Recursive State Estimation for a Set-Membership Description of Uncertainty

DIMITRI P. BERTSEKAS AND IAN B. RHODES, MEMBER, IEEE

Abstract—This paper is concerned with the problem of estimating the state of a linear dynamic system using noise-corrupted observations, when input disturbances and observation errors are unknown except for the fact that they belong to given bounded sets. The cases of both energy constraints and individual instantaneous constraints for the uncertain quantities are considered. In the former case, the set of possible system states compatible with the observations received is shown to be an ellipsoid, and equations for its center and weighting matrix are given, while in the latter case, equations describing a bounding ellipsoid to the set of possible states are derived. All three problems of filtering, prediction, and smoothing are examined by relating them to standard tracking problems of optimal control theory. The resulting estimators are similar in structure and comparable in simplicity to the corresponding stochastic linear minimum-variance estimators, and it is shown that they provide distinct advantages over existing schemes for recursive estimation with a set-membership description of uncertainty.

I. Introduction

THIS PAPER is concerned with the estimation of the state of a linear dynamic system when there is uncertainty in the initial state, disturbances in the system dynamics, and errors in the measurement of the system output. The most common approach to such problems is to model the initial state as a random vector and the dynamics and measurement noises as stochastic processes. Under these circumstances, all information about the system state at any time that is provided by the measurements of the system output is contained in the probability density function (or distribution function) of the state conditioned on these measurements. This probability density function is then used, explicitly or implicitly, to determine an estimate of the system that is best in some prescribed sense.

In this paper, the uncertain quantities are not modeled as random variables or stochastic processes, but are considered instead to be unknown except that they belong to given sets in appropriate vector spaces. In this case, all information about the system state that is provided by the observations of the system output may be summarized by the set of states consistent with both the observations received and the constraints on the uncertain quantities. The estimation problem then becomes one of characterizing this set of possible states. This approach to the estimation problem was first taken by Witsenhausen [1] in the framework of a more general problem. An ellipsoidal approximation algorithm with certain computational advantages was later given by Schweppe [2], [3]. In this algorithm, the observations are used to calculate recursively a bounding ellipsoid to the set of possible states, under the assumption that the sets containing the initial condition and the input and observation noises are, or can be approximated by, ellipsoids. For related work also see [4]–[7].

Attention in this paper will be directed primarily to the case of a continuous-time linear dynamic system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

to which there are available noise-corrupted measurements

$$y(t) = C(t)x(t) + v(t) \tag{2}$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^r$ is an input disturbance, and $v(t) \in R^m$ is measurement noise. The corresponding results for discrete-time systems are summarized in Appendix I.

Two distinct types of contraints on the uncertain quantities will be considered. The first is the energy-type constraint

$$[x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t_1} u'(t)Q^{-1}(t)u(t) + v'(t)R^{-1}(t)v(t)\,dt \leq 1 \tag{3}$$

where x_0 is a given n vector, Ψ, $Q(t)$, and $R(t)$ are given positive-definite matrices, and t_0 and t_1 are fixed initial and final times, respectively. For this constraint, we show that the set of possible states at any time consistent with the output observations is an ellipsoid whose center and weighting matrix are generated by equations identical to those associated with the best linear estimator (Kalman filter) for a certain stochastic estimation problem. We examine the filtering problem, the smoothing problem, and the prediction problem. In the filtering and prediction cases, we solve the problem by reducing it to a well known optimal control problem (the tracking or servomechanism problem) in which time is reversed. In the smoothing case, the problem is reduced to a combination of two tracking problems, one in forward time and one with time reversed. This leads to an estimator involving two systems, one operating in forward time and one with time reversed, as in the corresponding stochastic smoothing problem.

The second type of constraint that is considered is the more practically important case where the uncertain quantities are constrained at each instant of time to lie in

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D. P. Bertsekas is with the Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Mass.

I. B. Rhodes is with the Control Systems Science and Engineering Laboratory, Washington University, St. Louis, Mo.

an ellipsoid, i.e.,

$$[x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] \le 1$$
$$u'(t)Q^{-1}(t)u(t) \le 1$$
$$v'(t)R^{-1}(t)v(t) \le 1.$$

In this case the set of states consistent with the observed output and these constraints is not an ellipsoid and it is not, in general, easily characterized by a finite set of numbers. However, an ellipsoidal bound to it can be determined by bounding the instantaneous constraints by an energy constraint and using the results derived for that constraint. The resulting estimator is similar to that proposed by Schweppe [3], but it has two important advantages: first, the gain matrix does not depend on the particular output observations received and is therefore precomputable; second, it reduces to a constant system as the final time becomes infinite. In all other respects it is comparable to that proposed by Schweppe.

In the next section we formulate more precisely an estimation problem with an energy constraint on the uncertain quantities. The filtering, prediction, and smoothing problems are then examined, in turn, in Sections III–V. Section VI contains a formulation of the estimation problem with instantaneous constraints on the uncertain quantities. A bound on the set of possible states for this problem is derived in Section VII, and the behavior of the resulting estimator as the final time becomes infinite is examined in Section VIII.

II. Formulation of the Problem With Energy Constraints

In this section we formulate a general estimation problem involving a continuous-time linear dynamic system and a combined energy constraint on the uncertain quantities. This problem includes as special cases the filtering, prediction, and smoothing problems.

Problem 1: Consider the linear continuous-time dynamic system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

to which there are available noise-corrupted measurements

$$z(t) = y(t) + v(t) = C(t)x(t) + v(t) \tag{2}$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^r$ is the input disturbance, $v(t) \in R^m$ is the measurement noise, and the matrices $A(t)$, $B(t)$, and $C(t)$ have the appropriate dimensions. The initial state $x(t_0)$ and the disturbances $u(\cdot)$ and $v(\cdot)$ are assumed unknown except that they satisfy the energy constraint

$$[x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t_1} u'(t)Q^{-1}(t)u(t) + v'(t)R^{-1}(t)v(t)\, dt \le 1 \tag{3}$$

where x_0 is a given n vector Ψ, $Q(t)$, and $R(t)$ are given positive-definite symmetric matrices, and t_0 and t_1 are fixed initial and final times. Let t be an arbitrary time in $[t_0, t_1]$, let T be a fixed real number, and let $Z(t)$ denote the measured system output function up to time t, i.e.,

$$Z(t) = \{(z(s), s) : s \in [t_0, t]\}. \tag{4}$$

Find the set $X(t + T|t)$ of possible system states $x(t + T)$ at time $t + T$ that are consistent with the constraint (3) and the output function $Z(t)$ observed up to time t.

If $T = 0$ this problem is usually called the filtering problem, if $T > 0$ it is called the prediction problem while if $T < 0$ it is called the smoothing problem. These problems are considered in turn in the next three sections. We begin with the filtering problem.

III. Filtering Problem With Energy Constraints

Given an observed output function $Z(t) = \{(z(s), s) : s \in [t_0, t]\}$, we have, by definition of $X(t|t)$, that $\xi \in X(t|t)$ if and only if there exist a vector $x(t_0)$ and functions $u(\cdot)$ and $v(\cdot)$ defined on $[t_0, t]$ such that

$$[x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t} u'(s)Q^{-1}(s)u(s) + v'(s)R^{-1}(s)v(s)\, ds \le 1 \tag{5}$$

$$x(t) = \xi \tag{6}$$

and, from (2),

$$v(s) = z(s) - C(s)x(s), \qquad t_0 \le s \le t \tag{7}$$

where $x(\cdot)$ is the trajectory of the system (1) corresponding to the initial state $x(t_0)$ and the input disturbance $u(\cdot)$. Notice that the integration in the constraint (5) is from t_0 to t, whereas that in the original constraint (3) is from t_0 to the final time t_1. These two constraints are entirely consistent and, insofar as the determination of $X(t|t)$ is concerned, they are equivalent, since it is possible that all the available disturbance energy has been used during the interval $[t_0 t]$. Substitution of (7) into (5) immediately gives that $\xi \in X(t|t)$ if and only if there exist a vector $x(t_0)$ and a function $u(\cdot)$ defined on $[t_0, t]$ such that

$$J[\xi, t; u, x(t_0)] \le 1 \tag{8}$$

subject to the system (1) and the constraint (6), where $J[\xi, t; u, x(t_0)]$ is defined by

$$J[\xi, t; u, x(t_0)] \triangleq [x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t} u'(s)Q^{-1}(s)u(s) + [z(s) - C(s)x(s)]'R^{-1}(s)[z(s) - C(s)x(s)]\, ds. \tag{9}$$

