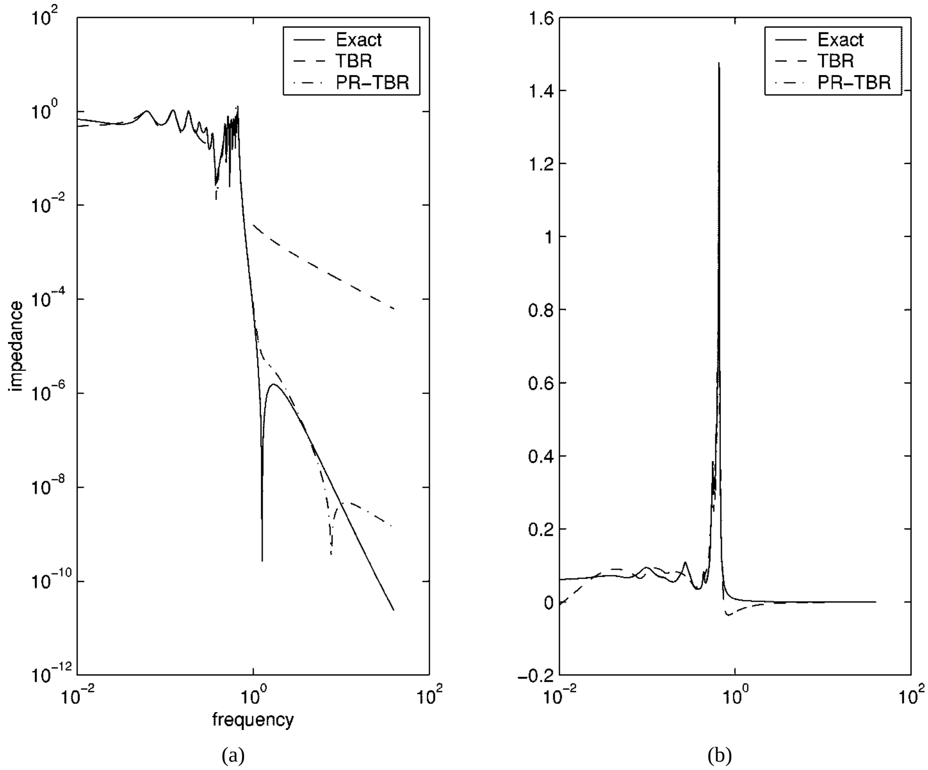


Fig. 5. (a) Magnitude of Y_{12} for *LC* line in normalized units. (b) Minimum eigenvalue of symmetric part of reduced model transfer function. Note that the minimum eigenvalue of the TBR model drops below zero for some frequencies, indicating nonpassivity.

at some frequencies which implies that the model is nonpassive and may produce nonphysical responses when used in time-domain simulations. In fact, on this example, *almost none* of the models produced by TBR were passive. Only very high order models exhibiting an almost exact match to the transfer function over the entire frequency axis were passive. In contrast, all the models produced by the PR-TBR method were found to be passive, as expected.

F. DC Accuracy Improvement

Our final example illustrates the behavior of the reciprocal system technique in preserving accuracy at low frequencies. An abstract system of order 64 was constructed, and the PR-TBR algorithm, both the "standard" and the "reciprocal" variants, were applied to generate reduced models of order 4. Since PR-TBR was used, both reduced models were guaranteed-passive. The results are shown in Fig. 6. Standard PR-TBR is very accurate at high frequencies, but not so accurate near dc. The reciprocal variant is very accurate near dc, but trades this for accuracy at high frequencies. Note that both methods show a fairly good match to the features around the sharp large-amplitude resonance. This is in agreement with [35], where it is shown that using the reciprocal transformation before and after the balanced-truncation procedure results in models that are exactly accurate at dc. In fact, it is possible to choose any frequency point s_0 such that the reciprocal systems can be computed and then reduced as an approximation at any frequency s_0, such that the reduced model will match the value of the original system at $s = s_0$.

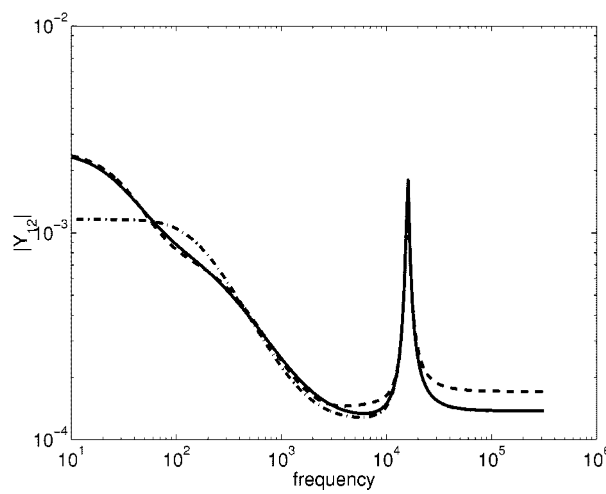


Fig. 6. Solid line: Original order-64 model. Dash–dotted line: order-6 reduced model from PR-TBR. Dash line: order-6 reduced model PR–TBR, reciprocal variant.

VII. CONCLUSION

In this paper, we presented a family of algorithms that can be used to compute guaranteed passive, reduced-order models of controllable accuracy for state-space systems with arbitrary internal structure.

The algorithms presented are similar to the well-known truncated balanced realization (TBR) techniques and share some of their advantages, such as computable error bounds. However, unlike standard TBR techniques, the algorithms presented have been shown to produce provably passive reduced-order models. In addition, unlike other techniques known to also produce passive reduction, the algorithms presented pose no constraints on the internal structure of the state-space. They are thus equally well applicable to systems that represent for instance Y or Z parameters as well as systems that represent S parameters. An hybrid algorithm was also presented where a TBR model is first computed, then checked for passivity and the passive-TBR algorithm is only used if that check fails. Our hybrid algorithm is more reliable than simply slightly changing the order of the produced model which can often produce passive systems, although not always. In addition we also examined a dc-accurate technique that can be used in conjunction with the algorithms presented in order to produce models that have accurate steady-state responses.

We have experimented with our techniques in a large number of settings and have shown that they can be used as standalone procedures or as part of second step reductions for systems with a large number of unknowns, perhaps replacing the usual TBR procedure. We have thus applied our method to obtain reduced models of various structures, namely the two-port impedance of a crystal filter, a spiral inductors, a large connector and an *RLC* line. All models were found to be accurate and passive. All previously known techniques failed to produce acceptable models in some of the examples used.

Further applications of the algorithms are possible, for example, balancing the Lyapunov observability Grammian versus the Lur'e controllability Grammian could be useful in obtaining passive models in situations (common in *RLC* interconnect analysis) where the number of outputs (e.g., from voltage observation points) exceeds the number of inputs (i.e., drivers).

ACKNOWLEDGMENT

The authors would like to acknowledge C. Coelho of INESC ID/IST for several discussions on various topics of relevance, and for making available preprints of some relevant publications. They would also like to thank M. Kamon of Coventor, N. Marques of Alcatel, and M. Tsuk of HP-Compaq for providing some of the example systems.

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