Stable Reduced Models for Nonlinear Descriptor Systems Through Piecewise-Linear Approximation and Projection

Bradley N. Bond, *Student Member, IEEE*, and Luca Daniel, *Member, IEEE*

***Abstract*—This paper presents theoretical and practical results concerning the stability of piecewise-linear (PWL) reduced models for the purposes of analog macromodeling. Results include proofs of input–output (I/O) stability for PWL approximations to certain classes of nonlinear descriptor systems, along with projection techniques that are guaranteed to preserve I/O stability in reduced-order PWL models. We also derive a new PWL formulation and introduce a new nonlinear projection, allowing us to extend our stability results to a broader class of nonlinear systems described by models containing nonlinear descriptor functions. Lastly, we present algorithms to compute efficiently the required stabilizing nonlinear left-projection matrix operators.**

***Index Terms*—Analog macromodeling, model order reduction (MOR), nonlinear systems, stability.**

I. Introduction

With the increasing complexity of analog systems, the automatic extraction of nonlinear macromodels has become an extremely important task to enable system design and optimization. Despite recent efforts in nonlinear macromodeling, existing methods have failed to gain widespread use due to a lack of rigorous statements concerning the accuracy of the resulting models. The presence of highly nonlinear elements in analog systems makes global verification of model accuracy and preservation of global qualitative aspects of the original system (such as stability) an extremely difficult task.

One popular approach to nonlinear macromodeling uses piecewise-linear (PWL) models. PWL models are desirable in part due to their ability to capture highly nonlinear effects. While there has recently been great interest in PWL model order reduction (MOR) as a method for nonlinear macromodeling for analog systems [1]–[6], there have been few results concerning the stability of the resulting PWL models [7], [8].

The construction of stable reduced order model (ROM) for *linear* systems has been thoroughly explored in recent years [9]–[13] and has yielded many useful results. For instance, there exist projection-based reduction methods and optimization-based fitting techniques that can reliably produce stable reduced models from originally stable large-order linear systems.

In a recent work [8], we presented preliminary results on the stability of PWL approximations of *nonlinear* systems. In this paper, we expand upon those results by extending them to nonlinear systems whose linearizations are described by descriptor systems (Section III). Results for this class of systems include guarantees of finite-gain stability for PWL models comprised of structured matrices, and a projection framework that is guaranteed to preserve input–output (I/O) stability. In Section IV, we propose a new model formulation that allows us to extend our stability results to a larger class of descriptor systems. Section V considers all other systems for which we cannot guarantee global stability or which are originally unstable. In these cases, we propose a nonlinear projection that is guaranteed to preserve stability for every stable local linear model, resulting in a guarantee of local stability. Section VI presents algorithms to compute efficiently the resulting nonlinear reduced models, with an emphasis on constructing the nonlinear left-projection functions. Finally, Section VII presents results from the proposed algorithms applied to several examples of nonlinear systems, including analog circuits and a microelectromechanical-system (MEMS) device.

II. Background

In this section, we summarize relevant background information about system stability and model reduction.

A. System Stability: Internal Stability

Consider the nonlinear dynamical system

$$E\dot{x} = f(x, u), \quad y = c^{\mathrm{T}} x \tag{1}$$

which may arise, for instance, when modeling analog circuits using modified nodal analysis. In this paper, $x \in \mathbb{R}^N$ is the state vector corresponding to node voltages and inductor currents in the circuit, $u \in \mathbb{R}^{m_i}$ represents inputs to the circuit, and $y \in \mathbb{R}^{m_o}$ corresponds to circuit outputs. Assume that we have separated the algebraic constraints such that the descriptor matrix E is nonsingular. Additionally, assume that the system has a unique equilibrium point x_{eq} such that

$$f(x_{eq}, 0) = 0.$$

Without loss of generality, we may transform the coordinate system such that $x_{eq} = 0$.

Manuscript received January 22, 2009; revised April 22, 2009. Current version published September 18, 2009. This work was supported in part by the Interconnect Focus Center, one of five research centers funded under the Focus Center Research Program, a DARPA and Semiconductor Research Corporation program. This paper was recommended by Associate Editor L. M. Silveira.

The authors are with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: bnbond@mit.edu; luca@mit.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCAD.2009.2030596

The internal stability of system (1) is a property of the equilibrium point x_{eq} and is defined by the behavior of solutions to the autonomous system

$$E\dot{x} = f(x, 0) \tag{2}$$

for various initial conditions x_0. Specifically, the equilibrium point is exponentially stable if all solutions starting from arbitrary initial conditions converge to the equilibrium point exponentially fast.

Definition 2.1: The equilibrium $x_{eq} = 0$ is said to be **exponentially stable** if there exist constants $r, a, b > 0$ such that

$$\|x(t_0 + t)\| \leq a\|x_0\|e^{-bt} \qquad \forall\, t,\ t_0 \geq 0,\ \forall\, x_0 \in \mathbb{B}_r \tag{3}$$

where $\mathbb{B}_r$ is a ball of radius r centered at x_{eq}. If $\mathbb{B}_r = \mathbb{R}^N$, then the equilibrium is said to be **globally exponentially stable**.

Exponential stability, along with various other types of internal stability, can be proven through Lyapunov functions.

Theorem 2.1 [14]: The equilibrium point $x_{eq} = 0$ of system (2) is exponentially stable if there exist constants $\lambda_1, \lambda_2, \lambda_3 > 0$ and a continuously differentiable Lyapunov function $L(x)$ such that

$$\lambda_1 x^{\mathrm{T}} x \leq L(x) \leq \lambda_2 x^{\mathrm{T}} x \tag{4}$$

$$\frac{\partial}{\partial t} L(x) \leq -\lambda_3 x^{\mathrm{T}} x \tag{5}$$

$\forall t \geq 0$ and $\forall x \in \mathbb{B}_r$. If $\mathbb{B}_r = \mathbb{R}^N$, then the equilibrium point is globally exponentially stable.

For proofs of the preceding theorem and additional stability results, see, for example, [14].

B. System Stability: External Stability

When considering external stability, we refer to the I/O system (1). Qualitatively, the system is said to be externally stable if the system's output $y(t)$ can be bounded in some measure by a linear function of the system's input $u(t)$ in that same measure.

Definition 2.2 [14]: System (1) is said to be **small-signal finite-gain** L_p **stable** if there exist constants $r_p > 0$ and $\gamma_p < \infty$ such that

$$\|y\|_p \leq \gamma_p \|u\|_p$$

for all $t > t_0$, given initial state $x(0) = 0$ and input $u(t)$ such that $\|u\|_\infty < r_p$. If $r_p = \infty$, then the system is **finite-gain** L_p **stable**.

Before presenting a method for proving external stability, it is useful to recall the notion of Lipschitz continuity.

Definition 2.3: A function $f(x, u)$ is **locally Lipschitz continuous** at (0, 0) if there exist finite positive constants k_f, r such that

$$\|f(x, u) - f(z, v)\| \leq k_f \left[\|x - z\| + \|u - v\|\right] \tag{6}$$

$\forall (x, u), (z, v) \in \mathbb{B}_r$, and $\forall t \geq 0$. If $\mathbb{B}_r = \mathbb{R}^N$, then the function is **Lipschitz continuous**.

Observation 2.1: A function $f(x, u)$ is locally Lipschitz continuous with Lipschitz constant κ in the ball $\mathbb{B}_r$ if

$$\left|\frac{\partial f_i(x, u)}{\partial x_j}\right| \leq \kappa \quad \left|\frac{\partial f_i(x, u)}{\partial u_j}\right| \leq \kappa \qquad \forall\, (x, u) \in \mathbb{B}_r$$

and for all i, j.

The external stability of system (1) can now be proven by exploiting its connection to internal stability, as shown in the following theorem.

Theorem 2.2 [14]: Suppose $x = 0$ is an exponentially stable equilibrium of system (1). If $f(x, u)$ is continuously differentiable and $f(x, u)$ is locally Lipschitz continuous at (0,0), then system (1) is small-signal finite-gain L_p stable. If $\mathbb{B}_r = \mathbb{R}^N$, then the system is finite-gain L_p stable.

For the remainder of this paper, we will consider small-signal finite-gain L_2 stability and refer to it simply as I/O stability.

C. System Stability: Linear Systems

A linear descriptor system

$$E\dot{x} = Ax + bu(t) \tag{7}$$

is said to be stable if the generalized eigenvalues of the pair (E, A) have negative real part. Equivalently, we say the pair (E, A) is Hurwitz or stable.

More generally, consider the matrix equation

$$E^{\mathrm{T}} PA + A^{\mathrm{T}} PE = -Q \tag{8}$$

where Q is a symmetric positive-definite (SPD) matrix. A positive-semidefinite matrix is one for which $x^{\mathrm{T}} Q x \geq 0$ for all x, and a positive-definite matrix adds the constraint that $x^{\mathrm{T}} Q x = 0 \Longleftrightarrow x = 0$.

Theorem 2.3 [15]: If system (7) is stable, i.e., the matrix pair (E, A) has all eigenvalues with negative real part, then for any SPD matrix Q, there exists a unique SPD matrix P which solves (8). Conversely, if there exist SPD matrices Q, P satisfying (8), then the matrix pair (E, A) has all eigenvalues with negative real part, and the system is stable. If the matrix E is singular, then there may not exist an SPD solution P for some SPD Q, and if there are solutions, they may not be unique.

To clarify the connection between Lyapunov functions and (8), we define the Lyapunov function $L(x) = x^{\mathrm{T}} E^{\mathrm{T}} PEx$ such that

$$\begin{aligned}
L(x) &= x^{\mathrm{T}} E^{\mathrm{T}} PEx = x^{\mathrm{T}} \tilde{P} x \geq \sigma_{\min}(\tilde{P}) x^{\mathrm{T}} x \\
L(x) &\leq \sigma_{\max}(\tilde{P}) x^{\mathrm{T}} x \\
\frac{\partial L(x)}{\partial t} &= 2x^{\mathrm{T}} E^{\mathrm{T}} PE\dot{x} = x^{\mathrm{T}} (E^{\mathrm{T}} PA + A^{\mathrm{T}} PE) x \\
&= -x^{\mathrm{T}} Q x \leq -\sigma_{\min}(Q) x^{\mathrm{T}} x
\end{aligned}$$

where $\sigma_{\min}(\tilde{P})$ and $\sigma_{\max}(\tilde{P})$ are the minimum and maximum singular values of $\tilde{P}$, respectively. Thus, conditions (4) and (5) are satisfied, proving exponential stability of the origin equilibrium point, if there exist SPD matrices P and Q that solve (8).