It is clear that there exist $u(\cdot)$ and $x(t_0)$ satisfying (6) and (8) if and only if

$$J^*[\xi, t] \triangleq \min_{u(\cdot)} J[\xi, t; u, x(t_0)] \le 1 \tag{10}$$

subject again to the system (1) and the boundary condition (6). We remark that, for the purposes of the minimization indicated in (10), the vector $x(t_0)$ is not chosen separately but is considered specified by the (unconstrained) choice of $u(\cdot)$ and the system (1) with the single boundary condition (6), viz., $x(t) = \xi$. It should be clear that this leads to a simpler minimization problem than would be obtained if $x(t_0)$ were somehow chosen separately and $u(\cdot)$ chosen subject to the constraint that it transfer the system between state $x(t_0)$ at time 0 and state ξ at time t. To emphasize the fact that $x(t_0)$ is considered to be determined by the system (1) with boundary condition (6) and the unconstrained choice of control $u(\cdot)$, we henceforth delete the explicit dependence of J upon $x(t_0)$ and write $J[\xi, t; ux(t_0)]$ as $J[\xi\ t; u]$. In fact, the minimization indicated in (10) subject to the system (1) and the boundary condition $x(t) = \xi$ is simply the tracking (or servomechanism) problem in which time is reversed. In other words, the cost $J[\xi, t; u]$ is to be minimized and the system (1) is viewed as starting at state ξ at time t and running backwards to time t_0. The problem of characterizing the set $X(t|t)$ of states at time t consistent with the energy constraint (3) and the observed output function $Z(t) = \{(z(s), s) : s \in [t_0, t]\}$ has thus been reduced to the following tracking (or servomechanism) problem of optimal control theory.

Problem 1′: Consider the linear system (1) with boundary condition $x(t) = \xi$. Consider also the quadratic cost functional $J[\xi, t; u]$ defined by (9), where Ψ, $Q(s)$, and $R(s)$ are symmetric positive-definite matrices, x_0 is a given n vector, and $z(\cdot)$ is a given m-vector-valued function defined on $[t_0, t]$. Find the set $X(t|t)$ of all states ξ at time t for which

$$J^*[\xi, t] \triangleq \min_u J[\xi, t; u] \leq 1. \tag{10}$$

As noted above, Problem 1′ is simply the standard tracking problem of optimal control theory in which the system operates in reverse time. The solution to the tracking problem is well known (see, e.g., [8]), and we need only make the appropriate identifications to obtain the solution to Problem 1. The relevant results are summarized in the following proposition.

Proposition 1: The solution to Problem 1′, and therefore the solution to Problem 1 in the filtering case, is the ellipsoid $X(t|t)$ given for all $t \in [t_0, t_1]$ by

$$X(t|t) = \{\xi : [\xi - \hat{x}(t|t)]'K(t)[\xi - \hat{x}(t|t)] \leq 1 - \delta^2(t)\} \tag{11}$$

where the $n \times n$ positive-definite matrix $K(t)$ is the solution to the Riccati equation

$$\dot{K}(s) = -A'(s)K(s) - K(s)A(s) - K(s)B(s)Q(s)B'(s)K(s) + C'(s)R^{-1}(s)C(s) \tag{12}$$

with boundary condition

$$K(t_0) = \Psi^{-1}. \tag{13}$$

The n vector $\hat{x}(t|t)$ is the solution to the linear differential equation

$$\dot{\hat{x}}(s|s) = A(s)\hat{x}(s|s) + K^{-1}(s)C'(s)R^{-1}(s)[z(s) - C(s)\hat{x}(s|s)] \tag{14}$$

with boundary condition

$$\hat{x}(t_0|t_0) = x_0 \tag{15}$$

and the positive real number $\delta^2(t)$ is given by

$$\delta^2(t) = \int_{t_0}^{t} [z(s) - C(s)\hat{x}(s|s)]'R^{-1}(s)[z(s) - C(s)\hat{x}(s|s)]\, ds. \tag{16}$$

Since the matrix $K(t)$ is positive definite for all $t \geq t_0$, the ellipsoid $X(t|t)$ may also be expressed as

$$X(t|t) = \{\xi : [\xi - \hat{x}(t|t)]'\Sigma^{-1}(t)[\xi - \hat{x}(t|t)] \leq 1 - \delta^2(t)\} \tag{17}$$

where the $n \times n$ positive-definite matrix $\Sigma(t)$ is given for all $t > t_0$ as the solution to the Riccati equation

$$\dot{\Sigma}(s) = A(s)\Sigma(s) + \Sigma(s)A'(s) - \Sigma(s)C'(s)R^{-1}(s)C(s)\Sigma(s) + B(s)Q(s)B'(s) \tag{18}$$

with boundary condition

$$\Sigma(t_0) = \Psi. \tag{19}$$

We remark that the weighting matrix $\Sigma(t)$ of the ellipsoid $X(t|t)$ does not depend on the particular observed output and therefore may be precomputed using the Riccati equation (18) with boundary condition (19). The center $\hat{x}(t|t)$ of the ellipsoid and the scalar $\delta^2(t)$ both depend on the particular output measurements and are computed on-line using (14)–(16). The existence and positive definiteness of $\Sigma(t)$ for all $t \in [t_0, t_1]$ is guaranteed by the existence and positive definiteness of the matrix $K(t)$ associated with the tracking problem. Furthermore, the observed output must correspond to some permissible $x(t_0)$, $u(\cdot)$, and $v(\cdot)$ and thus $X(t|t)$ must contain at least one point; consequently, it follows from (17) that $\delta^2(t)$ must be less than or equal to unity.

It should be noted that the linear differential equation (14) with initial condition (15) describing the center $\hat{x}(t|t)$ of the ellipsoid $X(t|t)$ and the Riccati equation (18) with boundary condition (19) for its weighting matrix are precisely those specifying the Kalman filter for the stochastic minimum-variance estimation problem involving the linear system (1) and the observations (2), where the initial state is a random vector with mean x_0 and covariance Ψ, the noises $u(\cdot)$ and $v(\cdot)$ are uncorrelated white zero-mean stochastic processes with covariances

$$\operatorname{cov}[u(t), u(\tau)] = Q(t)\delta(t - \tau)$$

$$\operatorname{cov}[v(t), v(\tau)] = R(t)\delta(t - \tau)$$

and the initial state is uncorrelated with $u(t)$ and $v(t)$ for all t. Thus, there is a one–one correspondence between filtering problems where the uncertain quantities satisfy an energy constraint and stochastic filtering problems of the form given above. This is not altogether surprising in view of the fact that we have reduced the former problem to a linear least-squares optimal control problem, and there is a well-known correspondence between such problems and linear least-squares stochastic estimation (filtering) problems [9]. Notice, however, that the correspondence used in this case is not the usual one involving the adjoint equation, but rather a second possible correspondence that is valid when conditions are such that the solution to the Riccati equation is positive definite.

IV. Prediction Problem With Energy Constraints

The solution to the prediction problem follows closely that for the filtering problem given in the preceding section. In fact, the prediction problem may be formally reduced to a filtering problem by suitably changing the weighting matrix $R(s)$, as we now show.

Given $t \in [t_0 t_1 - T]$ and given an observed output function $Z(t) = \{(z(s), s) : s \in [t_0, t]\}$ it follows by the definition of $X(t + T|t)$ that $\xi \in X(t + T|t)$ if and only if there exist a vector $x(t_0)$, a function $u(\cdot)$ defined on $[t_0, t + T]$, and a function $v(\cdot)$ defined on $[t_0, t]$ such that

$$[x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t+T} u'(s)Q^{-1}(s)u(s)\,ds + \int_{t_0}^{t} v'(s)R^{-1}(s)v(s)\,ds \leq 1 \quad (20)$$

$$x(t + T) = \xi \quad (21)$$

and

$$v(s) = z(s) - C(s)x(s), \qquad t_0 \leq s \leq t. \quad (22)$$

Substitution of (22) into (20) yields that $\xi \in X(t + T|t)$ if and only if there exist a vector $x(t_0)$ and a function $u(\cdot)$ defined on $[t_0, t + T]$ with the property that

$$J_1[\xi, t + T; u] \leq 1$$

subject to the system equation (1) with boundary condition $x(t + T) = \xi$, where $J_1[\xi, t + T; u]$ is defined by

$$J_1[\xi, t + T; u] = [x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t+T} u'(s)Q^{-1}(s)u(s)\,ds + \int_{t_0}^{t} [z(s) - C(s)x(s)]'R^{-1}(s)[z(s) - C(s)x(s)]\,ds. \quad (23)$$

If $R_1^{-1}(s)$ is defined over $[t_0, t + T]$ by

$$R_1^{-1}(s) = \begin{cases} R^{-1}(s), & t_0 \leq s \leq t \\ 0, & t < s \leq t + T. \end{cases} \quad (24)$$

The expression (23) for $J_1[\xi, t + T; u]$ becomes

$$J_1[\xi, t + T; u] = [x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t+T} u'(s)Q^{-1}(s)u(s) + [z(s) - C(s)x(s)]'R_1^{-1}(s)[z(s) - C(s)x(s)]\,ds,$$

which is identical to that for $J[\xi, t + T; u, x(t_0)]$ given by (9), when $R^{-1}(s)$ is replaced by $R_1^{-1}(s)$. We remark that the matrix $R^{-1}(s)$ defined by (24) is nonnegative definite rather than positive definite, while the matrix $R^{-1}(s)$ in (3) has been assumed positive definite. It is easily seen, however, that Proposition 1 still holds for $R^{-1}(s)$ nonnegative definite, since the solution of the equivalent tracking problem (Problem 1′) requires only that the weighting matrix $R^{-1}(s)$ on output deviations in the cost functional (9) be nonnegative definite. Hence no difficulty is created in what follows by the nonnegative definiteness of the matrix $R_1^{-1}(s)$.

