Parametrized Variational Inequality Approaches to the Generalized Nash Equilibrium Problem

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Talk Outline

- Generalized Nash Equilibrium with Shared Constraints
- VI Approach to Finding Some GNEs
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Generalized Nash Equilibrium with Shared Constraints

N players. Each player ν has a “cost” function $\theta_\nu : \Re^n \to \Re$ and a “feasible” set $X_\nu \in \Re^{n_\nu}$, $\nu = 1, \ldots, N$ ($n_1 + \cdots + n_N = n$).

There is a shared constraint function $g : \Re^n \to \Re^m$.

A Generalized Nash Equilibrium (GNE) is any $x^* = (x_1^*, \ldots, x_N^*) \in \Re^n$ satisfying

$$\boxed{x_\nu^* \in \operatorname*{arg\,min}_{x_\nu \in X_\nu} \left\{ \theta_\nu(x_\nu, x_{-\nu}^*) \mid g(x_\nu, x_{-\nu}^*) \le 0 \right\}, \quad \nu = 1, \ldots, N}$$

Assumption:

- Each X_ν is nonempty, closed, convex.
- $\theta_\nu(\cdot, x_{-\nu})$ is convex for each ν and $x_{-\nu} \in X_{-\nu}$.
- $g = (g_1, \ldots, g_m)^T$, with each g_i differentiable, convex.

Under suitable CQ at x^*, there exist $\lambda_\nu^* \in \Re^m$, $\nu = 1, \ldots, N$, satisfying the GNE-KKT condition:

$$\boxed{\begin{array}{ll} 0 \in \nabla_{x_\nu} \theta_\nu(x^*) + \nabla_{x_\nu} g(x^*) \lambda_\nu^* + N_{X_\nu}(x_\nu^*), & \nu = 1, \ldots, N \\ 0 \le \lambda_\nu^* \perp g(x^*) \le 0, \qquad x_\nu^* \in X_\nu, & \end{array}}$$

GNE has many interesting applications. Facchinei, Fischer, Fukushima, Kanzow, Krawczyk, Pang, Robinson, Rosen, Smeers, Uryasev, Wei, ...

Existing approaches, based on penalty method Fukushima, Pang or VI formulation Facchinei, Fischer, Piccialli, Smeers, Wei or Nikaido-Isoda function von Heusinger, Kanzow, Krawczyk, Uryasev, find only one or some GNEs, but not all GNEs.

Like to find all GNEs or, at least, a “sufficiently rich” subset of GNEs.

VI Approach to Finding Some GNEs

Define

$$\begin{aligned} F(x) &:= \left(\nabla_{x_\nu}\theta_\nu(x)\right)_{\nu=1}^{N} \\ X &:= \{x \in X_1 \times \cdots \times X_N \mid g(x) \le 0\} \end{aligned}$$

$$x^* \text{ is a soln of } \mathrm{VI}\,(F, X) \quad \Longleftrightarrow \quad x^* \in \operatorname*{arg\,min}_{x \in X} \langle F(x^*), x \rangle$$

Theorem 0: Wei & Smeers, Facchinei et al.

If x^* and $\lambda_0^* \in \Re^m$ satisfy KKT condition for $\mathrm{VI}\,(F, X)$, then x^* and $\lambda_\nu^* = \lambda_0^*$, $\nu = 1, \ldots, N$, satisfy GNE-KKT condition.

Parameterized VI Approach to Finding "All" GNEs

1. Price-Directed Parametrization:

For any $\pi_\nu \in \Re^m_+$, $\nu = 1, \ldots, N$, let $\pi = (\pi_\nu)_{\nu=1}^N$ and define

$$F^\pi(x) := F(x) + (\nabla_{x_\nu} g(x)\pi_\nu)_{\nu=1}^N$$

Theorem 1: Fix any $\pi = (\pi_\nu)_{\nu=1}^N \in \Re^{mN}_+$. If x^* and $\lambda_0^* \in \Re^m$ satisfy KKT condition for $\mathrm{VI}\,(F^\pi, X)$, then a sufficient condition for x^* and $\lambda_\nu^* = \lambda_0^* + \pi_\nu$, $\nu = 1, \ldots, N$, to satisfy GNE-KKT condition is

$$\langle g(x^*), \pi_\nu \rangle = 0, \quad \nu = 1, \ldots, N$$

If x^* satisfies GNE-LICQ, this condition is also necessary.