D. System Stability: Nonlinear Descriptor Functions

When modeling analog circuits containing nonlinear capacitances, such as those containing transistors, the resulting dynamical systems will contain a nonlinear descriptor function $q(x)$

$$\frac{d}{dt}[q(x)] = f(x,u), \quad y = c^{\mathrm{T}}x. \tag{9}$$

If the function $q(x)$ is invertible, then it is possible to obtain a system of the form (1) through a nonlinear change of coordinates. However, the functions $q(x)$ are often not invertible, making it difficult to make statements about global properties of the system. These cases will be considered in more detail in Section IV.

E. Model Reduction: Stable MOR for Linear Systems

Macromodeling typically employs some form of reduced-complexity modeling to decrease the computational costs of simulation. In the case of order reduction, the existence of an accurate low-order model relies on the assumption that the important dynamics of the original system are confined to a low-dimensional subspace. For example, projection-based methods are a common order-reduction approach for state-space models of linear systems. One such projection technique, which is presented in [16] and [17], that preserves stability in the reduced model is as follows.

Theorem 2.4 [16], [17]: Consider a linear descriptor system

$$E\dot{x} = Ax + bu \tag{10}$$

where (E, A) is a Hurwitz pair. Let P and Q be SPD matrices that solve (8) and let V be an orthonormal projection matrix such that $x = Vz$ and $z \in \mathbb{R}^q$, where $q \ll N$. If U is defined by

$$U^{\mathrm{T}} = (V^{\mathrm{T}}E^{\mathrm{T}}PEV)^{-1}V^{\mathrm{T}}E^{\mathrm{T}}P$$

and $\hat{A} = U^{\mathrm{T}}AV$, then $\hat{E} = U^{\mathrm{T}}EV = I$ and the reduced-order system

$$\dot{z} = \hat{A}z + \hat{b}u \tag{11}$$

is stable.

An alternative approach to stable model reduction for linear systems, first presented in [18], involves solving an optimization problem for an optimal stabilizing projection framework. Given linear system (10) and a right-projection matrix V constructed for accuracy, a stabilizing left-projection matrix U can be found by solving the following optimization problem:

$$\begin{aligned} &\min_U g(U) \\ &U^{\mathrm{T}}EV \succ 0 \\ &U^{\mathrm{T}}AV + V^{\mathrm{T}}AU \prec 0 \end{aligned} \tag{12}$$

where $g(U)$ is a convex cost function. Using techniques described in [18], the problem can be formulated with $O(q^4)$ complexity. This approach provides a relatively cheap alternative to the method described in Theorem 2.4 in the case where no efficient method is available for computing the matrix P.

While there exist other stability-preserving techniques for linear systems, such as a Galerkin projection for symmetric definite systems and balanced truncation for indefinite systems, the earlier approaches are particularly appealing because they do not require symmetry or definiteness of the system matrices (E, A) and additionally work for arbitrary right-projection matrices V.

F. Model Reduction: Trajectory Piecewise Linear (TPWL) MOR for Nonlinear Systems

For nonlinear systems, complexity reduction can be achieved through both state-variable reduction (projection) and function approximation. Consider a nonlinear system of order N

$$\frac{\partial}{\partial t}[q(x)] = f(x) + bu \quad y = c^{\mathrm{T}}x$$

whose nonlinearities $f(x)$, $q(x)$ can be approximated in some important regions of the state space by a convex combination of affine functions

$$f(x) \approx \sum_i w_i(x)[A_i x + k_i]$$
$$q(x) \approx \sum_i w_i(x)[E_i x + h_i]$$

where

$$A_i = \left.\frac{\partial f(x)}{\partial x}\right|_{x_i} \qquad k_i = f(x_i) - A_i x_i$$
$$E_i = \left.\frac{\partial q(x)}{\partial x}\right|_{x_i} \qquad h_i = q(x_i) - E_i x_i$$

are linearizations of $f(x)$, and $w_i(x)$ are weighting functions such that $w_i \in [0, 1]$ and $\sum_i w_i = 1$. Two examples of possible weighting functions are [1], [5]

$$w_i(x) = \frac{\exp\left(\frac{-\beta\|x-x_i\|_2^2}{\min_k \|x-x_k\|_2^2}\right)}{\sum_j \exp\left(-\frac{\beta\|x-x_j\|_2^2}{\min_k \|x-x_k\|_2^2}\right)}$$

or

$$w_i(x) = \frac{\left(e^{-\beta\|x-x_i\|_2^2}\right)^{-k}}{\sum_j \left(e^{-\beta\|x-x_j\|_2^2}\right)^{-k}}.$$

The final result is a PWL system of the form [19]

$$\frac{d}{dt}\left[\sum_i w_i(x)(E_i x + h_i)\right] = \sum_i w_i(x)(A_i x + k_i) + bu. \tag{13}$$

It is now possible to introduce a linear projection $x = Vz$, where $z \in \mathbb{R}^q$ and $q \ll N$, such that each linear system is projected into the subspace spanned by the columns of V, creating the PWL reduced-order model [19]

$$\frac{d}{dt}\left[\sum_i w_i(z)(\hat{E}_i z + \hat{h}_i)\right] = \sum_i w_i(z)[\hat{A}_i z + \hat{K}_i] + \hat{b}u$$

in which we have defined $\hat{E}_i = V^{\mathrm{T}} E_i V$, $\hat{A}_i = V^{\mathrm{T}} A_i V$, $\hat{k}_i = V^{\mathrm{T}} k_i$, $\hat{h}_i = V^{\mathrm{T}} h_i$, $\hat{b} = V^{\mathrm{T}} b$, and $\hat{c} = V^{\mathrm{T}} c$. The model complexity has been reduced by both approximating the nonlinear functions $f(x)$, $q(x)$ and by reducing the number of state variables.

In standard trajectory PWL methods [19], the linearization points x_i are chosen as states along trajectories that solve system (1) when driven by some typical inputs $u(t)$. There are many possible options for generating the columns of the projection matrix V. For instance, one can use Krylov vectors of the individual linearized systems [1], [19], dominant singular vectors from simulated trajectories [20], or truncated balanced realization vectors [2].

It is important to notice here that the standard TPWL procedure may potentially produce unstable reduced models from originally stable systems. Such instabilities may arise in three places.

1) Jacobian matrix pairs (E_i, A_i) of stable nonlinear systems are not guaranteed to be Hurwitz.
2) The pair $(V^{\mathrm{T}} E_i V, V^{\mathrm{T}} A_i V)$ is not guaranteed to be Hurwitz even if (E_i, A_i) is Hurwitz.
3) Convex combinations of Hurwitz matrix pairs $(\sum_i w_i E_i, \sum_i w_i A_i)$ are not guaranteed to be Hurwitz.

III. Stability of PWL Systems With Constant Descriptor Matrix

In general, PWL models created from stable nonlinear systems are not stable. This is because linearizations of an arbitrary stable nonlinear system are not necessarily stable, and interpolating between arbitrary stable linear models will not necessarily produce a stable model. However, there exist many nonlinear systems for which we can guarantee both that linearizations will always be stable and that convex combinations of the resulting stable linear systems will also be stable. In this section, we examine nonlinear systems that provably generate structured and stable linearizations, and can be formulated, either directly or through a change of coordinates as described in Section II-D, to possess a constant descriptor matrix. In these cases, we obtain finite-gain stability guarantees for the large-order PWL models as well as a stability-preserving linear-projection framework.

A. Relaxing the Model

To begin, we introduce a new notation to concisely represent the PWL model. Define the matrix-valued functions

$$A_p(x) = \sum_i w_i(x) A_i, \quad E_p(x) = \sum_i w_i(x) E_i$$

and the vector-valued functions

$$k_p(x) = \sum_i w_i(x) k_i, \quad h_p(x) = \sum_i w_i(x) h_i$$

such that the large-order PWL approximation becomes

$$\begin{cases} \frac{d}{dt}\left[E_p(x)x + h_p(x)\right] = A_p(x)x + k_p(x) + bu \\ y = c^{\mathrm{T}} x \end{cases} \tag{14}$$

where $E_p(x)$ and $A_p(x)$ are nonlinear matrix-valued functions that interpolate between the local matrices.

In addition, note that the PWL model (14) can be rewritten in a more general form as

$$\frac{d}{dt}\left[E_p(x)x + h_p(x)\right] = A_p(x)x + B_p(x,u) \quad y = c^{\mathrm{T}} x \tag{15}$$

where

$$B_p(x,u) = bu_1 + k_p(x)u_2 \tag{16}$$

is a state-dependent input matrix. In this formulation, we are treating the constant offset vectors resulting from the linearizations as additional input vectors with the new input $u_2(t)$. System (14) is obtained by selecting $u_2 = 1$ for all $t > 0$. Thus, systems of the form (14) are a subset of systems of the form (15), and any stability results that apply to the latter will also apply to the former.

B. Stability From Structured Matrices

We first consider systems described by models containing a constant descriptor matrix

$$E\dot{x} = A_p(x)x + B_p(x,u), \quad y = c^{\mathrm{T}} x. \tag{17}$$

Recall from Section II-A that internal stability can be proven through the existence of Lyapunov functions. Finding Lyapunov functions for arbitrary nonlinear systems is difficult. However, often, a PWL system's Jacobian matrices A_i will all share some nice structure because they are all linearizations of the same nonlinear function, and in those cases, it may be possible to find a Lyapunov function that proves internal stability of the autonomous PWL system

$$\begin{cases} E\dot{x} = A_p(x)x \\ A_p(x) = \sum_i w_i(x) A_i. \end{cases} \tag{18}$$

For example, a Lyapunov function that proves stability for each individual linear system, and thus also for an interpolation of the systems, would suffice and is specified by the following proposition.

Proposition 3.1 (Exponential Stability): If $w_i(x) : \mathbb{R}^N \mapsto [0,1]$ are continuously differentiable functions such that $\sum_i w_i = 1$ and there exists an SPD matrix $P \succ 0$ such that the matrices

$$Q_i = -\left(E^{\mathrm{T}} P A_i + A_i^{\mathrm{T}} P E\right) \tag{19}$$

are SPD for all i, then $L(x) = x^{\mathrm{T}} E^{\mathrm{T}} P E x$ is a Lyapunov function for system (18), and system (18) has a globally exponentially stable equilibrium at the origin.