Thus the prediction problem, that of characterizing $X(t + T|t)$, has been reduced, by the appropriate definition of $R_1^{-1}(s)$, to a tracking problem. This tracking problem is, in turn, equivalent to the filtering problem of Section III over the interval $[t_0, t + T]$. The solution of the prediction problem is thus given by Proposition 1 with, of course, t replaced by $t + T$ and $R^{-1}(s)$ replaced by $R_1^{-1}(s)$. The solution may, however, be expressed more conveniently by substituting the expression (24) for $R_1^{-1}(s)$ into the equations corresponding to (14)–(19) under the indicated identifications, and splitting the interval $[t_0, t + T]$ into the two intervals $[t_0, t]$ and $[t, t + T]$. The following proposition then follows after some straightforward manipulations.

Proposition 2: The solution to Problem 1 in the prediction case, is the ellipsoid $X(t + T|t)$ given for all $t \in [t_0, t_1 - t]$ by

$$X(t + T|t) = \{\xi : [\xi - \hat{x}(t + T|t)]'\Sigma^{-1}(t + T|t) \times [\xi - \hat{x}(t + T|t)] \leq 1 - \delta^2(t)\}$$

where the $n \times n$ positive-definite matrix $\Sigma(t + T|t)$ is given by

$$\Sigma(t + T|t) = \Phi(t + T, t)\Sigma(t)\Phi'(t + T, t) + \int_t^{t+T} \Phi(t + T, s)B(s)Q(s)B'(s)\Phi'(t + T, s)\,ds \quad (25)$$

where $\Phi(t, s)$ is the transition matrix corresponding to the matrix $A(t)$ and the matrix $\Sigma(t)$ is given as the solution of the Riccati equation (18) with the boundary condition (19). The n vector $\hat{x}(t + T|t)$ is given by

$$\hat{x}(t + T|t) = \Phi(t + T, t)\hat{x}(t|t) \quad (26)$$

where the vector $\hat{x}(t|t)$ is the solution of the linear differential equation (14) with the boundary condition (15), and the positive scalar term $\delta^2(t)$ is given by (16).

Thus the solution to the prediction problem may be readily obtained from the solution to the filtering problem. The center of the ellipsoid $X(t + T|t)$ is obtained from the center of the ellipsoid $X(t|t)$ simply by multiplying the latter by the transition matrix $\Phi(t + T, t)$ as indicated in (26). The weighting matrix of the ellipsoid $X(t + T|t)$ is easily obtained from that of $X(t|t)$ simply by propagating the former through the system as indicated in (25), while the scaling term $1 - \delta^2(t)$ does not change since no measurements from t to $t + T$ are available. These, of course, are precisely the operations that are performed to obtain the solution to the stochastic prediction problem from that of the stochastic filtering problem. Thus, as in the filtering case, there is a one–one correspondence between prediction problems where the uncertain quantities satisfy an energy constraint and stochastic prediction problems of the type indicated in the preceding section.

V. Smoothing Problem With Energy Constraints

The solution to the smoothing problem follows similar lines to that of the filtering and prediction problems. As in the corresponding stochastic smoothing problem, however, the solution is slightly more complicated in that two dynamic systems, one operating in forward time and the other in reverse time, are required for a solution.

Given $T > 0, t \in [t_0 + T, t_1]$ and an observed output function $Z(t) = \{(z(s), s): s \in [t_0, t]\}$, it follows directly from the definition of $X(t - T|t)$ that $\xi \in X(t - T|t)$ if and only if there exist a vector $x(t_0)$, a function $u(\cdot)$ defined on $[t_0, t]$, and a function $v(\cdot)$ defined on $[t_0, t]$ such that

$$J[\xi, t - T; u] + J_2[\xi, t - T; u] \leq 1 \tag{27}$$

where $J[\xi, t - T; u]$ is given by (9), $J_2[\xi, t - T; u]$ is defined by

$$J_2[\xi, t - T; u] = \int_{t-T}^{t} u'(s)Q^{-1}(s)u(s) + [z(s) - C(s)x(s)]'R^{-1}(s)[z(s) - C(s)x(s)]\, ds \tag{28}$$

and $x(\cdot)$ is the trajectory of the system (1) under the input $u(\cdot)$ with boundary condition

$$x(t - T) = \xi. \tag{29}$$

Thus a necessary and sufficient condition for $\xi \in X(t - T|t)$ is that there exists $u(\cdot)$ defined on $[t_0, t]$ such that

$$\min_{u(\cdot)} \{J[\xi, t - T; u] + J_2[\xi, t - T; u\} \leq 1. \tag{30}$$

Since, for fixed ξ and $t - T$, $J[\xi, t - T; u]$ depends only on the portion of $u(\cdot)$ defined over $[t_0, t - T]$ and $J_2[\xi, t - T; u]$ depends only on the section of $u(\cdot)$ defined on $[t - T, t]$, we may write (30) as

$$\min_{u(\cdot)} J[\xi, t - T; u] + \min_{u(\cdot)} J_2[\xi, t - T; u] \leq 1. \tag{31}$$

The minimization of $J[\xi, t - T; u]$ is simply the tracking problem with time reversed that was obtained in Section III (with, of course, $t - T$ replacing t). The minimization of $J_2[\xi, t - T; u]$ is just a tracking problem in forward time from $t - T$ to t. Since t was chosen arbitrarily in $[t_0 + T, t_1]$, we may therefore use Proposition 1 and the standard results on tracking problems to write down a complete characterization of $X(t - T|t)$, and therefore give a complete solution to the smoothing problem with energy constraints.

Proposition 3: The solution to Problem 1 for the smoothing case is the ellipsoid $X(t - T|t)$ given for all $t \in [t_0 + T, t_1]$ by

$$X(t - T|t) = \{\xi : [\xi - \hat{x}(t - T)]'K(t - T)[\xi - \hat{x}(t - T)] + [\xi - \hat{x}_2(t - T)]'K_2(t - T)[\xi - \hat{x}_2(t - T)] \leq 1 - \delta^2(t - T) - \delta_2^2(t - T)\} \tag{32}$$

where the $n \times n$ positive-definite matrix $K(t - T)$ is given by the solution to the Riccati equation (12) with initial condition (13) and the $n \times n$ nonnegative-definite matrix $K_2(t)$ satisfies the Riccati equation

$$\dot{K}_2(s) = -A'(s)K_2(s) - K_2(s)A(s) + K_2(s)B(s)Q(s)B'(s)K_2(s) - C'(s)R^{-1}(s)C(s) \tag{33}$$

with terminal boundary condition

$$K_2(t) = 0. \tag{34}$$

The n vector $\hat{x}(t - T)$ is the solution to the linear differential equation (14) with initial condition (15), the n vector $\hat{x}_2(t - T)$ satisfies

$$\dot{\hat{x}}_2(s) = A(s)\hat{x}_2(s) - K_2^{-1}(s)C'(s)R^{-1}(s) \cdot [z(s) - C(s)\hat{x}_2(s)] \tag{35}$$

with the terminal boundary condition

$$\hat{x}_2(t) = 0 \tag{36}$$

and the positive scalar terms $\delta^2(t - T)$, $\delta_2^2(t - T)$ are given by

$$\delta^2(t - T) = \int_{t_0}^{t-T} [z(s) - C(s)\hat{x}(s)]'R^{-1}(s)[z(s) - C(s)\hat{x}(s)]\, ds$$

$$\delta_2^2(t - T) = \int_{t-T}^{t} [z(s) - C(s)\hat{x}_2(s)]'R^{-1}(s)[z(s) - C(s)\hat{x}_2(s)]\, ds. \tag{38}$$

