Reinforcement Learning and Optimal Control

ASU, CSE 691, Winter 2019

Dimitri P. Bertsekas
dimitrib@mit.edu

Lecture 6

Outline

Recall the Approximation in Value Space Framework for Finite Horizon Problems

First Step — "Future"

Approximate Min

$$\min_{u_k} E\Big\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}(x_{k+1}) \Big\}$$

Approximate $E\{\cdot\}$

- Certainty equivalence
- Adaptive simulation
- Monte Carlo tree search

Approximate Cost-to-Go $\tilde{J}_{k+1}$

- Problem approximation
- Rollout, Model Predictive Control
- Parametric approximation
- Neural nets
- Aggregation

Parametric Approximation in Value Space

An approximation architecture is a class of functions $\tilde{J}(x, r)$ that depend on x and a vector $r = (r_1, \ldots, r_m)$ of m "tunable" scalar parameters (or weights).

Issues and terminology

- Aim: Choose r to make $\tilde{J}(x, r)$ close to some target cost function $J(x)$.
- Training algorithm chooses r. It typically uses least squares optimization (regression) to fit $\tilde{J}(x, r)$ to a data set of state-cost pairs.
- An architecture is called linear if $\tilde{J}(x, r)$ is linear in r.
- It is called feature-based if it depend on x via a feature vector $\phi(x)$,

$$\tilde{J}(x, r) = \hat{J}\big(\phi(x), r\big),$$

  where $\hat{J}$ is some function. Idea: Features capture dominant nonlinearities.
- A linear feature-based architecture:

$$\tilde{J}(x, r) = \sum_{\ell=1}^{m} r_\ell \phi_\ell(x) = r' \phi(x),$$

  where r_ℓ and $\phi_\ell(x)$ are the ℓth components of r and $\phi(x)$.

Training of Architectures

Least squares regression

- Collect a set of state-cost training pairs (x^s, β^s), $s = 1, \ldots, q$, where β^s is equal to the target cost $J(x^s)$ plus some "noise".
- r is determined by solving the problem

$$\min_r \sum_{s=1}^{q} \big(\tilde{J}(x^s, r) - \beta^s\big)^2$$

- Sometimes a quadratic regularization term $\gamma\|r\|^2$ is added to the least squares objective, to facilitate the minimization (among other reasons).

Training of linear feature-based architectures can be done exactly

- If $\tilde{J}(x, r) = r'\phi(x)$, where $\phi(x)$ is the m-dimensional feature vector, the training problem is quadratic and can be solved in closed form.
- The exact solution of the training problem is given by

$$\hat{r} = \left(\sum_{s=1}^{q} \phi(x^s)\phi(x^s)'\right)^{-1} \sum_{s=1}^{q} \phi(x^s)\beta^s$$

- This requires a lot of computation for a large m and data set; may not be best.

Training of Nonlinear Architectures

The main training issue

How to exploit the structure of the training problem

$$\min_r \sum_{s=1}^{q} \big(\tilde J(x^s, r) - \beta^s\big)^2$$

to solve it efficiently.

Key characteristics of the training problem

- Possibly nonconvex with many local minima, horribly complicated graph of the cost function (true when a neural net is used).
- Many terms in the least least squares sum; standard gradient and Newton-like methods are essentially inapplicable.
- Incremental iterative methods that operate on a single term $\big(\tilde J(x^s, r) - \beta^s\big)^2$ at each iteration have worked well enough (for many problems).

Incremental Gradient Methods

Generic sum of terms optimization problem

Minimize

$$f(y) = \sum_{i=1}^{m} f_i(y)$$

where each f_i is a differentiable scalar function of the n-dimensional vector y (this is the parameter vector in the context of parametric training).

The ordinary gradient method generates y^{k+1} from y^k according to

$$y^{k+1} = y^k - \gamma^k \nabla f(y^k) = y^k - \gamma^k \sum_{i=1}^{m} \nabla f_i(y^k)$$

where $\gamma^k > 0$ is a stepsize parameter.