Converse Fact: If x^* and $\lambda^* = (\lambda_\nu^*)_{\nu=1}^N$ satisfy the GNE-KKT condition, then x^* is a soln of $\mathrm{VI}\,(F^{\lambda^*}, X)$.

GNE-LICQ:

$$\left\{ \begin{array}{c} 0 \in \nabla_{x_\nu} g(x^*)\lambda_\nu + N_{X_\nu}(x^*_\nu) + (-N_{X_\nu}(x^*_\nu)) \\ \lambda_\nu \perp g(x^*) \leq 0 \end{array} \right\} \quad \Longrightarrow \quad \lambda_\nu = 0, \quad \nu = 1, \ldots, N.$$

Fact: Assume $g(x) = \sum_{\nu=1}^{N} g_\nu(x_\nu)$.

If F is (strongly) monotone on X, then F^π is (strongly) monotone on X for any $\pi \in \Re^{mN}_+$.

2. Resource-Directed Parametrization:

Assume $\quad g(x) = \sum_{\nu=1}^N g_\nu(x_\nu)$.

For any $\beta_\nu \in \Re^m$, $\nu = 1, \ldots, N$, satisfying $\sum_{\nu=1}^N \beta_\nu = 0$, let $\beta := (\beta_\nu)_{\nu=1}^N$,

$$X^\beta := X_1^{\beta_1} \times \cdots \times X_N^{\beta_N} \quad \text{with} \quad X_\nu^{\beta_\nu} := \{x_\nu \in X_\nu \mid g_\nu(x_\nu) \leq \beta_\nu\}$$

Theorem 2: Fix any $\beta = (\beta_\nu)_{\nu=1}^N$ satisfying $\sum_{\nu=1}^N \beta_\nu = 0$. If x^* is a soln of $\mathrm{VI}\,(F, X^\beta)$, then a sufficient condition for x^* to be a GNE is that

for each $i = 1, \ldots, m$, $\left\{ \begin{array}{lll} \text{either} & g_{\nu,i}(x_\nu^*) = \beta_{\nu,i} & \nu = 1, \ldots, N \\ \text{or} & g_{\nu,i}(x_\nu^*) < \beta_{\nu,i} & \nu = 1, \ldots, N \end{array} \right\}$

Converse Fact: If x^* is a GNE, then x^* is a soln of $\mathrm{VI}\,(F, X^\beta)$, where $\beta = (\beta_\nu)_{\nu=1}^N$, $\beta_\nu = g_\nu(x_\nu^*) - \alpha_\nu g(x^*)$, with $\alpha_\nu > 0$ satisfying $\sum_{\nu=1}^N \alpha_\nu = 1$.

Some Numerical Results

1. Harker's example: $N = 2$, $n = 2$, $m = 1$.

$$x_1^* = \operatorname*{arg\,min}_{x_1 \in [0,10]} \left\{ x_1^2 + \frac{8}{3} x_1 x_2^* - 34 x_1 \mid x_1 + x_2^* \leq 15 \right\}$$

$$x_2^* = \operatorname*{arg\,min}_{x_2 \in [0,10]} \left\{ x_2^2 + \frac{5}{4} x_1^* x_2 - 24.25 x_2 \mid x_1^* + x_2 \leq 15 \right\}$$

$$F(x) = \begin{bmatrix} 2x_1 + \frac{8}{3}x_2 - 34 \\ \frac{5}{4}x_1 + 2x_2 - 24.25 \end{bmatrix}$$

$$X = \{(x_1, x_2) \mid 0 \leq x_1, x_2 \leq 10,\ x_1 + x_2 \leq 15\}$$

$\mathrm{VI}\,(F, X)$ is strongly monotone, affine. Soln $(5, 9)$.