Proof: Consider the candidate Lyapunov function $L = x^{\mathrm{T}} E^{\mathrm{T}} P E x$. Since E is nonsingular and P is SPD, then

$E^{\mathrm{T}}PE$ is also SPD because it is a congruence transform of an SPD matrix, and condition (4) is satisfied

$$x^{\mathrm{T}}x\left(\sigma_{\min}(E^{\mathrm{T}}PE)\right) \leq L(x) \leq x^{\mathrm{T}}x\left(\sigma_{\max}(E^{\mathrm{T}}PE)\right).$$

Similarly, condition (5) is satisfied

$$\begin{aligned}\dot{L}(x) &= 2x^{\mathrm{T}}E^{\mathrm{T}}PE\dot{x} = x^{\mathrm{T}}\left(E^{\mathrm{T}}PA_p(x) + A_p(x)^{\mathrm{T}}PE\right)x \\ &= \sum_i w_i(x)x^{\mathrm{T}}\left(E^{\mathrm{T}}PA_i + A_i^{\mathrm{T}}PE\right)x \\ &= -\sum_i w_i(x)x^{\mathrm{T}}Q_i x \leq -x^{\mathrm{T}}x\min_i\{\sigma_{\min}(Q_i)\}.\end{aligned}$$

Thus, $L(x)$ is a Lyapunov function, and by Theorem 2.1, the system is globally exponentially stable. ■

It is now possible, using the results from Proposition 3.1, to prove I/O stability for system (17).

Proposition 3.2 (I/O Stability): If system (18) is globally exponentially stable with Lyapunov function $L(x) = x^{\mathrm{T}}E^{\mathrm{T}}PEx$ for some SPD matrix P (e.g., the assumptions of Proposition 3.1 hold), $w_i(x) : \mathbb{R}^N \mapsto [0,1]$ are continuously differentiable functions such that $\sum_i w_i = 1$, then system (17) is I/O stable, and therefore the PWL system

$$E\dot{x} = A_p(x)x + k_p(x) + bu, \quad y = c^{\mathrm{T}}x \tag{20}$$

is I/O stable.

Proof: This can be proven using Theorem 2.2. By assumption, the autonomous system is globally exponentially stable, and $w_i(x)$ are all continuously differentiable, and therefore, $f(x,u)$ is also continuously differentiable. To prove Lipschitz continuity, we examine the partial derivatives of $f(x,u) = A_p(x)x + B_p(x,u)$

$$\begin{aligned}\frac{\partial f_j}{\partial x_k} &= \sum_i\left[w_i(x)a_{jk}^i + \frac{\partial w_i(x)}{\partial x_k}\sum_m\left(a_{jm}^i x_m\right)\right] \\ &\quad + \sum_i\left[\frac{\partial w_i(x)}{\partial x_k}k_i u_2\right] \\ \frac{\partial f_j}{\partial u_1} &= b \\ \frac{\partial f_j}{\partial u_2} &= \sum_i w_i(x)k_i\end{aligned}$$

where a_{jk}^i is the element of A_i in the jth row and kth column. By assumption, $\partial f_j/\partial u_1$ is bounded because it is constant, and $\partial f_j/\partial u_2$ is bounded for all x because w_i is bounded. Similarly, since $w_i(x)$ are Lipschitz, the derivatives $\partial w_i/\partial x_k$ are bounded. Thus, the Jacobian is locally bounded for all x, u, the functions are Lipschitz continuous by Observation 2.1, and the system is I/O stable by Theorem 2.2. ■

To obtain global finite-gain stability, we must add the additional constraint that $(\partial w_i(x)/\partial x)x$ is bounded for all x. This constraint is not restrictive, as it merely requires that the weights converge to some uniform value when the state becomes sufficiently large, rather than oscillate back and forth indefinitely. Practically, the PWL model is comprised of a finite number of linearizations that are locally accurate; therefore, for x sufficiently far away from all local models, the interpolation is no longer accurate regardless of the weighting functions, and thus, the constraint will not affect the accuracy.

Examples of systems for which stability may be guaranteed through Propositions 3.1 and 3.2 are those that produce negative-definite Jacobian matrices. These include analog circuits comprised of monotonic elements such as inductors, capacitors, linear and nonlinear resistors, and diodes. One such example is presented in Section VII-A.

Note that the finite-gain stability results are based solely on the existence of the quadratic Lyapunov function and do not explicitly require any special structure in the matrices A_i. Structured matrices, such as negative-definite matrices, are a sufficient condition for the existence of such a Lyapunov function but are not a necessary condition.

C. Stability-Preserving Projection

In the previous section, we presented conditions under which large-order PWL systems are both internally stable and finite-gain stable. In this section, we present a projection framework that preserves these two stability properties in the reduced model.

Consider the PWL model system (17) and approximate the solution x in a low-dimensional subspace as $x = Vz$, such that

$$EV\dot{z} = A_p(Vz)Vz + B_p(Vz,u). \tag{21}$$

A left-projection matrix U is next chosen to reduce the number of equations, resulting in the reduced-order model

$$U^{\mathrm{T}}EV\dot{z} = U^{\mathrm{T}}A_p(Vz)Vz + U^{\mathrm{T}}B_p(Vz,u), \quad y = c^{\mathrm{T}}Vz. \tag{22}$$

By proper selection of the matrix U, it is possible to preserve internal stability in the reduced-order system

$$\begin{cases}\hat{E}\dot{z} = \hat{A}_p(z)z \\ \hat{A}_p(z) = \sum_i w_i(z)U^{\mathrm{T}}A_iV \\ \hat{E} = U^{\mathrm{T}}EV.\end{cases} \tag{23}$$

Proposition 3.3 (Preservation of Lyapunov Functions): If $L(x) = x^{\mathrm{T}}E^{\mathrm{T}}PEx$ is a Lyapunov function for system (18) (e.g., the assumptions of Proposition 3.1 hold), then given any right-projection matrix V, if we define a left-projection matrix $U^{\mathrm{T}} = V^{\mathrm{T}}E^{\mathrm{T}}P$, then $\hat{L}(z) = z^{\mathrm{T}}\hat{E}z$ is a Lyapunov function for system (23), where $\hat{E} = U^{\mathrm{T}}EV$.

Proof: To begin, note that the proposed Lyapunov function $\hat{L}(z)$ satisfies

$$\begin{aligned}\hat{L}(z) &= z^{\mathrm{T}}V^{\mathrm{T}}E^{\mathrm{T}}PEVz = L(Vz) \\ \dot{\hat{L}}(z) &= 2z^{\mathrm{T}}\hat{E}\dot{z} = 2z^{\mathrm{T}}\hat{A}(z)z \\ &= 2z^{\mathrm{T}}V^{\mathrm{T}}E^{\mathrm{T}}PA(Vz)z = \dot{L}(Vz).\end{aligned}$$

By assumption, $L(x)$, and therefore $L(Vz)$, satisfies (4) and (5). Thus, $\hat{L}(z)$ satisfies conditions (4) and (5) and is a Lyapunov function for system (23). ■

Given the existence of a quadratic Lyapunov function for the reduced model, it is now possible to apply the results of Section III-B to obtain guarantees for the various notions of stability for the reduced model.

Corollary 3.1: If $L = x^{\mathrm{T}} E^{\mathrm{T}} P E x$ is a Lyapunov function for system (18), V is a right-projection matrix, and $w_i(x) : \mathbb{R}^N \mapsto [0,1]$ are continuously differentiable functions such that $\sum_i w_i = 1$, then if we define the left-projection matrix $U = V^{\mathrm{T}} E^{\mathrm{T}} P$, the reduced-order PWL model (23) is globally exponentially stable, and system (22) is I/O stable.

Proof: By Proposition 3.3, $\hat{L}(z) = z^{\mathrm{T}} V^{\mathrm{T}} E P E V z$ is a Lyapunov function for system (23), and therefore, the reduced model is globally exponentially stable. Exponential stability combined with Proposition 3.2 yield finite-gain stability for the reduced model. ■

IV. Stability of PWL Systems With Nonlinear Descriptor Functions

A. Difficulties With Nonlinear Descriptor Functions

In the previous section, we considered only systems described by models possessing a constant descriptor matrix E. In this section, we extend the results to the case where the descriptor function $q(x)$ is nonlinear.

It is more difficult to prove stability for systems with nonlinear descriptor functions because the quadratic Lyapunov function approach from Section III-B does not directly apply. Additionally, even if the large-order PWL system is stable, we cannot directly apply the approach of Section III-C to preserve stability in the reduced model.

For example, consider the PWL model with nonlinear descriptor function whose state is approximated in the reduced space as $x = Vz$

$$\frac{d}{dt}\left[E_p(Vz)Vz+h_p(Vz)\right]=A_p(Vz)Vz+B_p(Vz,u). \quad (24)$$

Attempting to preserve stability by preserving Lyapunov functions as done in Section III-C, i.e., to ensure that $\hat{L}(z) = L(Vz)$, requires the selection of a nonlinear left-projection matrix

$$U(z)^{\mathrm{T}} = V^{\mathrm{T}} E_p(Vz) P. \quad (25)$$

Applying a nonlinear left-projection $U(z)$ to (24) results in

$$\begin{aligned} U(z)^{\mathrm{T}} \frac{d}{dt}&\left[E_p(Vz)Vz + h(Vz)\right] \\ &= U(z)^{\mathrm{T}} A_p(Vz)Vz + U(z)^{\mathrm{T}} B_p(Vz,u) \end{aligned} \quad (26)$$

which is not a reduced-order system in the typical sense. The expression on the left cannot be explicitly multiplied out because the time dependence in $U(z)$ prevents it from passing directly through the time-derivative operators. As a result, systems of the form (26) require $O(N)$ computations to evaluate and are not desirable for the purpose of simulation.

B. Alternative Formulations

To avoid the projection problems resulting from nonlinear descriptor functions, we will rewrite the system in a manner that separates nonlinearities from the time-derivative operator. First assume that there is no explicit time dependence in $q(x)$. This allows for the nonlinear descriptor system

$$\frac{d}{dt}\left[q(x)\right] = f(x)$$

to be rewritten as

$$Q(x)\dot{x} = f(x)$$

where $Q(x) = (\partial q(x)/\partial x)$ is a nonlinear matrix-valued function. Additionally, the system can be rewritten as

$$\dot{x} = Q(x)^{-1} f(x) = g(x). \quad (27)$$

Note that no approximations have been made so far.