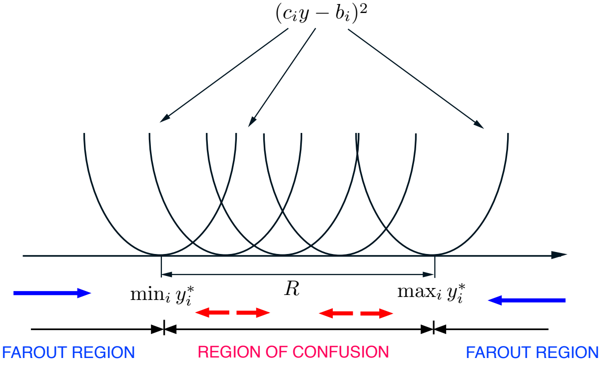
The incremental gradient counterpart

Choose an index i_k and iterate according to

$$y^{k+1} = y^k - \gamma^k \nabla f_{i_k}(y^k)$$

where $\gamma^k > 0$ is a stepsize parameter.

The Advantage of Incrementalism: An Interpretation from the NDP Book



Minimize $f(y) = \frac{1}{2} \sum_{i=1}^{m} (c_i y - b_i)^2$

Compare the ordinary and the incremental gradient methods in two cases

- When far from convergence: Incremental gradient is as fast as ordinary gradient with $1/m$ amount of work.
- When close to convergence: Incremental gradient gets confused and requires a diminishing stepsize for convergence.

Incremental Aggregated and Stochastic Gradient Methods

Incremental aggregated method aims at acceleration

- Evaluates gradient of a single term at each iteration.
- Uses previously calculated gradients as if they were up to date

$$y^{k+1} = y^k - \gamma^k \sum_{\ell=0}^{m-1} \nabla f_{i_{k-\ell}}(y^{k-\ell})$$

- Has theoretical and empirical support, and it is often preferable.

Stochastic gradient method (also called stochastic gradient descent or SGD)

- Applies to minimization of $f(y) = E\{F(y,w)\}$ where w is a random variable
- Has the form

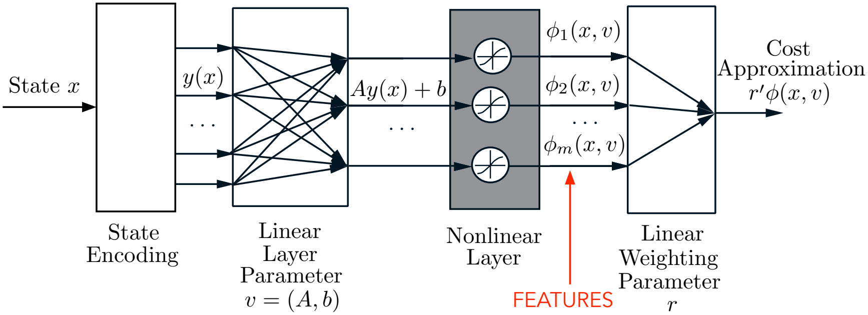
$$y^{k+1} = y^k - \gamma^k \nabla_y F(y^k, w^k)$$

  where w^k is a sample of w and $\nabla_y F$ denotes gradient of F with respect to y.
- The incremental gradient method with random index selection is the same as SGD [convert the sum $\sum_{i=1}^m f_i(y)$ to an expected value, where i is random with uniform distribution].

Implementation Issues of Incremental Methods - Alternative Methods

- How to pick the stepsize γ^k (usually $\gamma^k = \frac{\gamma}{k+1}$ or similar).
- How to deal (if at all) with region of confusion issues (detect being in the region of confusion and reduce the stepsize).
- How to select the order of terms to iterate (cyclic, random, other).
- Diagonal scaling (a different stepsize for each component of y).
- Alternative methods (more ambitious): Incremental Newton method, extended Kalman filter (see the textbook and references).