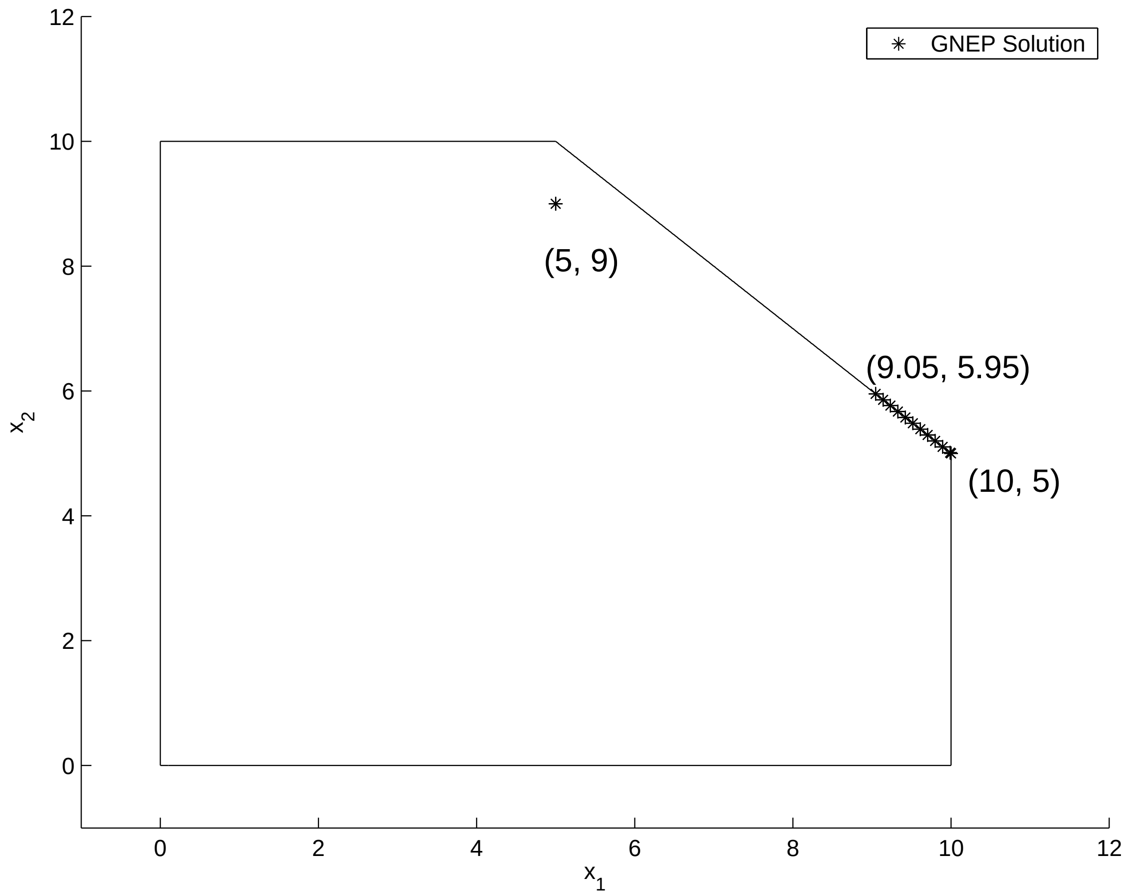
$g(x) = x_1 + x_2 - 15$ is separable.

Price-directed decomp:

Can restrict to $\pi_1 \in [0, 2], \pi_2 = 0$ or $\pi_1 = 0, \pi_2 \in [0, 2]$. Use 256 grid pts for each.