A linearization of system (27) at state x_i yields the local linear model

$$\dot{x} = A_i x + k_i \quad (28)$$

with system matrices

$$\begin{aligned} A_i &= Q(x_i)^{-1} J(x_i) - Q(x)^{-1} \frac{\partial Q(x)}{\partial x} Q(x)^{-1} f(x_i) \\ k_i &= Q(x_i)^{-1} f(x_i) - A_i x_i \end{aligned} \quad (29)$$

where $J(x) = \partial f/\partial x$. If the function $q(x)$ is known explicitly, then $Q(x)$ and $\partial Q/\partial x$ can also be computed, resulting in an accurate constant-descriptor PWL model

$$\begin{cases} \dot{x} = A_p(x)x \\ A_p(x) = \sum_i w_i(x) A_i \end{cases} \quad (30)$$

where A_i are defined in (29), and each linear model is accurate to first order in $f(x)$ and to first order in $Q(x)$.

However, the function $q(x)$ is not always available analytically. Often, only samples of $q(x)$ and $Q(x)$ are available. In this case, it is possible to ignore the derivative of $Q(x)$, simplifying the linearizations to

$$A_i = Q(x_i)^{-1} J(x_i), \quad k_i = Q(x_i)^{-1} f(x_i) - A_i x_i. \quad (31)$$

The resulting PWL system has the form of system (30), where the system matrices A_i are defined in (31), and each linear model is accurate to first order in $f(x)$ and zeroth order in $Q(x)$. A piecewise-constant approximation of $Q(x)$ is not a poor approximation, because, in general, the function $q(x)$ must be well behaved simply to ensure that a unique solution to the nonlinear system exists. In addition, if $Q(x)$ changes sharply, the accuracy of the approximation can always be increased by increasing the number of linearization points.

C. Stable PWL Systems From Nonlinear Descriptor Functions

For system (30), regardless of whether using system matrices (29) or (31), we can directly apply Proposition 3.1 to obtain an exponential stability guarantee because of the constant descriptor matrix $E = I$. However, we must consider one additional factor before applying Proposition 3.2 to obtain a finite-gain stability guarantee. As a result of the reformulation of the equations, the system now possesses additional state dependence in the input function $B_p(x,u)$

$$\begin{cases} \dot{x} = A_p(x)x + B_p(x,u) \\ A_p(x) = \sum_i w_i(x)A_i \\ B_p(x,u) = \sum_i w_i(x)[b_i u_1 + k_i u_2] \end{cases} \tag{32}$$

where, for example, $b_i = Q_i^{-1}b$. All of the other assumptions of Proposition 3.2 hold, so it merely needs to be shown that the input function $B_p(x,u)$ is still Lipschitz continuous.

Proposition 4.1 (I/O Stability): If system (30) is globally exponentially stable with Lyapunov function $L(x) = x^{\mathrm{T}}Px$ for some SPD matrix P (e.g., the assumptions of Proposition 3.1 hold) and $w_i(x) : \mathbb{R}^N \mapsto [0,1]$ are continuously differentiable functions such that $\sum_i w_i = 1$, then system (32) is I/O stable.

Proof: Following the proof of Proposition 3.2, we simply need to show that $B_p(x,u)$ is Lipschitz continuous. The partial derivatives are

$$\begin{aligned} \frac{\partial B_p}{\partial x_k} &= \sum_i \left[\frac{\partial w_i(x)}{\partial x_k}(b_i u_1 + k_i u_2)\right] \\ \frac{\partial B_p}{\partial u_1} &= \sum_i w_i(x) b_i \\ \frac{\partial B_p}{\partial u_2} &= \sum_i w_i(x) k_i \end{aligned}$$

which are all locally bounded by assumption. Thus, by Theorem 2.2, system (32) is I/O stable. ■

Additionally, given a right-projection matrix V, the left-projection matrix U can be chosen such that the reduced model resulting from application of U and V to system (32) is I/O stable.

Corollary 4.1: If $L = x^{\mathrm{T}}Px$ is a Lyapunov function for system (30), V is a right-projection matrix, and $w_i(x) : \mathbb{R}^N \mapsto [0,1]$ are continuously differentiable functions such that $\sum_i w_i = 1$, then if we define the left-projection matrix $U = V^{\mathrm{T}}P$, the reduced-order PWL model

$$\begin{cases} \hat{E}\dot{z} = \hat{A}_p(z)z + \hat{B}_p(z,u) \\ \hat{A}_p(z) = \sum_i w_i(z)U^{\mathrm{T}}A_i V \\ \hat{B}_p(z) = \sum_i w_i(z)U^{\mathrm{T}}[b_i u_1 + k_i u_2] \\ \hat{E} = U^{\mathrm{T}}V \end{cases} \tag{33}$$

is I/O stable.

Proof: Proposition 3.3 guarantees that $\hat{L}(z) = z^{\mathrm{T}}V^{\mathrm{T}}PVz$ is a Lyapunov function for the reduced model, and Proposition 4.1 applied to system (33) guarantees I/O stability. ■

Note that the reduced-model terms such as $U^{\mathrm{T}}Q_i^{-1}J_iV$ can be efficiently computed by first solving the linear system $Q_i^{\mathrm{T}}M = U$ for the matrix $M \in \mathbb{R}^{N\times q}$, and subsequently evaluating $(U^{\mathrm{T}}Q_i^{-1})J_iV = M^{\mathrm{T}}J_iV$.

For systems with complicated and unstructured descriptor functions, it becomes difficult to prove stability with quadratic Lyapunov functions. These issues will be addressed in the following section.

V. Unstructured and Unstable PWL Systems

All of the results presented up to this point have relied on the assumption that the large-order PWL system is stable and that there exists a quadratic Lyapunov function. However, in general, it may not be easy, or even possible, to find a quadratic Lyapunov function for a stable PWL system. Additionally, a stable nonlinear system may produce an unstable PWL model. In these cases, we will try both to "eliminate" as much instability as possible from the large-order PWL system through equation reformulation and to utilize a projection that preserves or regenerates stability in as many of the linear models as possible.

A. Stability Through Reformulation

Although the reformulation in the previous section permits the application of the results from Section III, it is possible that an alternative reformulation may be more useful in some situations. Consider the case where $Q(x)$ is approximated by a zeroth-order expansion and interpolate the descriptor matrices on the left side of the equation directly. The resulting system

$$\begin{cases} E_p(x)\dot{x} = A_p(x)x \\ A_p(x) = \sum_i w_i(x)A_i, \qquad A_i = J(x_i) \\ E_p(x) = \sum_i w_i(x)E_i, \qquad E_i = Q(x_i) \end{cases} \tag{34}$$

has a nonlinear descriptor matrix, is comprised of local linear models that are accurate to zeroth order in $Q(x)$ and first order in $f(x)$, and can be efficiently reduced with a nonlinear projection operator.

One possible benefit of this formulation is that the system matrices E_i and A_i in system (34) are much more likely to have a nice structure, such as symmetry or definiteness, than the system matrices (29) and (31). In general, structured system matrices make it easier to find Lyapunov functions as described in the previous section. One circuit example for which reformulation of the equations improves stability of PWL approximations is presented in Section VII-B.

B. Stability From Projection

Finally, we consider the case where the PWL system is not stable and some of the linear models (E_i, A_i) are unstable. Given the large-order PWL system and reduced-state approximation $x = Vz$, we may reduce the number of equations with a weighted piecewise-constant left-projection function

$$U(z) = \sum_{k=1}^{\kappa} \mu_k(z)U_k \tag{35}$$

where $\mu_k(z) \in [0,1]$, $\sum_k \mu_k(z) = 1$, and $U_k \in \mathbb{R}^{N\times q}$. Consider system (34) evaluated at $x = Vz$ and left multiplied by $U(z)^{\mathrm{T}}$, leading to

$$\sum_k \mu_k(z)U_k^{\mathrm{T}} \sum_i w_i(z)E_iV\dot{z} = \sum_k \mu_k(z)U_k^{\mathrm{T}} \left(\sum_i w_i(z)[A_iVz] + B(z)u\right)$$

which can be rearranged as

$$\sum_k \sum_i \mu_k(z) w_i(z) U_k^{\mathrm{T}} E_i V \dot{z} = \sum_k \sum_i \mu_k(z) w_i(z) U_k^{\mathrm{T}} A_i V z + \sum_k \mu_k(z) U_k^{\mathrm{T}} B_k(z) u.$$

To simplify the notation, we define

$$\begin{aligned} \hat{B}_{ki} &= U_k^{\mathrm{T}}[b_i, k_i], & \hat{B}_p(z,u) &= \sum_i \sum_k \mu_k(z) w_i(z) \hat{B}_{ki} u \\ \hat{E}_{ki} &= U_k^{\mathrm{T}} E_i V, & \hat{E}_p(z) &= \sum_i \sum_k \mu_k(z) w_i(z) \hat{E}_{ki} \\ \hat{A}_{ki} &= U_k^{\mathrm{T}} A_i V, & \hat{A}_p(z) &= \sum_i \sum_k \mu_k(z) w_i(z) \hat{A}_{ki} \end{aligned}$$

resulting in the final reduced nonlinear descriptor system

$$\hat{E}_p(z)\dot{z} = \hat{A}_p(z) z + \hat{B}_p(z) u. \tag{36}$$

We wish to select the matrices U_k such that the reduced models $(\hat{E}_{kk}, \hat{A}_{kk})$ are stable for all k. Two possible available methods for computing such U_k were summarized in background Section II-E, as found in [16]–[18]. For example, if we define the matrices

$$U_k^{\mathrm{T}} = \begin{cases} \left(V^{\mathrm{T}} E_k^{\mathrm{T}} P_k E_k V\right)^{-1} V^{\mathrm{T}} E_k^{\mathrm{T}} P_k, & \text{if } k \in \mathbb{I} \\ V^{\mathrm{T}}(V^{\mathrm{T}} E_k V)^{-1}, & \text{if } k \notin \mathbb{I} \end{cases} \tag{37}$$

where

$$\mathbb{I} = \{k \in \{1, \ldots, \kappa\} | (E_k, A_k) \text{ is Hurwitz}\} \tag{38}$$

and P_k solves

$$E_k^{\mathrm{T}} P_k A_k + A_k^{\mathrm{T}} P_k E_k = -Q_k \prec 0 \tag{39}$$

for (E_k, A_k), then $\hat{E}_{kk} = U_k^{\mathrm{T}} E_k V = I$, and $\hat{A}_{kk}$ is Hurwitz for all k.

To completely specify the reduced-order system (36), we must specify a set of left-projection weights $\mu_k(z)$. One possible choice of $\mu_k(z)$ that simplifies the model is

$$\mu_k(z) = \begin{cases} 1, & \text{if } z_k = \arg\min_i \|z - z_i\| \\ 0, & \text{otherwise} \end{cases}$$

resulting in reduced-model terms

$$\begin{aligned} \hat{E}_p(z) &= \sum_i w_i(z) \hat{E}_{ii} \\ \hat{A}_p(z) &= \sum_i w_i(z) \hat{A}_{ii} \\ \hat{B}_p(z,u) &= \sum_i w_i(z) \hat{B}_i u. \end{aligned}$$

Note that, by our choice of U_k, we also obtain $\hat{E}_p(z) = I$.