Alternatively the ellipsoid (32) can be written as

$$X(t - T|t) = \{\xi : [\xi - \hat{x}(t - T|t)]'\Sigma^{-1}(t - T) \cdot [\xi - \hat{x}(t - T|t)] \leq 1 - \delta^2(t)\} \tag{39}$$

where the $n \times n$ positive-definite matrix $\Sigma(t - T)$ is given as the solution of the equation

$$\dot{\Sigma}(s) = [A(s) + B(s)Q(s)B'(s)K(s)]\Sigma(s) + \Sigma(s)[A(s) + B(s)Q(s)B'(s)K(s)]' - B(s)Q(s)B'(s) \tag{40}$$

with the boundary condition

$$\Sigma(t) = K^{-1}(t). \tag{41}$$

The n vector $\hat{x}(t - T|t)$ is the solution of the differential equation

$$\frac{d\hat{x}(s|t)}{ds} = \dot{\hat{x}}(s|t) = A\hat{x}(s|t) + B(s)Q(s)B'(s)K(s)[\hat{x}(s|t) - \hat{x}(s)] \tag{42}$$

with the boundary condition

$$\hat{x}(t|t) = \hat{x}(t) \tag{43}$$

and the scalar term $\delta^2(t)$ is given by

$$\delta^2(t) = \int_{t_0}^{t} [z(s) - C(s)\hat{x}(s)]'R^{-1}(s)[z(s) - C(s)\hat{x}(s)]\, ds \tag{44}$$

where $K(s)$ and $\hat{x}(s)$ are given for all s by (12) and (14) with boundary conditions (13) and (15).

Proof: Equations (32)–(38) follow directly from the reformulation of the problem as two tracking problems, one in forward time and one in reverse time, with common boundary condition $x(t-T)=\xi$. The equivalence of the two expressions (32) and (39) for $X(t-T|t)$ when $\Sigma(t)$, $\hat{x}(t-T|t)$, and $\delta^2(t)$ are given by (40)–(44) involves straightforward but tedious manipulations using the identifications

$$\Sigma^{-1}(t-T)=K(t-T)+K_2(t-T)$$

$$\Sigma^{-1}(t-T)\hat{x}(t-T|t)=K(t-T)\hat{x}(t-T)+K_2(t-T)\hat{x}_2(t-T)$$

$$\begin{aligned}\delta^2(t)&+\hat{x}'(t-T|t)\Sigma^{-1}(t-T)\hat{x}(t-T|t)\\&=\delta^2(t-T)+\hat{x}'(t-T)K(t-T)\hat{x}(t-T)\\&\quad+\delta_2^2(t-T)+\hat{x}_2'(t-T)K_2(t-T)\hat{x}_2(t-T),\end{aligned}$$

which as is readily seen by expanding out (32) and (39), are sufficient for equality of the two expressions for $X(t-T|t)$. The first two of these identifications are also used in [10] where the corresponding stochastic smoothing result is proved.

We remark that, as long as the system (1) is completely observable from $y(t)=C(t)x(t)$, the matrix $K_2(s)$ is positive definite for all $s<t$. The presence of $K_2^{-1}(s)$ in (35) thus poses a potential difficulty only at $s=t$ when, by (34), $K_2(t)=0$. As shown in [10], this difficulty may be removed by using (33) and (35) to write

$$\frac{d}{ds}[K_2(s)\hat{x}_2(s)]=-[A(s)-B(s)Q(s)B'(s)K_2(s)]'\cdot[K_2(s)\hat{x}_2(s)]-C'(s)R^{-1}(s)z(s)$$

with, from (34) and (36), $K_2(t)\hat{x}_2(t)=0$, from which $\hat{x}_2(s)$ can be determined for all $s<t$.

Note that the solution to the smoothing problem may be generated by a combination of two filters, one operating forward in time (which corresponds to the tracking problem with time reversed), and the other operating backwards in time (which corresponds to the tracking problem in forward time). A similar interpretation of the solution to the stochastic minimum-variance smoothing problem is well known [10]. In fact, (42) and (40) for the center $\hat{x}(t-T|t)$ and weighting matrix $\Sigma(t-T)$ of the ellipsoid $X(t-T|t)$ are precisely those specifying the best estimate and error covariance for the stochastic minimum-variance smoothing problem of [11] with the identifications for the noise covariances described earlier in Section III.

In Sections III, IV, and V a complete solution has been given to Problem 1. Entirely analogous derivations and results can be given for the discrete system counterpart or Problem 1. Due to space limitations, we will only state the result for the filtering case for both energy constraints and instantaneous constraints (to be discussed in the next section) in Appendix I.

VI. Formulation of the Continuous-Time Problem With Instantaneous Constraints

While the preceding sections show it to be of theoretical interest, the model for the uncertainty described by the energy constraint (3) is of limited use as far as practical applications are concerned. From a practical viewpoint, a far more natural model for uncertainty is that in which the uncertain quantities are individually constrained at each point in time. In this section we formulate such a problem, which is then examined in Section VII using the results of the preceding sections. In particular, we bound the instantaneous constraints by a single combined energy constraint and apply the results of Section III. The resulting estimator is shown to be simpler but otherwise comparable to that proposed by Schweppe [3], with the additional advantage that it permits a steady-state solution.

Problem 2: Consider Problem 1 in which the single energy constraint (3) on the uncertain quantities is replaced by the three individual instantaneous constraints

$$\begin{aligned}&[x(t_0)-x_0]'\Psi^{-1}[x(t_0)-x_0]\le 1\\&u'(s)Q^{-1}(s)u(s)\le 1,\qquad \forall s\in[t_0,t_1]\\&v'(s)R^{-1}(s)v(s)\le 1,\qquad \forall s\in[t_0,t_1]\end{aligned}\tag{45b}$$

where Ψ, $Q(s)$, and $R(s)$ are symmetric positive-definite matrices and x_0 is a given n vector. As in Problem 1, find the set $X(t+T|t)$ of system states at time t that are consistent with both the measured output function $Z(t)=\{(z(s),s):s\in[t_0,t]\}$ up to time t and the constraints (45).

VII. Filtering Problem With Instantaneous Constraints

Contrary to the case of energy constraints, it is very difficult to obtain the exact solution of Problem 2. The energy constraint (3) is an ellipsoid in the space $R^n\times L_2^r[t_0,t_1]\times L_2^m[t_0,t_1]$ where $L_2^p[t_0,t_1]$ is the space of Lebesgue-square-integrable p-vector-valued functions on $[t_0,t_1]$. Since any measured output function $Z(t)$ defines a linear variety in this space and since the intersection of an ellipsoid with a linear variety is also an ellipsoid, the set of possible system states $X(t+T|t)$, obtained by a linear transformation on this ellipsoid intersection, is also an ellipsoid, as found in Sections III–V. The individual instantaneous constraints (45) do not, on the other hand, define an ellipsoid and thus the intersection of the linear variety defined by any observed output with the set in $R^n\times L_2^r[t_0,t_1]\times L_2^m[t_0,t_1]$ satisfying (45) is not, in general, an ellipsoid. Consequently, the set of system states at time t consistent with the observed output function is not, in general, an ellipsoid: it is a convex set that, in contrast to the ellipsoidal case, cannot be characterized generally by a finite set of numbers.

Alternatively, it is possible to cast Problem 2 as an optimization problem, as was done with Problem 1. This optimization problem, however, will involve control constraints as well as state constraints and does not admit a closed form solution, with the result that practical on-line algorithms cannot be devised.

Thus, one is forced to seek approximate solutions to Problem 2. The approach taken by Schweppe [2], [3] is to compute a bounding ellipsoid to the set $X(t+T|t)$. Since an ellipsoid in R^n is completely characterized by an n vector (its center) and an $n \times n$ weighting matrix, the storage problem is reduced to more manageable proportions. Schweppe considered the filtering and prediction problems for a discrete system in [2], and gave a recursive algorithm for the center and weighting matrix of a bounding ellipsoid to the set of possible states. The approach used was to bound recursively the set of possible states at each time instant by an ellipsoid. This algorithm was later extended to the continuous system case [3] using a discrete-to-continuous limiting argument. The following lemma gives the filtering algorithm that is presented in [3].