For each grid pt $\pi = (\pi_1, \pi_2)$, $\mathrm{VI}\,(F^\pi, X)$ is strongly monotone, affine. Using Thm 1, soln x^* (found by PATHLCP.M Ferris, Munson) is declared GNE if

$$|\pi_\nu g(x^*)| < 10^{-6}, \quad \nu = 1, 2.$$


GNEP Solution
(5, 9)
(9.05, 5.95)
(10, 5)
x1
x2
0
2
4
6
8
10
12

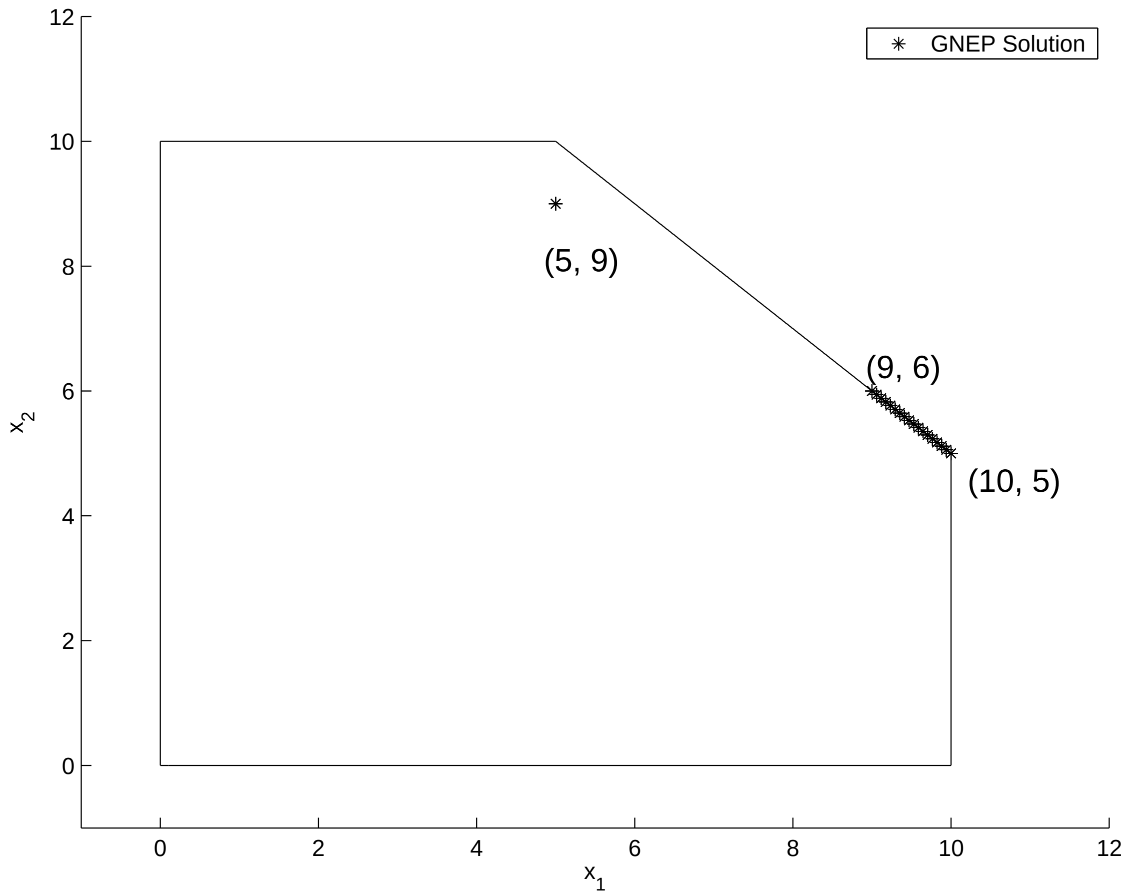
Resource-directed decomp:

Choose $g_1(x_1) = x_1 - \dfrac{15}{2}, \quad g_2(x_2) = x_2 - \dfrac{15}{2}.$

Can restrict to $\beta_1 + \beta_2 = 0, |\beta_1| \leq \frac{15}{2}$. Use 256 grid pts.