While we cannot guarantee global stability of the resulting reduced model through the existence of a Lyapunov function, our projection guarantees that stability will be preserved for all stable local linear models and, additionally, that the equilibrium point of the reduced model will be at least locally stable because there will always be a stable local model at the equilibrium point. In our experience, reduced models created with the proposed stabilizing projection have always produced stable outputs in response to typical inputs of interest, even though the models are not provably globally stable. Several examples using this stabilizing-projection scheme will be presented in Sections VII-C and VII-D.

VI. Implementation

For nonlinear systems producing unstructured and unstable Jacobian matrices, the stabilizing nonlinear left-projection technique presented in Section V-B must be used to create stable reduced models. Constructing the nonlinear projection can be extremely computationally expensive, as it requires computing a stabilizing left-projection matrix for every local linear model. The left projection matrices defined in (37) are particularly expensive, as they require solving Lyapunov matrix equations for each linear system. Although there exist methods [21], [22] for solving Lyapunov equations that perform better than $O(N^3)$, this matrix-equation solution is the dominant computational cost in creating the reduced models using (37). In this section, we present one approach to reduce the computational costs of solving Lyapunov equations for the linearized systems as well as present our full model-reduction algorithm.

A. *Reusability of Lyapunov Equation Solutions*

We first consider the nonlinear left-projection function $U(z)$, as defined in (35), with U_k as defined in (37). Constructing $U(z)$ is expensive because we assume that there does not exist one matrix $P \succ 0$ that satisfies (39) for *all* k, and thus, we have to solve Lyapunov matrix equations for every local linear model. However, there may be a matrix $P \succ 0$ that satisfies (39) for *some* set of k. That is, given a solution P_k to (39) for a single k, there may exist $j \neq k$ such that

$$E_j^{\mathrm{T}} P_k A_j + A_j^{\mathrm{T}} P_k E_j \prec 0.$$

In this case, we may "reuse" the Lyapunov matrix-equation solution to also prove stability for linear model j. Additionally, we may also reuse the matrix P_k for constructing the stability-preserving local left-projection matrix U_j for linear model j, such that $U_j = P_k E_j V (V^{\mathrm{T}} E_j^{\mathrm{T}} P_k E_j V)^{-1}$ preserves stability for linear system (E_j, A_j). Since all local models are linearizations of the same physical system, it is likely that some matrix-equation solutions P_k may be reused to satisfy other matrix equations.

Fig. 1(a) shows the reusability of Lyapunov equation solutions for linearizations of a system, as discussed in detail in Section VII-D, described by a model of the form

$$Q(x)\dot{x} = f(x) + b(x)u. \tag{40}$$

A dark dot in location (j, k) shown in Fig. 1(a) signifies that the matrix P_k (which was constructed to satisfy $E_k^{\mathrm{T}} P_k A_k + A_k^{\mathrm{T}} P_k E_k \prec 0$) satisfies $E_j^{\mathrm{T}} P_k A_j + A_j^{\mathrm{T}} P_k E_j \prec 0$. Note that the plot is not symmetric because $A_i \neq A_i^{\mathrm{T}}$ in this example.

From the near-periodic structure of this plot, it appears that there is a correlation between reusability of Lyapunov equation

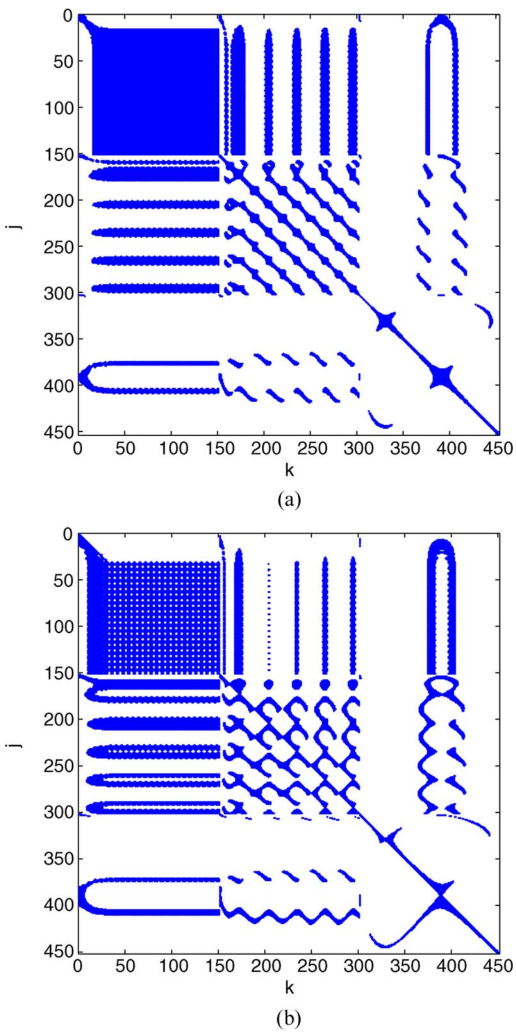

Fig. 1. Comparison of reusability of Lyapunov matrix-equation solutions with proximity of linearization points. (a) Reusability of Lyapunov solutions—all pairs $\{P_k, (E_j, A_j)\}$ that satisfy $E_j^{\mathrm{T}} P_k A_j + A_j^{\mathrm{T}} P_k E_j \prec 0$. The system matrices (E_i, A_i) were obtained by training system (47) with three different sinusoidal inputs. (b) All linearization point pairs (x_j, x_k) that satisfy $\|x_j - x_k\| < \epsilon$. The linearization points were obtained by training system (47) with three different sinusoidal inputs.

solutions and state-space location of the linearization points from which the linear models are created. Fig. 1(b) shows the proximity of linearization points in the state space. In this plot, a dark dot in location (j, k) indicates that the distance between linearization points x_j and x_k is less than some small tolerance: $\|x_j - x_k\| < \epsilon$.

The correlation between the patterns shown in Fig. 1(a) and (b) can be explained by considering the perturbed linear system

$$(E_k + \Delta E)\dot{x} = (A_k + \Delta A)x.$$

This system is stable if there exist SPD matrices P_k, Q_k such that

$$E_k^{\mathrm{T}} P_k A_k + A_k^{\mathrm{T}} P_k E_k = -Q_k - \tilde{Q} \tag{41}$$

where

$$\tilde{Q} = (\Delta E^{\mathrm{T}} P \Delta A + \Delta A^{\mathrm{T}} P \Delta E) + (\Delta E^{\mathrm{T}} P A + A^{\mathrm{T}} P \Delta E) + (E^{\mathrm{T}} P \Delta A + \Delta A^{\mathrm{T}} P E).$$

Assume that there exist SPD matrices P_k, Q_k that satisfy (41) for $\Delta E = 0$ and $\Delta A = 0$. Next, define $\Delta E = E_j - E_k$ and $\Delta A = A_j - A_k$ such that the perturbed system is actually linear system j. If $\sigma_{\max}(\tilde{Q}) < \sigma_{\min}(Q)$, then we may take $P_j = P_k$ and $Q_j = Q_k - \tilde{Q}$ to satisfy the Lyapunov equation for linear system (E_j, A_j). Thus, for ΔE and ΔA small enough, the Lyapunov solutions are reusable. Note that $\tilde{Q} \to 0$ as $\Delta E \to 0$ and $\Delta A \to 0$. From their definitions

$$\Delta A = A_j - A_k = \frac{\partial f(x_j)}{\partial x} - \frac{\partial f(x_k)}{\partial x}$$
$$\Delta E = E_j - E_k = \frac{\partial q(x_j)}{\partial x} - \frac{\partial q(x_k)}{\partial x}$$

and by the smoothness of $f(x)$ and $q(x)$, we find that the perturbations go to zero as $x_j \to x_k$, and thus, the Lyapunov equation solutions will be reusable for models arising from sufficiently close linearization points. If linearization points are too close and the models are too similar, then the model is redundant and not needed in the PWL approximation. However, in our experience, even after removing all redundant models, we have found that it is often still possible to reduce the number of required Lyapunov equation solutions by at least 50% by reusing Lyapunov matrix-equation solutions.

Algorithm 1 Reusability of Matrix Equation Solutions

Given a linear model pair (E_j, A_j) and linearization point x_j, a set $\mathcal{P} = \{P_k\}$ of SPD matrices P_k, a set of linearization points $X = \{x_k\}$, and an orthonormal right-projection matrix V
Compute $V_{aj} \in \mathbb{R}^{N \times q}$, $V_{ej} \in \mathbb{R}^{N \times q}$

$$V_{aj} = A_j V \quad V_{ej} = E_j V \tag{42}$$

for $m = 1 : \tau$ **do**
  Find $x_k = \arg\min_i \|x_j - x_i\|$ where $x_i \in X$
  Compute $\hat{Q}_{jk} = V_{ej}^{\mathrm{T}} P_k V_{aj} + V_{aj}^{\mathrm{T}} P_k V_{ej}$
  if $\lambda_{\max}(\hat{Q}_{jk}) \prec 0$ **then**
    Define $U_j = P_k V_{ej} (V_{ej}^{\mathrm{T}} P_k V_{ej})^{-1}$
    break for
  else
    Remove x_k from X and try again
  end if
end for
if no reusable solution found **then**
  Solve for $P_j : E_j^{\mathrm{T}} P_j A_j + A_j^{\mathrm{T}} P_j E_j = -I$
  Define $U_j = P_j V_{ej} (V_{ej}^{\mathrm{T}} P_j V_{ej})^{-1}$
  Add P_j to $\mathcal{P}$: $\mathcal{P} = \{\mathcal{P}, P_j\}$
end if

An algorithm to exploit this fact might first search through existing solutions to Lyapunov equations corresponding to nearby linearized models, and then test those existing solutions on the given model before solving a new Lyapunov equation for the given model. Although this procedure will require fewer Lyapunov equations solutions, it is still expensive because it requires matrix–matrix products and eigendecompositions for matrices in $\mathbb{R}^{N \times N}$. However, since it is only required that

$\hat{Q}_{jk} = V^{\mathrm{T}} E_j^{\mathrm{T}} P_k A_j V + V^{\mathrm{T}} A_j^{\mathrm{T}} P_k E_j V \in \mathbb{R}^{q \times q}$ be a symmetric negative-definite matrix, it is possible instead to check if this smaller term is negative definite. The eigendecomposition is now performed on a size $q \times q$ matrix instead of a size $N \times N$ matrix, and the cost of matrix–matrix products is reduced from $O(N^3)$ to $O(N^2 q)$. An example of this procedure is presented in Algorithm 1, where the parameter τ defines the maximum number of existing solutions P_k that will be tested (in our experiments, $\tau = 25$ has produced good results).