Neural Nets: An Architecture that Automatically Constructs Features

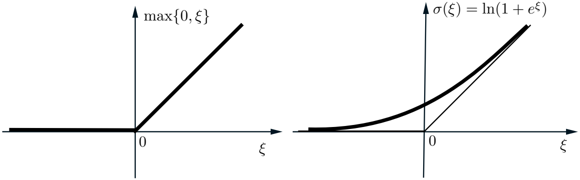


Given a set of state-cost training pairs (x^s, β^s), $s = 1, \ldots, q$, the parameters of the neural network (A, b, r) are obtained by solving the training problem

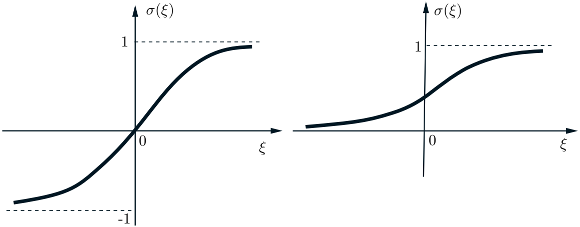
$$\min_{A,b,r} \sum_{s=1}^{q} \left(\sum_{\ell=1}^{m} r_\ell \sigma \left(\left(Ay(x^s) + b\right)_\ell \right) - \beta^s \right)^2$$

- Incremental gradient is typically used for training.
- Universal approximation property.

Rectifier and Sigmoidal Nonlinearities

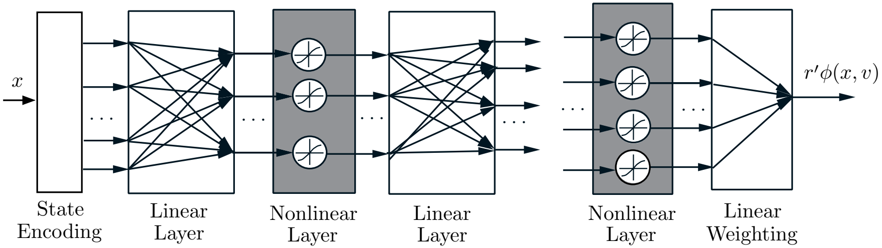


The rectified linear unit $\sigma(\xi) = \ln(1 + e^{\xi})$. It is the rectifier function $\max\{0, \xi\}$ with its corner "smoothed out."



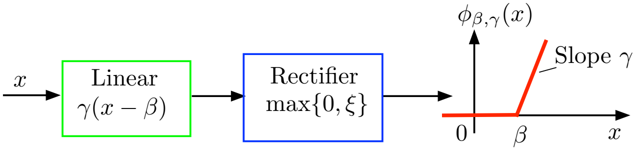
Sigmoidal units: The hyperbolic tangent function $\sigma(\xi) = \tanh(\xi) = \frac{e^{\xi} - e^{-\xi}}{e^{\xi} + e^{-\xi}}$ is on the left. The logistic function $\sigma(\xi) = \frac{1}{1+e^{-\xi}}$ is on the right.

Deep Neural Networks

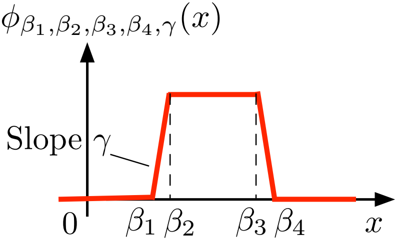


- The multilayer network provides a hierarchy of features (each set of features being a function of the preceding set of features).
- We may use matrices A with a special structure that encodes special linear operations such as convolution.
- When such structures are used, the training problem may become easier, because the number of parameters in the linear layers is drastically decreased.
- They have been found more effective than shallow neural nets for some problems.
- Incremental gradient is still used for training. The algorithm is based on an intelligent way of using the chain rule to calculate the incremental gradient at each iteration.