For each grid pt $\beta = (\beta_1, \beta_2)$, $\mathrm{VI}\,(F, X^\beta)$ is strongly monotone, affine.
Using Thm 2, soln x^* is declared GNE if

$$\begin{array}{ll} \text{either} & |g_\nu(x_\nu^*) - \beta_\nu| < 10^{-6}, \quad \nu = 1, 2 \\ \text{or} & g_\nu(x_\nu^*) - \beta_\nu < -10^{-6}, \quad \nu = 1, 2. \end{array}$$


GNEP Solution
(5, 9)
(9, 6)
(10, 5)
x_1
x_2
0
2
4
6
8
10
12

2. River basin pollution Krawczyk, Uryasev: $N = 3$, $n = 3$, $m = 2$.

$$x_\nu^* = \arg\min_{x_\nu \geq 0} \left\{ (\alpha_\nu x_\nu + 0.01(x_\nu + x_{-\nu}^*) - \chi_\nu) x_\nu | g(x_\nu, x_{-\nu}^*) \leq 0 \right\}, \quad \nu = 1, 2, 3$$

with $\alpha_1 = \alpha_3 = 0.01$, $\alpha_2 = 0.05$, $\chi_1 = 2.9$, $\chi_2 = 2.88$, $\chi_3 = 2.85$, and

$$g(x) = \begin{bmatrix} 3.25x_1 + 1.25x_2 + 4.125x_3 - 100 \\ 2.2915x_1 + 1.5625x_2 + 2.8125x_3 - 100 \end{bmatrix}$$

$$F(x) = \begin{bmatrix} 0.04 & 0.01 & 0.01 \\ 0.01 & 0.12 & 0.01 \\ 0.01 & 0.01 & 0.04 \end{bmatrix} x - \begin{bmatrix} 2.9 \\ 2.88 \\ 2.85 \end{bmatrix}$$

$$X = \{x \in \Re_+^3 \mid g(x) \leq 0\}$$

$\mathrm{VI}\,(F, X)$ is strongly monotone, affine. Soln $= (21.14.., 16.03.., 2.73..)$

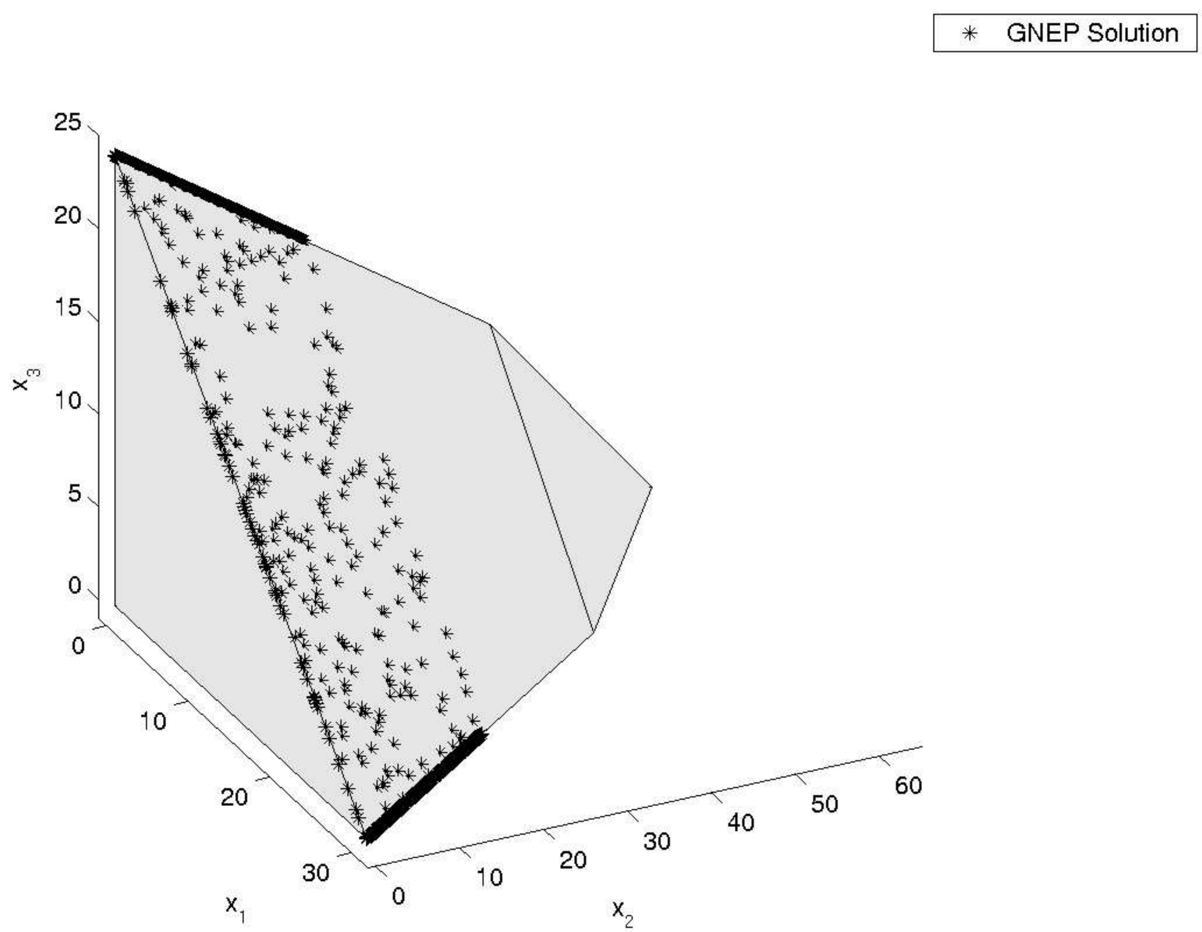
$g(x)$ is separable.