B. Algorithm

In this section, we present a routine to create stable PWL reduced models from originally stable nonlinear systems by using the nonlinear-projection methodology described in Section V-B. Our procedure, summarized in Algorithm 2, is described as follows.

Algorithm 2 STPWL: Stabilizing Trajectory Piecewise Linear

1: Train System (43) to obtain κ linear model pairs (E_i, A_i) with corresponding linearization points x_i
2: Construct orthonormal right-projection matrix V
3: Set $\mathcal{P} = \{\}$
4: **for** $j = 1 : \kappa$ **do**
5: Compute stabilizing U_j for model (E_j, A_j) using, for instance, Algorithm 1 or optimization problem (12)
6: **if** no solution found **then**
7: Unstable model: Define $U_j = V(V^{\mathrm{T}} E_j^{\mathrm{T}} V)^{-1}$
8: **end if**
9: Project system j with U_j

$$\hat{A}_j = U_j^{\mathrm{T}} A_j V, \quad \hat{k}_j = U_j^{\mathrm{T}} k_j, \quad \hat{b}_j = U_j^{\mathrm{T}} b$$

10: Project linearization points $z_j = V^{\mathrm{T}} x_j$
11: **end for**
12: Obtain ROM of the form

$$\dot{z} = \sum_{j=1}^{\kappa} w_j(z) \hat{A}_j z + \hat{B}(z) u.$$

Given a stable nonlinear descriptor system

$$\frac{d}{dt}[q(x)] = f(x) + bu \tag{43}$$

a training procedure is used to obtain κ linear models. Information from the trajectories and linear models is then used to construct an orthonormal projection matrix V, using, for example, Krylov vectors. Details on methods for training and constructing V can be found in [1], [2], [4], and [6]. At this point, stabilizing left-projection matrices U_m are computed for each Hurwitz matrix pair (E_m, A_m). Such U_m can be computed using the technique described in Algorithm 1, or by solving (12). In the case of unstable linear models, we cannot guarantee stability in the corresponding local reduced model through projection, so we define the projection matrix as $U_m = V(V^{\mathrm{T}} E_m^{\mathrm{T}} V)^{-1}$ to ensure that $U_m^{\mathrm{T}} E_m V = I$. The resulting local left-projection matrices U_m are used to project the local models (E_m, A_m). The final result is a collection of stable linear models that are the basis of a reduced-order PWL model.

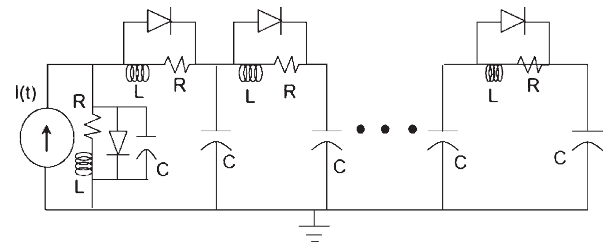

Fig. 2. Nonlinear analog circuit containing monotonic devices.

The storage cost and simulation cost of the final reduced model is the same as that of a model created with the traditional projection approach, i.e., $U = V$. All of the additional computational costs occur offline as part of model generation. Additionally, using stabilizing left-projection matrices obtained from optimization, problem (12) can be much cheaper computationally than the alternative proposed approach. For example, when reducing a linear system of size $N = 1500$ to reduced order $q = 15$, we have found that solving (12) is more than ten times faster than solving the corresponding Lyapunov matrix equation.

VII. Examples

In this section, we will examine several nonlinear systems whose PWL approximations exhibit the properties considered in Sections III–V and present results from the proposed reduction algorithms applied to such systems. All of the model generation and simulation results for both the large and reduced systems were performed in MATLAB on a desktop computer with a 3.33-GHz Intel dual-core processor and 4 GB of RAM. Additional simulation speedups of the PWL reduced models could be obtained by using previously reported techniques in [4], such as fast nearest-neighbor searches, without altering any of the stability results obtained from projection. Additionally, solving Lyapunov equations for constructing the left-projection matrices can be performed much faster using recent algorithms for solving matrix equations [21], [22] instead of MATLAB solvers.

A. Example of Systems With Constant Descriptor Matrix

We first consider nonlinear systems described by models with constant descriptor matrices and structured Jacobian matrices such that the system satisfies the assumptions of Proposition 3.1 in Section III-B. One such example is a system whose Jacobian matrices A_i are negative definite and whose descriptor matrix E is an SPD matrix. In this case, select $P = E^{-1}$, which is also SPD, leading to the Lyapunov function $L(x) = x^{\mathrm{T}} E x$ and left-projection matrix $U = V$. Such systems with structured Jacobian matrices are encountered, for instance, when using PWL approximations on nonlinear circuits comprised of monotonic nonlinear elements such as nonlinear resistors and diodes.

For example, consider the following nonlinear analog circuit, shown in Fig. 2 and first considered in [2], satisfying the previous criteria. The circuit contains the monotonic elements

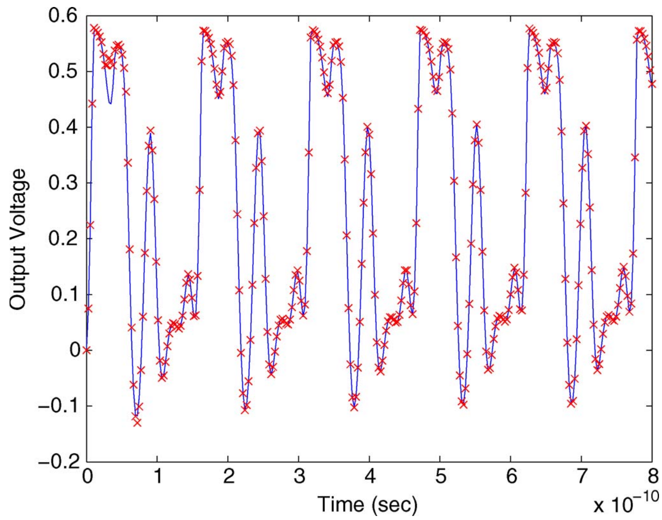

Fig. 3. Comparison of outputs from (crosses) 15th-order stable reduced model and (solid line) 500th-order stable full model resulting from a multitone input to the circuit shown in Fig. 2.

such as resistors, capacitors, inductors, and diodes. The current conservation law applied to this circuit produces the conservation equations

$$\begin{aligned} C_n \frac{dV_n}{dt} &= I_n - I_{n+1} + I_0 \left(e^{\alpha(V_{n-1}-V_n)} - 1 \right) \\ &\quad - I_0 \left(e^{\alpha(V_n - V_{n+1})} - 1 \right) \\ L_n \frac{dI_n}{dt} &= V_{n-1} - V_n - I_n R_n \end{aligned}$$

at each node, leading to the state-space model

$$E\dot{x} = Gx + f(x) + bu \tag{44}$$

where E is a constant SPD matrix, G is a constant stable (but not symmetric) matrix, and f is a nonlinearity whose Jacobians are always negative definite. For this system, we find that $A_i + A_i^{\mathrm{T}}$ is always negative definite, and thus, $L(x) = x^{\mathrm{T}} E x$ is a Lyapunov function for the PWL system. Thus, Proposition 3.1 guarantees internal stability.

Additionally, Proposition 3.2 in Section III-B guarantees I/O stability for the large-order PWL model, and Proposition 3.3, together with Corollary 3.1 from Section III-C, guarantees that the reduced model created with the left-projection matrix $U = V$ will also be I/O stable for any right-projection matrix V. Fig. 3 shows several outputs of a stable reduced model of the nonlinear transmission line created with the projection $U = V$, where V was constructed to match moments. The original system has order $N = 500$ and was trained with sinusoidal inputs of varying amplitude and frequency around 1 GHz. The resulting reduced model has order $q = 15$ and consists of approximately 2000 local linear models, resulting in a simulation speedup factor of about 15.

B. Reformulation for Systems With Nonlinear Descriptor Functions

In Section V-A, we considered reformulating nonlinear descriptor systems such that the resulting Jacobian matrices, while less accurate, were more likely to be structured and stable. To illustrate this point, we consider a nonlinear transmission line used for signal shaping, as shown in Fig. 4, containing distributed nonlinear capacitors. A thorough analysis of this line can be found in [23].

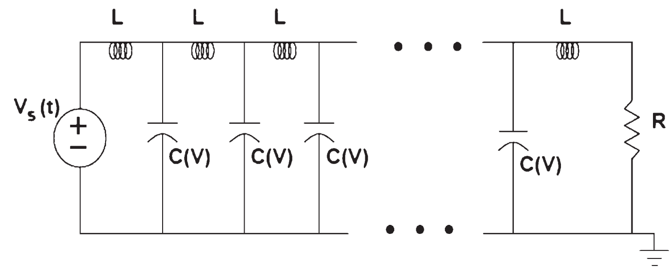

Fig. 4. Pulse narrowing transmission-line circuit containing nonlinear capacitors [23].

The nonlinearity arises from the voltage dependence of the capacitors, which is approximated as $C_n = C(V_n) \approx C_0(1 - b_c V_n)$. Setting the system state to the node voltages and inductor currents, system equations can be derived using Kirchoff's current law and nodal analysis. The input is an ideal voltage source $u(t) = V_s(t)$, and the output is the voltage at some node m along the line, $y(t) = V_m(t)$. Using this formulation, the system equations for an interior node n using the traditional constant-descriptor formulation would be of the form

$$\frac{dV_n}{dt} = \frac{I_n - I_{n+1}}{C_n(V_n)}, \quad \frac{dI_n}{dt} = \frac{V_{n-1} - V_n}{L_n}$$

resulting in a model of the form

$$\dot{x} = f(x) + b(x)u. \tag{45}$$

If, on the other hand, one were to allow the system to possess a nonlinear descriptor matrix

$$C_n(V_n)\frac{dV_n}{dt} = I_n - I_{n+1}, \quad L_n \frac{dI_n}{dt} = V_{n-1} - V_n$$

the state-space model becomes

$$Q(x)\dot{x} = Ax + bu, \quad y = c^{\mathrm{T}} x. \tag{46}$$

Although the nonlinear descriptor formulation (46) will produce less accurate local models, the PWL interpolation of the collection of models is still sufficiently accurate. Fig. 5 shows a comparison of the highly nonlinear outputs of a large-order PWL model (with order $N = 200$) created from system (46) and the original nonlinear system in response to a sinusoidal input. In this example, the PWL system was created by training with sinusoidal inputs and consists of approximately 3000 local models.