A Working Break: Challenge Question

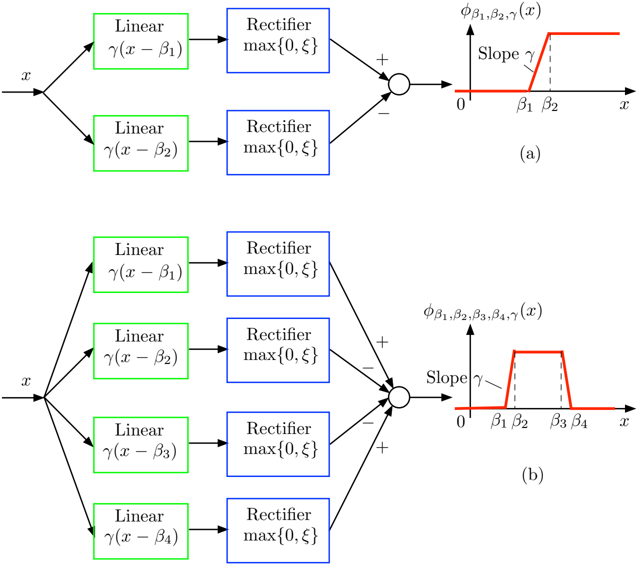


How can we use linear and rectifier units to construct the “pulse" feature below?



- What are the features that can be produced by neural nets?
- Why do neural nets have a “universal approximation" property?

Answer



Using the pulse feature as a building block, any feature can be approximated

Sequential DP Approximation - A Parametric Approximation at Every Stage (Also Called Fitted Value Iteration)

Start with $\tilde{J}_N = g_N$ and sequentially train going backwards, until $k = 0$

- Given a cost-to-go approximation $\tilde{J}_{k+1}$, we use one-step lookahead to construct a large number of state-cost pairs (x_k^s, β_k^s), $s = 1, \ldots, q$, where

$$\beta_k^s = \min_{u \in U_k(x_k^s)} E\Big\{g(x_k^s, u, w_k) + \tilde{J}_{k+1}\big(f_k(x_k^s, u, w_k), r_{k+1}\big)\Big\}, \qquad s = 1, \ldots, q$$

- We "train" an architecture $\tilde{J}_k$ on the training set (x_k^s, β_k^s), $s = 1, \ldots, q$.

Typical approach: Train by least squares/regression and possibly using a neural net

We minimize over r_k

$$\sum_{s=1}^{q} \big(\tilde{J}_k(x_k^s, r_k) - \beta^s\big)^2$$

Sequential Q-Factor Approximation

- Consider sequential DP approximation of Q-factor parametric approximations

$$\tilde{Q}_k(x_k, u_k, r_k) = E\Big\{g_k(x_k, u_k, w_k) + \min_{u \in U_{k+1}(x_{k+1})} \tilde{Q}_{k+1}(x_{k+1}, u, r_{k+1})\Big\}$$

(Note a mathematical magic: The order of $E\{\cdot\}$ and min have been reversed.)

- We obtain $\tilde{Q}_k(x_k, u_k, r_k)$ by training with many pairs $\big((x_k^s, u_k^s), \beta_k^s\big)$, where β_k^s is a sample of the approximate Q-factor of (x_k^s, u_k^s). [No need to compute $E\{\cdot\}$.]
- Note: No need for a model to obtain β_k^s. Sufficient to have a simulator that generates state-control-cost-next state random samples

$$\big((x_k, u_k), (g_k(x_k, u_k, w_k), x_{k+1})\big)$$

- Having computed r_k, the one-step lookahead control is obtained on-line as

$$\overline{\mu}_k(x_k) \in \arg \min_{u \in U_k(x_k)} \tilde{Q}_k(x_k, u, r_k)$$

without the need of a model or expected value calculations.

- Important advantage: The on-line calculation of the control is simplified.

About the Next Lecture

We will cover:

- Infinite horizon DP problems: Stochastic shortest path and discounted problems
- Analysis, Bellman's equation, optimality conditions
- Algorithms: Value iteration, policy iteration
- We will likely need more than one lecture

PLEASE READ AS MUCH OF SECTIONS 4.1-4.5 AS YOU CAN

APPENDIX OF CHAPTER 4 CONTAINS PROOFS; TAKE A CRACK AT THEM

PLEASE DOWNLOAD THE LATEST VERSIONS FROM MY WEBSITE