Lemma 1: A bounding ellipsoid to the set of system states $X(t|t)$ of Problem 2, is given for all $t \in [t_0, t_1]$ by

$$X^*(t|t) = \{\xi : [\xi - \hat{x}(t)]'\Sigma^{-1}(t)[\xi - \hat{x}(t)] \leq 1\} \quad (46)$$

where the $n \times n$ positive-definite matrix $\Sigma(t)$ is the solution of the equation

$$\dot{\Sigma}(s) = A(s)\Sigma(s) + \Sigma(s)A'(s) - \rho(s)\Sigma(s)C'(s)R^{-1}(s)C(s)\Sigma(s) + \beta^{-1}(s)B(s)Q(s)B'(s) + [\beta(s) + \rho(s) - \delta^2(s)]\Sigma(s) \quad (47)$$

with the boundary condition

$$\Sigma(t_0) = \Psi. \quad (48)$$

The n vector $\hat{x}(t)$ is the solution to the differential equation

$$\dot{\hat{x}}(s) = A(s)\hat{x}(s) + \rho(s)\Sigma(s)C'(s)R^{-1}(s)[z(s) - C(s)\hat{x}(s)] \quad (49)$$

with the boundary condition

$$\hat{x}(t_0) = x_0 \quad (50)$$

and the positive real number $\delta^2(s)$ is given for all s by

$$\delta^2(s) = \rho(s)[z(s) - C(s)\hat{x}(s)]'R^{-1}(s)[z(s) - C(s)\hat{x}(s)] \quad (51)$$

and $\beta(s)$, $\rho(s)$ are any real-valued time functions with $0 < \beta(s)$, $0 < \rho(s)$ for all $s \in [t_0, t_1]$.

The structure of the estimator of the above lemma is shown in Fig. 1, and it can be seen to have the basic structure of the stochastic Kalman filter. It should be noted, however, that the gain matrix $\{\rho(t)\Sigma(t)C'(t)R^{-1}(t)\}$ depends on the observations made at a particular run and must therefore be calculated by integrating the nonlinear matrix differential (47) on-line. Furthermore, even for a time-invariant system, this estimator does not possess a steady-state structure due to the fact that the solution of (47) does not converge to a steady state as time increases.

These disadvantages are avoided in the estimator we now derive. The approach is again to bound the set of possible states consistent with the observations by an ellipsoid. In contrast to [2], we do this indirectly by bounding the instantaneous constraints (45) with an energy constraint of the form (3) and then using the results of Sections III and V to produce an ellipsoidal bound on $X(t|t)$. For simplicity, we concentrate our attention on the filtering problem; entirely analogous results can be derived for the prediction and smoothing problems.

An energy bound for the instantaneous constraints (45) is given in the following lemma.

Lemma 2: The set $U_t \subset R^n \times L_2^r[t_0, t] \times L_2^m[t_0, t]$ where

$$U_t = \{x(t_0), u(s), v(s), t_0 \leq s \leq t : [x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] \leq 1, u'(s)Q^{-1}(s)u(s) \leq 1, v'(s)R^{-1}(s)v(s) \leq 1\} \quad (52)$$

is contained in the set

$$U_t^* = \{x(t_0), u(s), v(s), t_0 \leq s \leq t : a_1[x(t_0) - x_0]'\Psi^{-1}[x(t_0) - x_0] + \int_{t_0}^{t} [a_2(s)u'(s)Q^{-1}(s)u(s) + a_3(s)v'(s)R^{-1}(s)v(s)]\, ds \leq 1\} \quad (53)$$

where a_1 is any positive constant and $a_2(\cdot)$, $a_3(\cdot)$ are any positive integrable real-valued functions (i.e., $a_1 > 0$, $a_2(s) > 0$, $a_3(s) > 0$, $t_0 \leq s \leq t$) such that

$$a_1 + \int_{t_0}^{t} [a_2(s) + a_3(s)]\, ds = 1. \quad (54)$$

Proof: Multiply (45a–c), respectively, by a_1, $a_2(s)$, and $a_3(s)$, integrate the last two from t_0 to t, add, and use (54).

Having bounded the instantaneous constraints (52) by the energy constraint (53), we are now in a position to apply the results of Section III to give a bounding ellipsoid to the set $X(t|t)$. The equations that result by application of Proposition 1 become simpler if we write $a_1, a_2(\cdot)$, and $a_3(\cdot)$ in the following form.

$$a_1 = \exp\left(-\int_{t_0}^{t} [\beta(\sigma) + \rho(\sigma)]\, d\sigma\right) \quad (55a)$$

$$a_2(s) = \beta(s)\exp\left(-\int_{s}^{t} [\beta(\sigma) + \rho(\sigma)]\, d\sigma\right) \quad (55b)$$

$$a_3(s) = \rho(s)\exp\left(-\int_{s}^{t} [\beta(\sigma) + \rho(\sigma)]\, d\sigma\right) \quad (55c)$$

where $\beta(\cdot)$ and $\rho(\cdot)$ are positive integrable real-valued functions (i.e., $0 < \beta(s)$, $0 < \rho(s)$, $t_0 \leq s \leq t$). It is easy to see that under the identifications (55) the condition (54) is satisfied.

By combining now Lemma 2 under the identifications (55) with Proposition 1 we have after straightforward manipulation the following solution to Problem 2.

Proposition 4: A bounding ellipsoid to the set of system states $X(t|t)$ of Problem 2, is given for all $t \in [t_0, t_1]$ by

$$X^*(t|t) = \{\xi : [\xi - \hat{x}(t)]'\Sigma^{-1}(t)[\xi - \hat{x}(t)] \leq 1 - \delta^2(t)\} \quad (56)$$

where the $n \times n$ positive-definite matrix $\Sigma(t)$ is the solution of the equation

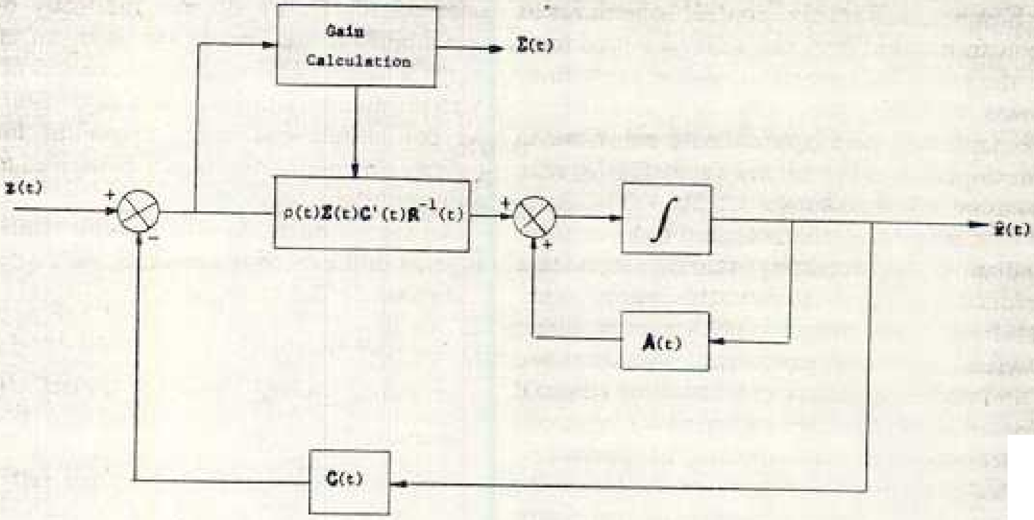
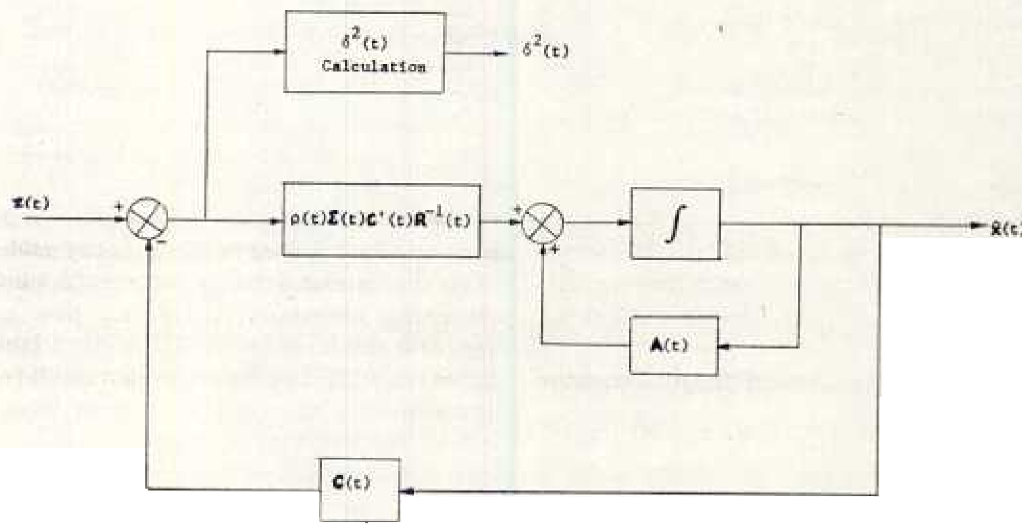


Fig. 1.



$$\dot{\Sigma}(s) = A(s)\Sigma(s) + \Sigma(s)A'(s) - \rho(s)\Sigma(s)C'(s)R^{-1}(s) \cdot C(s)\Sigma(s) + \beta^{-1}(s)B(s)Q(s)B'(s) + [\beta(s) + \rho(s)]\Sigma(s) \quad (57)$$

with the boundary condition

$$\Sigma(t_0) = \Psi. \quad (58)$$

The n vector $\hat{x}(t)$ is the solution to the linear differential equation

$$\dot{\hat{x}}(s) = A(s)\hat{x}(s) + \rho(s)\Sigma(s)C'(s)R^{-1}(s)[z(s) - C(s)\hat{x}(s)] \quad (59)$$

with the boundary condition

$$\hat{x}(t_0) = x_0 \quad (60)$$

and the positive real number $\delta^2(t)$ is the solution to the differential equation

$$\dot{\delta}^2(s) = -[\beta(s) + \rho(s)]\delta^2(s) + \rho(s)[z(s) - C(s)\hat{x}(s)]'R^{-1}(s)[z(s) - C(s)\hat{x}(s)] \quad (61)$$

with the boundary condition

$$\delta^2(t_0) = 0 \quad (62)$$

and $\beta(s)$, $\rho(s)$ are any positive real-valued time functions on $[t_0, t_1]$.