Now, consider PWL approximations to these two nonlinear systems, each of which is comprised of linear models created at the same set of linearization points. These two sets of linear models will be different. Table I compares the number of unstable linear models generated by linearizations of the two nonlinear systems, as well as the number of unstable reduced-order linear models created by a Galerkin projection framework $(U = V)$ when projected down to reduced order $q = 40$.

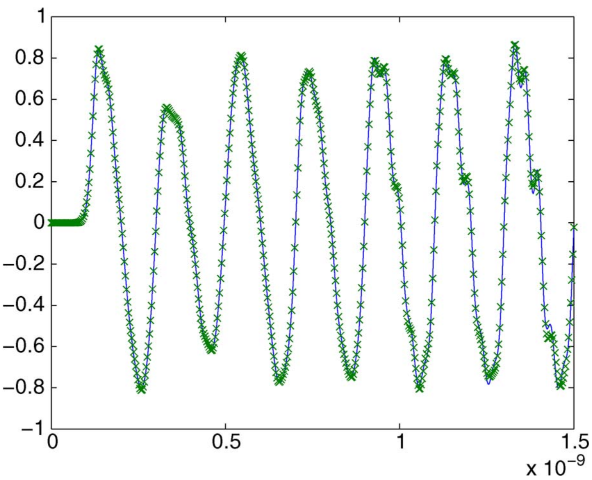

Fig. 5. Comparison of (solid line) full nonlinear system output with output of (crosses) reformulated nonlinear descriptor-form large-order PWL model for system (46).

TABLE I
NUMBER OF UNSTABLE LINEAR MODELS GENERATED FROM 3001 UNIQUE LINEARIZATION POINTS

	Non-descriptor form Equation (45)	Descriptor form Equation (46)
Large PWL	2927	1391
Reduced PWL	1882	0

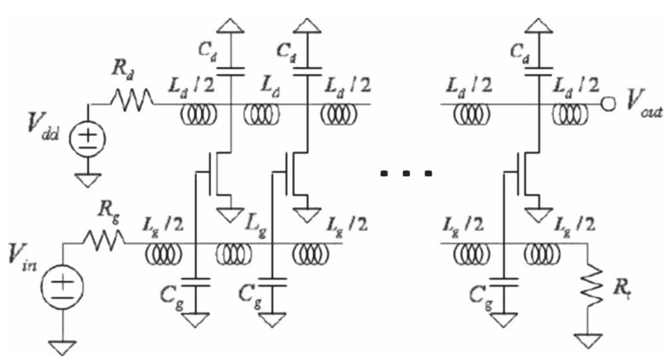

Fig. 6. Distributed amplifier circuit.

For this example, the piecewise-constant descriptor formulation (46) produces fewer unstable large-order linear models and zero unstable reduced-order linear models, while the more accurate constant-descriptor formulation (45) produces many unstable linear models for both large and reduced orders. This result is due to the improved structure of the Jacobian matrices resulting from a reformulation of the system equations into nonlinear descriptor form. Thus, the reformulation allows us to create stable reduced local models without employing the more expensive nonlinear stabilizing projection. In this case, the stable reduced-model provided a speedup of about four times over the original large system.

C. *Unstructured Analog Circuit*

To illustrate the nonlinear left-projection technique proposed in Section V-B for unstable and unstructured large-order PWL models, and Algorithm 1 for reusing matrix-equation solutions, we consider a distributed amplifier circuit shown in Fig. 6. It is not uncommon for analog circuits, such as this one, to produce PWL models that do not contain symmetric or sign-definite system matrices, making it difficult to guarantee stability for the PWL system through the use of a quadratic Lyapunov function. For this example, the transconductances in the small-signal transistor models do not appear symmetrically in the linearized system matrices A_i, making it necessary to utilize the nonlinear left-projection function $U(z)$, as defined in (35), to preserve stability.

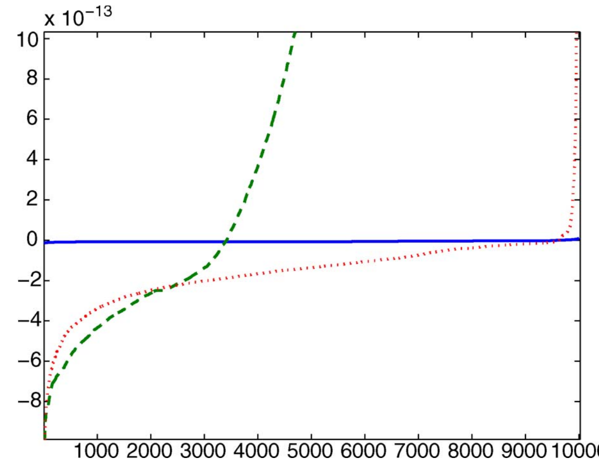

Fig. 7. Maximum real part of the eigenvalues of (solid line) linear models for the large system, containing only 368 unstable linear models, (dotted line) reduced models created with the proposed stabilizing nonlinear left-projection function, also resulting in 368 unstable models, and (dashed line) reduced system created with the traditional constant left-projection matrix, resulting in over 6000 unstable local models.

A collection of about 10 000 linear models of original order $N = 106$ were created by training the system with multitone inputs of the form

$$u(t) = \alpha_1 \sin(2\pi f_1 t) + \alpha_2 \sin(2\pi f_2 t)$$

while varying the amplitudes α_1, α_2 and the frequencies f_1, f_2, which were near 1 GHz. To examine the stability of the local linear models, the maximum real part of the eigenvalues for each large-order linear model is shown in Fig. 7 as the solid line. In this figure, a point with a positive value corresponds to an unstable linear model. Out of a total of almost 10 000 linear models, only 368 are unstable.

From these large linear systems, two different sets of reduced models (with reduced order $q = 10$) were created: the first using a constant left-projection matrix $U = V$ and the second using the nonlinear left-projection function described in (37). Algorithm 1 was used to reduce the number of matrix-equation solutions required to construct the stabilizing-projection matrix. The maximum real part of the resulting local reduced models, also shown in Fig. 7, compares the stability of the two resulting sets of reduced-order linear-model pairs by plotting the sorted maximum real part of the eigenvalues for each linear model. The dashed line corresponds to the reduced models created with the constant left-projection matrix $(U = V)$. Note that, for this model, about two-thirds of the 10 000 reduced linear models are unstable. The reduced models created with the proposed nonlinear left-projection are represented by the dotted line. Although it is not easy to see in the figure, for this reduced model,

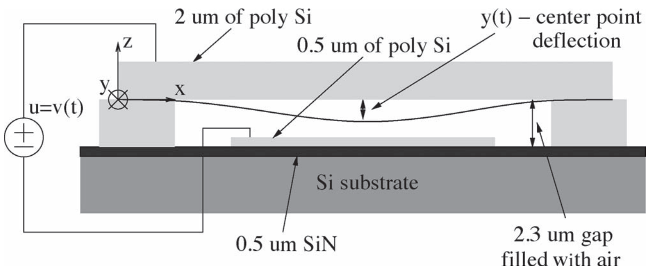

Fig. 8. Micromachined-switch MEMS device [1], [2].

there are precisely 368 unstable models, which correspond exactly to the original 368 unstable large-order linear models.

The stabilizing-projection matrices were computed using the procedure described in Algorithm 1, which solved approximately 4000 matrix equations to generate the approximately 10 000 local projection matrices, all of which took less than 5 min.

D. Unstructured MEMS

To illustrate the full stabilizing nonlinear-model-reduction algorithm from Section VI-B, we consider a nonlinear descriptor system that produces unstructured Jacobian pairs (E_i, A_i) that are not all stabilizable by a constant left projection. This example is a micromachined-switch MEMS device, shown in Fig. 8, which is well documented in [1] and [2].

The physical system is described by the pair of nonlinear partial differential equations

$$EIh^3w\frac{\partial^4 z}{\partial x^4} - Shw\frac{\partial^2 z}{\partial x^2} = F_e + \int_0^w (P - P_a)dy - \rho hw\frac{\partial^2 z}{\partial t^2}$$

$$\nabla \cdot \left((1+6K)z^3 P\nabla P\right) = 12\mu\frac{\partial (Pz)}{\partial t}.$$

After discretizing the device into m sections lengthwise and n sections widthwise, this model can be written in the form

$$Q(x)\dot{x} = f(x) + b(x)u \tag{47}$$

where the state variable $x \in \mathbb{R}^{mn+2m}$ is chosen to contain the vertical positions of the beam, the pressure beneath the beam, and a quantity related to the rate of change in pressure, all on the discretized grid. A detailed description of these functions can be found in [7].

To test Algorithm 2 from Section VI-B, the nonlinear descriptor system (47) was trained with a series inputs of the form

$$u(t) = (\alpha_1 \sin(2\pi f_1 t) + \alpha_2 \sin(2\pi f_2 t))^2 \tag{48}$$

with frequencies near 30 MHz to obtain a set of κ linear models, and then, a right-projection matrix V was constructed with a moment matching approach. From this point, two separate reduced models were created—one using the traditional TPWL projection technique with a constant left-projection matrix $(U = V)$, referred to as the TPWL-ROM, and one generated using Algorithm 2. The original large-order system has order $N = 360$, while both reduced models have order $q = 20$ and are comprised of approximately 1100 local models created from the same set of linearization points. For this example, the stabilizing projection matrices were created using Algorithm 1, which solved 375 matrix equations for the approximately 1100 local models. The entire reduction process was completed in under 15 min.

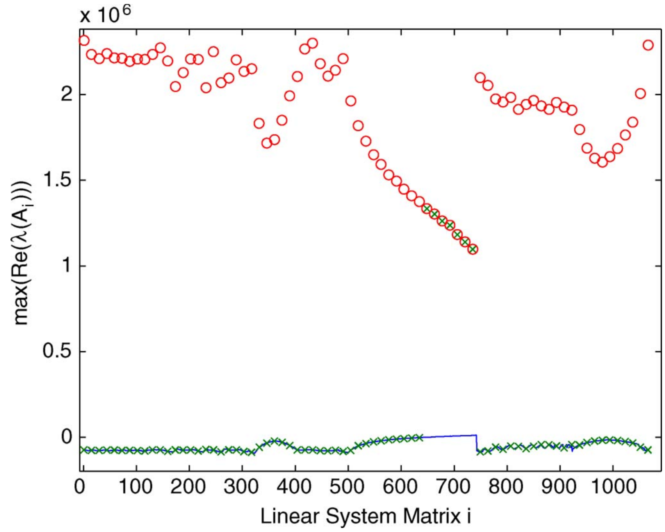

Fig. 9. Maximum real part of (solid line) eigenvalues for individual linear models comprising the large-order PWL model and two reduced-order PWL models. The crosses correspond to the reduced models created with the proposed projection method in (37), while the circles correspond to reduced models created with the traditional constant left-projection matrix.