Price-directed decomp:

Restrict to $\pi_1, \pi_2, \pi_3 \in [0, 2]^2$ such that the ith component is zero in one or three of them, $i = 1, 2$. Randomly generate 1000 pts for each of 16 cases except the all-zero case. Total of 15,001 pts.

For each pt $\pi = (\pi_1, \pi_2, \pi_2)$, $\mathrm{VI}\,(F^\pi, X)$ is strongly monotone, affine. Using Thm 1, soln x^* (found by PATHLCP.M) is declared GNE if

$$|\langle \pi_\nu, g(x^*) \rangle| < 10^{-6}, \quad \nu = 1, 2, 3.$$


GNEP Solution
25
20
15
10
5
0
x_3
0
10
20
30
x_1
0
10
20
30
40
50
60
x_2

Resource-directed decomp:

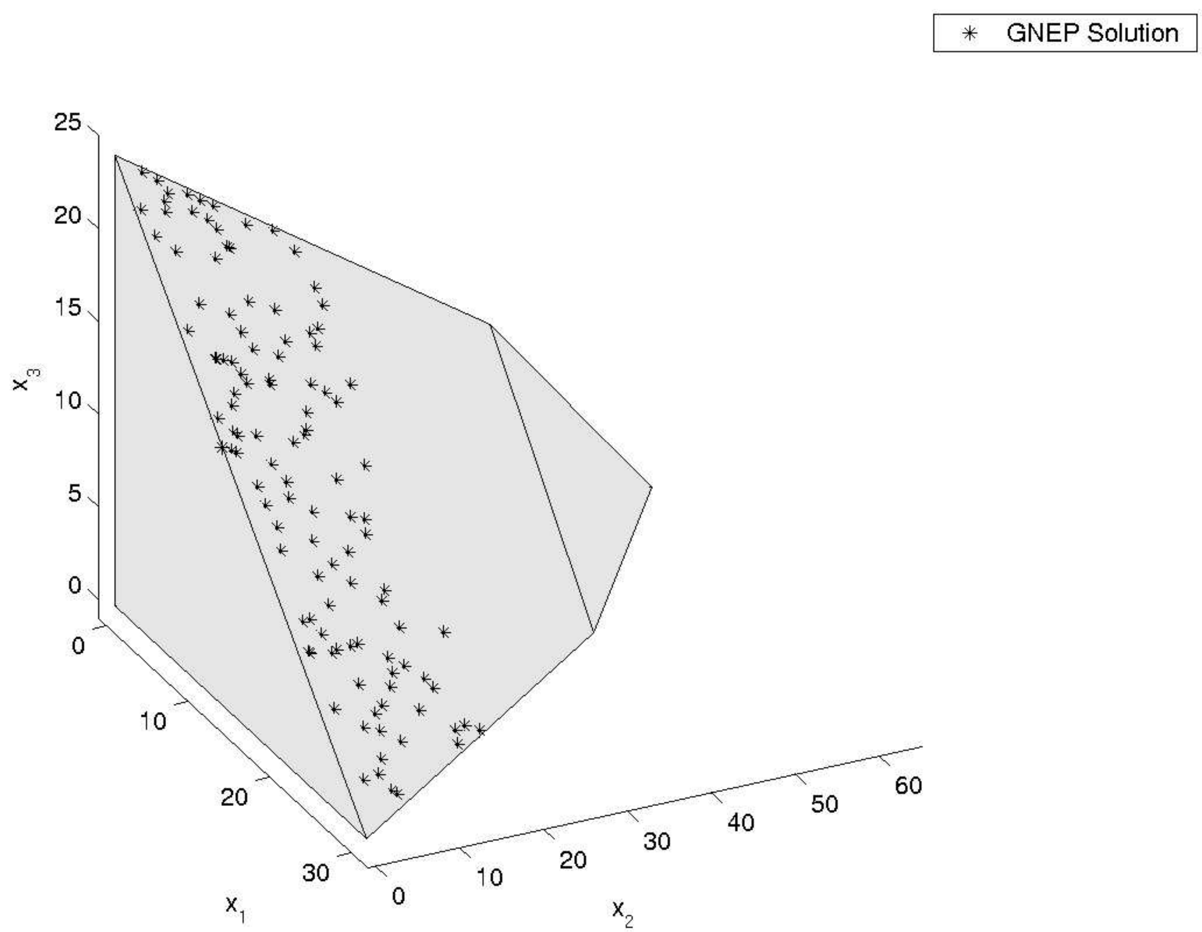
Choose

$$g_1(x_1) = \begin{bmatrix} 3.25x_1 - \frac{100}{3} \\ 2.2915x_1 - \frac{100}{3} \end{bmatrix}, g_2(x_2) = \begin{bmatrix} 1.25x_2 - \frac{100}{3} \\ 1.5625x_2 - \frac{100}{3} \end{bmatrix}, g_3(x_3) = \begin{bmatrix} 4.125x_3 - \frac{10}{3} \\ 2.8125x_3 - \frac{1}{} \end{bmatrix}$$

Can restrict to $\beta_1 + \beta_2 + \beta_3 = 0$, $\beta_1, \beta_2 \geq -\begin{bmatrix} \frac{100}{3} \\ \frac{100}{3} \end{bmatrix}$, $\beta_1 + \beta_2 \leq \begin{bmatrix} \frac{100}{3} \\ \frac{100}{3} \end{bmatrix}$. Use 3700 randomly generated pts.