The structural form of the estimator of Proposition 4 is shown in Fig. 2. It can be seen that it has a similar structure to the estimator of Fig. 1. However, it has the advantage that the gain matrix $\{\rho(t)\Sigma(t)C'(t)R^{-1}(t)\}$ is precomputable once the time functions $\beta(\cdot)$, $\rho(\cdot)$ are selected. Furthermore, as will be discussed in the next section, for a time-invariant system the estimator of Proposition 4 can be implemented as a time-invariant system if the final time t_1 approaches infinity. In practical applications this last advantage can be of extreme importance.

A vital question concerns the comparison of the quality of approximation to the set of possible states provided by the two estimators. It turns out that the approximation is comparable in the following sense. Let $\beta'(\cdot)$, $\rho'(\cdot)$ be the time functions used in the estimator of Lemma 1, and let $\beta(\cdot)$, $\rho(\cdot)$ be the time functions in the estimator of Proposition 4. Then, as was first shown by Schlaepfer [7], if we select for all $s \in [t_0, t_1]$

$$\beta'(s) = [1 - \delta^2(s)]^{-1}\beta(s)$$
$$\rho'(s) = [1 - \delta^2(s)]^{-1}\rho(s)$$

where $\delta^2(s)$ is the observation dependent term of (61) in Proposition 4, the estimate ellipsoids provided by the two estimators are identical for all time.

In the estimator of Proposition 4 it is important that the time functions $\beta(\cdot)$ and $\rho(\cdot)$ be selected judiciously. An algorithm has been derived by the authors for selection of these time functions so that the trace of the weighting matrix $\Sigma(t_1)$ is minimum at the final time t_1. Due to space limitations the presentation of this algorithm will be deferred to a future publication. A question of importance also is how closely the bounding ellipsoid $X^*(t|t)$ of Proposition 4 approximates the exact set of possible states $X(t|t)$. This is a very difficult question to answer in general, particularly for the continuous time case, and more research is required in this area. Some partial answers are available for the discrete system case, and they will be reported in a future publication.

It should be noted that the result of Proposition 4 can be easily modified for the case where the ellipsoids (45b), (45c) of the input and observation noise are not centered at the origin, as well as for the case where a control input is present. As in the stochastic case, the effect of any deterministic inputs can be superimposed by the linearity of the system.

We remark that similar estimators to the one of Proposition 4 can be derived for the smoothing and prediction problems, by making use of Lemma 2 and the results of Sections IV and V. The resulting estimators are likewise linear and are similar in structure to the corresponding stochastic estimators. The resulting predictor has similar advantages over the predictor of [2], [3] to the ones mentioned in connection with the filtering problem. The smoothing case of Problem 2 has not been treated previously in the literature.

It should also be noted that a problem analogous to Problem 2 can be stated for a discrete-time system, and results that are similar to those reported in this section can be derived. We will not repeat the derivations but instead, for purposes of easy reference, we will state in Appendix I the result for the filtering case and outline its derivation.

Finally, we remark that a problem falling in some sense between Problems 1 and 2 is that in which the initial state, the input disturbance function, and the observation disturbance function are individually bounded by separate energy constraints of the form

$$[\boldsymbol{x}(t_0) - \boldsymbol{x}_0]'\boldsymbol{\Psi}^{-1}[\boldsymbol{x}(t_0) - \boldsymbol{x}_0] \le 1$$
$$\int_{t_0}^{t_1} \boldsymbol{u}'(t)\boldsymbol{Q}^{-1}(t)\boldsymbol{u}(t)\,dt \le 1$$
$$\int_{t_0}^{t_1} \boldsymbol{v}'(t)\boldsymbol{R}^{-1}(t)\boldsymbol{v}(t)\,dt \le 1$$

where the vector $\boldsymbol{x}_0$ and the matrices $\boldsymbol{\Psi}$, $\boldsymbol{Q}^{-1}(t)$, and $\boldsymbol{R}^{-1}(t)$ are as defined in Problem 1. This problem can be solved using an approach that is analogous to that used in solving Problem 2. The three separate energy constraints are first bounded by a single energy constraint of the form (53), where the multipliers a_2 and a_3 are in this case constants rather than functions. The solution to Problem 1 can then be used to give at each time an upper bound on the set of possible system states that are consistent with the observed output function and the above individual energy constraints. The details are straightforward and are not included here.

VIII. Constant Systems and Infinite Time Intervals

In this section we consider the special case of Problem 2 where the system and the disturbance ellipsoids are constant, i.e., $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ are constant matrices. If we select the real-valued functions $\beta(\cdot)$ and $\rho(\cdot)$ to be also constant (i.e., $\beta(t) \equiv \beta > 0$, $\rho(t) \equiv \rho > 0$), (57) for the matrix $\Sigma(t)$ in Proposition 4 becomes

$$\dot{\Sigma}(s) = \boldsymbol{A}\Sigma(s) + \Sigma(s)\boldsymbol{A}' - \rho\Sigma(s)\boldsymbol{C}'\boldsymbol{R}^{-1}\boldsymbol{C}\Sigma(s) + \beta^{-1}\boldsymbol{BQB}' + (\beta + \rho)\Sigma(s) \quad (63)$$

with initial condition $\Sigma(0) = \boldsymbol{\Psi}$. This equation can be put into the usual Riccati equation form

$$\dot{\Sigma}(s) = \boldsymbol{A}^*\Sigma(s) + \Sigma(s)\boldsymbol{A}^{*\prime} - \Sigma(s)\boldsymbol{C}'\boldsymbol{R}^{*-1}\boldsymbol{C}\Sigma(s) + \boldsymbol{BQ}^*\boldsymbol{B}' \quad (64)$$

by defining the matrices $\boldsymbol{A}^*$, $\boldsymbol{Q}^*$, $\boldsymbol{R}^*$ as

$$\boldsymbol{A}^* = \boldsymbol{A} + \tfrac{1}{2}(\beta + \rho)\boldsymbol{I},\ \boldsymbol{R}^* = \rho^{-1}\boldsymbol{R},\ \boldsymbol{Q}^* = \beta^{-1}\boldsymbol{Q} \quad (65)$$

where $\boldsymbol{I}$ is the identity matrix. It is well known that the solution of (64) converges to a positive-definite steady-state solution Σ_∞ as $s \to \infty$ if the pair $(\boldsymbol{A}^*, \boldsymbol{C})$ is completely observable and the pair $(\boldsymbol{A}^*, \boldsymbol{B})$ is completely controllable. The pair $(\boldsymbol{A}^*, \boldsymbol{B})$ is completely controllable if and only if the pair $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable, i.e., the constant system (1) is completely controllable. This can be seen by the fact that if $\boldsymbol{b}$ is a column vector of the matrix $\boldsymbol{B}$, the subspace spanned by the vectors $(\boldsymbol{b}, \boldsymbol{Ab}, \boldsymbol{A}^2\boldsymbol{b}, \cdots, \boldsymbol{A}^{n-1}\boldsymbol{b})$ is the same as the subspace spanned by the vectors $[\boldsymbol{b}, (\boldsymbol{A} + \boldsymbol{I})\boldsymbol{b}, (\boldsymbol{A} + \boldsymbol{I})^2\boldsymbol{b}, \cdots, (\boldsymbol{A} + \boldsymbol{I})^{n-1}\boldsymbol{b}]$ which, in view of (65), is the same as the subspace spanned by the vectors $(\boldsymbol{b}, \boldsymbol{A}^*\boldsymbol{b},$

$A^{*2}b, \cdots, A^{*n-1}b$). Similarly, the pair (A^*, C) is completely observable if and only if the pair (A, C) is completely observable. Thus, for a completely controllable and observable time-invariant system, the gain $\Sigma(t)C'R^{*-1}$ in the estimator of Proposition 4 after an initial transient will converge to the steady-state constant gain $\Sigma_\infty C'R^{*-1}$. For practical reasons, one would like to implement the estimator as a time-invariant system using the steady-state gain for the whole time interval, i.e., starting at the initial time $t_0 = 0$. This is possible since, as we will prove below, the approximation that results if we neglect the initial transient vanishes as time goes to infinity.