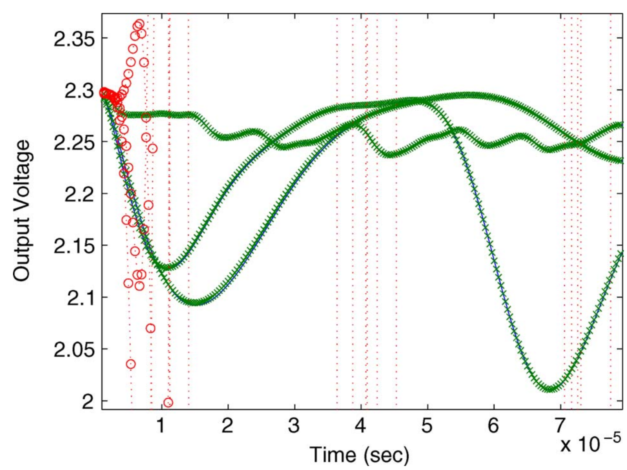

Fig. 10. Output responses of the full nonlinear system and two reduced models to three different inputs. (Solid line) Full nonlinear system and (crosses matching the solid line underneath) PWL reduced model created by Algorithm 2 are both stable, while (dotted line with circles) PWL model created using the traditional constant left-projection matrix is unstable.

Fig. 9 shows the maximum real part of the eigenvalues of the linear models for each large-order linearized model and the two reduced models. Despite most of the large matrix pairs (E_m, A_m) being stable (represented by the solid line), Fig. 9 shows that, in every case, the reduced models created with the constant left-projection matrix (circles) are unstable. The models created from the nonlinear left-projection (crosses), however, preserve stability in the local models in all cases where the original models were stable.

The two reduced-order models were then simulated with a set of inputs of the form (48) with different frequency and amplitude from the training inputs from which the linearized models were created. Fig. 10 shows the output of the full nonlinear system and the two reduced models for several different inputs. The output of the PWL-ROM created with the traditional constant left-projection matrix (dotted line with circles) grows unboundedly because the reduced model is unstable, while the PWL-ROM created with Algorithm 2 (crosses) is both accurate and stable. Additionally, the stable ROM is simulated approximately 25 times faster than the full nonlinear model.

VIII. Conclusion

In this paper, we have presented results addressing the issue of stability for PWL reduced models. These results include both theoretical results, in the form of theorems guaranteeing finite-gain stability for certain classes of PWL models, and practical results, in the form of efficient algorithms for constructing stable PWL reduced models. Special emphasis is placed on systems described by models with nonlinear descriptor functions, which often arise when modeling analog circuits. The results presented in this paper improve the reliability of PWL reduced models as a tool for design and optimization of analog systems and can be used synergistically together with any recent and future results on open issues regarding the efficient creation and simulation of PWL models (such as optimal selection of training inputs and weighting functions).

References

[1] M. Rewienski and J. White, "A trajectory piecewise-linear approach to model order reduction and fast simulation of nonlinear circuits and micromachined devices," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 22, no. 2, pp. 155–170, Feb. 2003.

[2] D. Vasilyev, M. Rewienski, and J. White, "A TBR-based trajectory piecewise-linear algorithm for generating accurate low-order models for nonlinear analog circuits and MEMS," in *Proc. IEEE/ACM Des. Autom. Conf.*, Jun. 2003, pp. 490–495.

[3] N. Dong and J. Roychowdhury, "Piecewise polynomial nonlinear model reduction," in *Proc. IEEE/ACM Des. Autom. Conf.*, Jun. 2003, pp. 484–489.

[4] S. Tiwary and R. Rutenbar, "Scalable trajectory methods for on-demand analog macromodel extraction," in *Proc. 42nd ACM/IEEE Des. Autom. Conf.*, Jun. 2005, pp. 403–408.

[5] S. Tiwary and R. Rutenbar, "Faster, parametric trajectory-based macromodels via localized linear reductions," in *Proc. IEEE/ACM Int. Conf. Comput.-Aided Des.*, Nov. 2006, pp. 876–883.

[6] B. Bond and L. Daniel, "A piecewise-linear moment-matching approach to parameterized model-order reduction for highly nonlinear systems," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 26, no. 12, pp. 2116–2129, Dec. 2007.

[7] M. Rewienski, "A trajectory piecewise-linear approach to model order reduction of nonlinear dynamical systems," Ph.D. dissertation, MIT Press, Cambridge, MA, Jun. 2003.

[8] B. Bond and L. Daniel, "Stabilizing schemes for piecewise-linear reduced order models via projection and weighting functions," in *Proc. IEEE/ACM Int. Conf. Comput.-Aided Des.*, San Jose, CA, 2007, pp. 860–867.

[9] K. J. Kerns, I. L. Wemple, and A. T. Yang, "Stable and efficient reduction of substrate model networks using congruence transforms," in *Proc. IEEE/ACM Int. Conf. Comput.-Aided Des.*, San Jose, CA, Nov. 1995, pp. 207–214.

[10] L. M. Silveira, M. Kamon, I. Elfadel, and J. K. White, "Coordinate-transformed Arnoldi algorithm for generating guarantee stable reduced-order models of RLC," *Comput. Methods Appl. Mech. Eng.*, vol. 169, no. 3/4, pp. 377–389, Feb. 1999.

[11] A. Odabasioglu, M. Celik, and L. T. Pileggi, "PRIMA: Passive reduced-order interconnect macromodeling algorithm," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 17, no. 8, pp. 645–654, Aug. 1998.

[12] Z. Bai, P. Feldmann, and R. W. Freund, "Stable and passive reduced order models based on partial Pade approximation via the Lanczos process," Bell Labs., Lucent Technol., Murray Hill, NJ, Tech. Rep. 97-3-10, Oct. 1997.

[13] K. C. Sou, A. Megretski, and L. Daniel, "A quasi-convex optimization approach to parameterized model order reduction," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 27, no. 3, pp. 456–469, Mar. 2008.

[14] M. Vidyagasar, *Nonlinear Systems Analysis*. New York: Prentice-Hall, 1978.

[15] T. Stykel, "Stability and inertia theorems for generalized Lyapunov equations," *Linear Algebra Appl.*, vol. 355, no. 1–3, pp. 297–314, Nov. 2002.

[16] A. Megretski, *Lecture Notes on Projection-Based Model Reduction*, Sep. 2004. [Online]. Available: http://web.mit.edu/6.242/www/syll.html

[17] B. Bond, A. Megretski, and L. Daniel, *Stable Model Reduction for Linear Systems Through Preservation of Dissipation*. Article in preparation.

[18] B. Bond and L. Daniel, "Guaranteed stable projection-based model reduction for indefinite and unstable linear systems," in *Proc. IEEE/ACM Int. Conf. Comput.-Aided Des.*, San Jose, CA, 2008, pp. 728–735.

[19] M. Rewienski and J. K. White, "A trajectory piecewise-linear approach to model order reduction and fast simulation of nonlinear circuits and micromachined devices," in *Proc. IEEE/ACM Int. Conf. Comput.-Aided Des.*, San Jose, CA, Nov. 2001, pp. 252–257.

[20] K. Willcox and J. Peraire, "Balanced model reduction via the proper orthogonal decomposition," in *Proc. 15th AIAA Comput. Fluid Dyn. Conf.*, Anaheim, CA, Jun. 2001.

[21] P. Benner, J. Li, and T. Penzl, "Numerical solution of large Lyapunov equations, Riccati equations, and linear-quadratic control problems," *Numer. Linear Algebra Appl.*, vol. 15, no. 9, pp. 755–777, Nov. 2008.

[22] P. Benner, H. Mena, and J. Saak, "On the parameter selection problem in the Newton–ADI iteration for large-scale Riccati equations," *Electron. Trans. Numer. Anal.*, vol. 29, pp. 136–149, 2008.

[23] E. Afshari and A. Hajimiri, "A non-linear transmission line for pulse shaping in silicon," *IEEE J. Solid-State Circuits*, vol. 40, no. 3, pp. 744–752, Mar. 2005.

Bradley N. Bond (S'07) received the B.S. degree in engineering science and mechanics from Penn State University, University Park, in 2004 and the M.S. degree in electrical engineering from the Massachusetts Institute of Technology, Cambridge, in 2006, where he is currently working toward the Ph.D. degree in the Computational Prototyping Group, Department of Electrical Engineering and Computer Science.

His research interests include nonlinear model order reduction.

Luca Daniel (S'98–M'04) received the Laurea degree (*summa cum laude*) in electronic engineering from the "Universita di Padova," Padova, Italy, in 1996 and the Ph.D. degree in electrical engineering from the University of California (UC Berkeley), Berkeley, in 2003.

In 1997, he collaborated with STMicroelectronics Berkeley Laboratories. In 1998, he was with HP Research Laboratories, Palo Alto. In 2001, he was with Cadence Berkeley Laboratories. He is currently an Associate Professor with the Electrical Engineering and Computer Science Department and a Principal Investigator of the RLE Computational Prototyping Group, Massachusetts Institute of Technology, Cambridge. His research interests include parameterized model order reduction of linear and nonlinear dynamical systems; mixed-signal, RF, and millimeter-wave circuit design and robust optimization; power electronics, microelectromechanical-system design and fabrication; and parasitic extraction and accelerated integral-equation solvers.

Dr. Daniel was the recipient of the "2003 ACM Outstanding Ph.D. Dissertation Award in Electronic Design Automation" and the Best Thesis Awards from both the Electrical Engineering and Computer Science and the Applied Math Departments at UC Berkeley for his Ph.D. degree thesis. He was also the recipient of four best paper awards in conferences, including the Design Automation Conference and the International Conference on Computer-Aided Design, and the "IEEE Power Electronic Society Prize Paper Award" for the best paper published in the IEEE Transactions on Power Electronics in the year 1999. He was also the recipient of the 2009 IBM Corporation Faculty Award, the 2008 Jonathan Allen RLE-MIT Junior Faculty Award, and the Emanuel E. Landsman Chair for Career Development from the MIT Department of Electrical Engineering in 2007–2009.