For each pt $\beta = (\beta_1, \beta_2, \beta_3)$, $\mathrm{VI}\,(F, X^\beta)$ is strongly monotone, affine.
Using Thm 2, soln x^* is declared GNE if

$$\text{for each } i = 1, 2, \quad \left\{ \begin{array}{lll} \text{either} & |g_{\nu,i}(x_\nu^*) - \beta_{\nu,i}| < 10^{-6}, & \nu = 1, 2, 3 \\ \text{or} & g_{\nu,i}(x_\nu^*) - \beta_{\nu,i} < -10^{-6}, & \nu = 1, 2, 3 \end{array} \right\}$$


GNEP Solution
x_3
x_1
x_2
0
5
10
15
20
25
30
40
50
60

Fact: If each θ_ν is quadratic, then $\{\mathrm{GNEs}\} \cap \mathrm{ri} Y$ is convex (possibly $\emptyset$) for each face Y of X. (So suffices to find extreme pts.)

Conclusions & Future Directions

1. More diverse GNEs can give more insight.

2. Price-directed decomposition does not require shared constraint function g be separable, but requires existence of multipliers. It samples in the space of prices. Sampling may not be "uniform" in the space of strategies.

3. Resource-directed decomposition does not require existence of multipliers but requires g be separable. Sampling appears more "uniform" in the space of strategies. Fewer samples are used.

4. More efficient ways to search the price/strategy space (beyond grid sampling or random sampling)? Restricting the search space?