Using the identifications (65), the estimator of Proposition 4 for a time-invariant system gives the estimate ellipsoid

$$X^*(t|t) = \{\xi : [\xi - \hat{x}(t)]'\Sigma^{-1}(t)[\xi - \hat{x}(t)] \le 1 - \delta^2(t)\} \tag{66}$$

$$\dot{\hat{x}}(s) = A\hat{x}(s) + \Sigma(s)C'R^{*-1}[z(s) - C\hat{x}(s)] \tag{67}$$

$$\dot{\Sigma}(s) = A^*\Sigma(s) + \Sigma(s)A^{*\prime} - \Sigma(s)C'R^{*-1}C\Sigma(s) + BQ^*B' \tag{68}$$

$$\dot{\delta}^2(s) = -(\beta + \rho)\delta^2(s) + [z(s) - C\hat{x}(s)]'R^{*-1}[z(s) - C\hat{x}(s)] \tag{69}$$

with

$$\hat{x}(0) = x_0, \Sigma(0) = \Psi, \delta^2(0) = 0. \tag{70}$$

If Σ_∞ is the steady-state solution of the Riccati equation (68) and we implement the estimator as a time-invariant system using the steady-state gain $\Sigma_\infty C'R^{*-1}$, the resulting estimate ellipsoid will be given by

$$Y(t|t) = \{\xi : [\xi - \hat{y}(t)]'\Sigma_\infty^{-1}[\xi - \hat{y}(t)] \le 1 - \bar{\delta}^2(t)\} \tag{71}$$

where

$$\dot{\hat{y}}(s) = A\hat{y}(s) + \Sigma_\infty C'R^{*-1}[z(s) - C\hat{y}(s)] \tag{72}$$

$$\dot{\bar{\delta}}^2(s) = -(\beta + \rho)\bar{\delta}^2(s) + [z(s) - C\hat{y}(s)]'R^{*-1}[z(s) - C\hat{y}(s)] \tag{73}$$

with

$$\hat{y}(0) = x_0, \bar{\delta}^2(0) = 0. \tag{74}$$

Using the fact that $\Sigma(t) \to \Sigma_\infty$ as $t \to \infty$, it will now be proved that $\hat{x}(t) \to \hat{y}(t)$ and $\delta^2(t) \to \bar{\delta}^2(t)$ as $t \to \infty$, i.e., that the estimate $X^*(t|t)$ of (66) converges to the set $Y(t|t)$ of (71) as $t \to \infty$. To this end, let $\Sigma(t|t) = \Sigma_\infty + H(t)$ where $H(t) \to 0$ as $t \to \infty$. Then from (67) and (72) we have

$$[\dot{\hat{x}}(t) - \dot{\hat{y}}(t)] = (A - \Sigma_\infty C'R^{*-1}C)[\hat{x}(t) - \hat{y}(t)] + H(t)C'R^{*-1}[z(t) - C\hat{x}(t)]. \tag{75}$$

Now note that the matrix $(A - \Sigma_\infty C'R^{*-1}C)$ is stable (has eigenvalues with negative real parts), since, by (65)

$$[A - \Sigma_\infty C'R^{*-1}C] = [A^* - \Sigma_\infty C'R^{*-1}C] - \tfrac{1}{2}(\beta + \rho)I$$

with $\beta, \rho > 0$, and the matrix $(A^* - \Sigma_\infty C'R^{-1})$ is stable by a well-known property of the Riccati equation. Furthermore, the driving term $H(t)C'R^{*-1}[z(t) - C\hat{x}(t)]$ goes to zero as $t \to \infty$ since $H(t) \to 0$ as $t \to \infty$ and $[z(t) - C\hat{x}(t)]$ is bounded. Therefore, the solution of (75) goes asymptotically to zero as $t \to \infty$ and therefore $\hat{x}(t) \to \hat{y}(t)$ as $t \to \infty$.

Also, from (69) and (73)

$$[\dot{\delta}^2(t) - \dot{\bar{\delta}}^2(t)] = -(\beta + \rho)[\delta^2(t) - \bar{\delta}^2(t)] + \epsilon(t) \tag{76}$$

where

$$\epsilon(t) = [z(t) - C\hat{x}(t)]'R^{*-1}[z(t) - C\hat{x}(t)] - [z(t) - C\hat{y}(t)]'R^{*-1}[z(t) - C\hat{y}(t)].$$

Since $\hat{x}(t) \to \hat{y}(t)$ as $t \to \infty$, we have $\epsilon(t) \to 0$ as $t \to \infty$, and since $(\beta + \rho) > 0$ the solution of (76) goes to zero as $t \to \infty$. Hence $\delta^2(t) \to \bar{\delta}^2(t)$ as $t \to \infty$.

Thus, in applications where the system is constant and the final time approaches infinity, one can use the steady-state time-invariant estimator and be assured that the error that results from neglecting the initial transient of the solution of the Riccati equation vanishes as time increases. An entirely analogous argument can be given for the discrete case, and similarly one obtains a time-invariant estimator for a constant system.

Finally, we note that in the infinite time case the parameter selection problem is greatly alleviated, as we now have to select only two constants β, ρ with $0 < \beta$, $0 < \rho$ in the continuous case, and $0 < \beta < 1$, $0 < \rho < 1$ in the discrete case. If, for example, we are interested in selecting β, ρ so that the trace of the matrix Σ_∞ is minimized, we can do this by a simple search in the discrete case. For a continuous system a steepest-descent method can be used, where the partial derivatives $(\partial/\partial\beta)\,\mathrm{tr}\,\Sigma_\infty$ and $(\partial/\partial\rho)\,\mathrm{tr}\,\Sigma_\infty$ can be calculated by differentiation of the algebraic Riccati equation.

IX. Conclusions

Attention has been given to the problem of estimating the state of a linear dynamic system from noisy measurements of the output, when the initial condition of the system and the input and observation noise vectors are unknown except for the fact that they lie in given sets. The cases of both energy constraints and individual instantaneous constraints for the uncertain quantities have been considered. In the former case, the set of possible states compatible with the observations received was shown to be an ellipsoid and equations for its center and weighting matrix were given. In the latter case equations describing a bounding ellipsoid to the set of possible states were derived. All three problems of filtering, prediction, and smoothing have been examined by relating them to standard tracking problems of optimal control theory. The estimators derived are similar in structure and comparable in simplicity to the corresponding stochastic linear minimum-variance estimators, and it was shown that they provide distinct advantages over existing schemes. The results reported in this paper can also be used in the solution of certain minimax control problems as has been shown elsewhere [12].

A problem that requires further consideration is the question of the quality of approximation resulting from the ellipsoidal bounding operation in the instantaneous constraint case. Some results pertinent to this question, an algorithm for the judicious selection of the parameter time functions in Proposition 4, and the special case where the observation noise is degenerate, will be considered in a future publication.

Appendix I
Discrete Time Estimators for the Filtering Problem

System

$$x_{k+1} = A_k x_k + B_k u_k, \qquad k = 0, \cdots, N-1$$

Compare (1).

$$z_k = C_k x_k + v_k, \qquad k = 0, \cdots$$

Compare (2).

Energy Constraint

$$[x_0 - \mu_0]'\Psi^{-1}[x_0 - \mu_0] + \sum_{i=0}^{N-1} (u_i'Q_i^{-1}u_i + v_{i+1}'R_{i+1}^{-1}v_{i+1}) \le 1.$$

Compare (3).

Filtering Algorithm for Energy Constraint

Compare Proposition 1

$$X_{k|k} = \{\xi : (\xi - \hat{x}_k)'\Sigma^{-1}(k|k)(\xi - \hat{x}_k) \le 1 - \delta^2(k)\}$$

$$\hat{x}_{k+1} = A_k\hat{x}_k + \Sigma(k+1|k+1)C_{k+1}'R_{k+1}^{-1} \cdot (z_{k+1} - C_{k+1}A_k\hat{x}_k)$$

$$\hat{x}_0 = \mu_0$$

$$\Sigma(k+1|k+1) = [\Sigma^{-1}(k+1|k) + C_{k+1}'R_{k+1}^{-1}C_{k+1}]^{-1}$$

$$\Sigma(k+1|k) = A_k\Sigma(k|k)A_k' + B_kQ_kB_k'$$

$$\Sigma(0|0) = \Psi$$

$$\delta^2(k) = \sum_{i=1}^{k} (z_i - C_iA_{i-1}\hat{x}_{i-1})'[C_i\Sigma \cdot (i|i-1)C_i' + R_i]^{-1}(z_i - C_iA_{i-1}\hat{x}_{i-1}).$$

Instantaneous Constraints

$$(x_0 - \mu_0)'\Psi^{-1}(x_0 - \mu_0) \le 1.$$

Compare (45a).

$$u_i'Q_i^{-1}u_i \le 1, \qquad i = 0, \cdots, N-1.$$

Compare (45b).

$$v_i'R_i^{-1}v_i \le 1, \qquad i = 1, \cdots, N.$$

Compare (45c).

Energy Constraint Bounding Instantaneous Constraint

Compare Lemma 2, (55).

$$a_0(x_0 - \mu)'\Psi^{-1}(x_0 - \mu_0) + \sum_{i=1}^{k} (a_{2i-1}u_{i-1}'Q_{i-1}^{-1}u_{i-1} + a_{2i}v_i'R_i^{-1}v_i) \le 1$$

$$a_0 = (1-\beta_0)(1-\rho_1)(1-\beta_1)\cdots(1-\beta_{k-1})(1-\rho_k)$$

$$a_1 = \beta_0(1-\rho_1)(1-\beta_1)\cdots(1-\beta_{k-1})(1-\rho_k)$$

$$a_2 = \rho_1(1-\beta_1)\cdots(1-\beta_{k-1})(1-\rho_k)$$

$$a_{2k-1} = \beta_{k-1}(1-\rho_k)$$

$$a_{2k} = \rho_k$$

$$0 < \beta_{i-1} < 1, \qquad 0 < \rho_i < 1, \qquad i = 1, \cdots, k.$$

Filtering Algorithm for Instantaneous Constraint

Compare Proposition 4.

$$X^*(k|k) = \{\xi : (\xi - \hat{x}_k)'\Sigma^{-1}(k|k)(\xi - \hat{x}_k) \le 1 - \delta^2(k)\}$$

$$\hat{x}_{k+1} = A_k\hat{x}_k + \rho_{k+1}\Sigma(k+1|k+1) \cdot C_{k+1}'R_{k+1}^{-1}(z_{k+1} - C_{k+1}A_k\hat{x}_k)$$

$$\hat{x}_0 = \mu_0$$

$$\Sigma(k+1|k+1) = [(1-\rho_{k+1})\Sigma^{-1}(k+1|k) + \rho_{k+1}C_{k+1}'R_{k+1}^{-1}C_{k+1}]^{-1}$$

$$\Sigma(k+1|k) = (1-\beta_k)^{-1}A_k\Sigma(k|k)A_k' + \beta_k^{-1}B_kQ_kB_k'$$

$$\Sigma(0|0) = \Psi$$

$$\delta^2(k) = (1-\beta_{k-1})(1-\rho_k)\delta^2(k-1) + (z_k - C_kA_{k-1}\hat{x}_{k-1})'[(1-\rho_k)^{-1}C_k\Sigma(k|k-1)C_k' + \rho_k^{-1}R_k]^{-1}(z_k - C_kA_{k-1}\hat{x}_{k-1})$$

$$\delta^2(0) = 0$$

$$0 < \beta_{k-1} < 1, \qquad 0 < \rho_k < 1, \qquad k = 1, \cdots, N.$$

References

[1] H. S. Witsenhausen, "Minimax control of uncertain systems," Elec. Syst. Lab., M.I.T., Rep. ESL-R-269, Cambridge, May 1966.
[2] F. C. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Trans. Automat. Contr.*, vol. AC-13, Feb. 1968, pp. 22–28.
[3] ——, "Stochastic dynamic systems," M.I.T. Notes, Cambridge, 1969.
[4] H. S. Witsenhausen, "Sets of possible states of linear systems given perturbed observations," *IEEE Trans. Automat. Contr.* (Short Paper), vol. AC-13, Oct. 1968, pp. 556–558.
[5] F. C. Schweppe and H. K. Knudsen, "Theory of amorphous cloud trajectory prediction," *IEEE Trans. Inform. Theory*, vol. IT-14, May 1968, pp. 415–427.

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-16, NO. 2, APRIL 1971

[6] E. Hnyilicza, "A set-theoretic approach to state estimation," M.S. thesis, Dep. Elec. Eng., M.I.T., Cambridge, June 1969.
[7] F. M. Schlaepfer, "Set theoretic estimation of distributed parameter systems," Elec. Syst. Lab., M.I.T. Rep. ESL-R-413, Cambridge, Jan. 1970.
[8] M. Athans and P. Falb, *Optimal Control.* New York: McGraw-Hill, 1966, p. 793.
[9] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Trans. ASME, J. Basic Eng.*, ser. D, vol. 82, 1960, pp. 35–44.
[10] D. C. Fraser and J. E. Potter, "Optimum linear smoother as a combination of two optimum linear filters," *IEEE Trans. Automat. Contr.*, (Short Paper), vol. AC-14, Aug. 1969, pp. 387–390.
[11] H. E. Rauch, F. Tung, and C. T. Striebel, "Maximum likelihood estimates of linear dynamic systems," *AIAA J.*, vol. 3, no. 8, 1965, pp. 1445–1450.
[12] D. P. Bertsekas and I. B. Rhodes, "On the minimax reachability of target sets and target tubes," *Automatica*, Mar. 1971.

Dimitri P. Bertsekas was born in Athens, Greece, in 1942. He received the Diploma in mechanical and electrical engineering from the National Technical University of Athens, Greece, in 1965, and the M.S.E.E. degree from George Washington University, Washington, D.C. in 1969. He is currently working towards the Ph.D. degree in electrical engineering at the Massachusetts Institute of Technology, Cambridge, Mass.

He has done research at the National Technical University of Athens from 1966 to 1967, and at the U.S. Army Research Laboratories, Ft. Belvoir, Va., from 1967 to 1969. His research interests lie in the areas of optimization and the estimation and control of uncertain systems.

Ian B. Rhodes (S'64–M'67) was born in Melbourne, Australia, on May 29, 1941. He received the B.E. and M.Eng.Sc. degrees from the University of Melbourne, Australia, in 1963 and 1965, respectively, and the Ph.D. degree in 1968 from Stanford University, Stanford, Calif., all in electrical engineering.

In the summer of 1967 he was a Research Engineer at Stanford Research Institute, Menlo Park, Calif. From 1968 to 1970 he was Assistant Professor of Electrical Engineering at the Massachusetts Institute of Technology, Cambridge, Mass. He is currently an Associate Professor of Engineering and Applied Science at Washington University, St. Louis, Mo. His current research interests are in the areas of optimization, estimation, and control.

Dr. Rhodes is a member of Sigma Xi and the Society for Industrial and Applied Mathematics.

Two-Level Form of the Kalman Filter

A. R. M. NOTON, SENIOR MEMBER, IEEE

Abstract—**Sequential estimation of the states of several high-order interconnected systems may be prohibitive on computer time and storage if the problem is formulated as for a single system. Therefore, multilevel systems theory has been applied to derive a coordination algorithm, with one-step convergence, for a number of subsystem Kalman estimators. The procedure may be computationally attractive for sparsely coupled subsystems with few stochastic inputs.**

I. Introduction

QUITE apart from its theoretical importance, the Kalman filter [1], [2] is now regarded as a highly practicable technique for state and parameter estimation [3]. For the linear system the theory is on a rigorous basis but an extension of the filter, by means of local linearization, has proved to be a useful approximation [4]–[7]. However, when the order of the system becomes high (e.g., 50 state variables) the on-line requirements on storage and computing time may become prohibitive. This paper is concerned with the application of multilevel systems theory to the problem, especially when the total system is composed of several subsystems and the manner of decomposition is evident from the system structure.

Multilevel systems theory has been under development in various forms since about 1962, primarily by Mesarovic, Lefkowitz, Pearson, Macko, and Takahara. Of the more recent references [8]–[12] the book [12] is by far the most comprehensive at the time of writing. The author is applying the theoretical developments due to Mesarovic, Macko, and Takahara. Multilevel theory has previously been applied to the problem of state estimation but Chen and Perlis [13] were using an earlier form of the theory (essentially as a constrained minimization) and they did not derive a recursive or sequential estimator (compare with the Kalman filter). Furthermore, the coordination process was simple and slowly convergent. On the other hand, using the so-called interaction prediction principle the essential form of the Kalman filter for each subsystem is preserved below. In addition, a coordination algorithm is derived having the property of one-step convergence.

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The author is with the University of Waterloo, Waterloo, Ont., Canada.