Reprinted from Journal of Economic Theory
Vol. 8, No. 2, June 1974
Printed in Belgium

Necessary and Sufficient Conditions for Existence of an Optimal Portfolio*

Dimitri P. Bertsekas

Department of Engineering-Economic Systems, Stanford University, Stanford, California

Received June 15, 1973

I. Introduction

This paper identifies necessary and sufficient conditions for existence of a solution to a class of optimization problems under uncertainty. This class includes certain problems of optimal portfolio selection when rates of return of risky assets are uncertain, as well as problems of optimal choice of inputs and outputs by a perfectly competitive firm facing uncertain prices.

The question of existence of a solution to a similar class of optimization problems has been examined earlier by Leland [1], who derived sufficient conditions for existence. Leland's conditions, while covering an important special case, are stronger than required and in particular they are not necessary conditions. In the present paper the exact necessary and sufficient conditions for existence of a solution are obtained. Furthermore, these conditions have an interesting and intuitively appealing economic interpretation. It is shown that whether the optimization problem has a solution depends on the magnitude of two scalars p^* and s^*. The scalar p^* is solely determined by the probability distributions of the uncertain quantities (rates of return, prices, etc.) and can be viewed as a measure of expected profitability of the gambling opportunities available. The scalar s^* depends entirely on the utility function of the decision maker and can be viewed as a measure of his risk aversion. It is shown that the optimization problem has a bounded solution set if and only if $p^* < s^*$. In terms of the portfolio problem the conditions proved imply that a decision maker will tend to invest in arbitrarily large amounts of certain risky assets by borrowing a correspondingly large amount of money if

* This work was carried out at the Department of Engineering-Economic Systems, Stanford University, Stanford, California, and supported by the National Science Foundation under Grant GK32870.

and only if the scalar measure p^* of profitability of the risky assets exceeds the scalar measure s^* of his risk aversion.

The paper is organized as follows. In the next section we formulate the class of optimization problems that we are considering. We derive the necessary and sufficient conditions for existence of an optimal solution, and we subsequently specialize our results to the case of the portfolio problem. In Section 3 we consider a dynamic version of the portfolio problem, with and without intermediate consumption, and we prove a sufficient condition for existence of a solution. The proofs of the basic propositions are given in an Appendix at the end of the paper. Predominant in our analysis is the theory of convex functions as authoritatively developed in the recent book by Rockafellar [2]. The notion from this theory which is very useful for proving existence results, and which is central to the development of our results, is that of a direction of recession of a concave function. This notion as well as other, not very widely known, facts from the theory of convex functions will be briefly explained as necessary in the paper.

II. Basic Results: Single Period Case

The problem that we will be concerned with is characterized by a feasible decision set X which is a subset of n-dimensional Euclidean Space (R^n), a probability measure P defined on the σ-algebra of Borel sets of R^n and a scalar utility function $U: R \to R$. We shall be interested in the question of existence of a solution to the optimization problem

$$\text{maximize} \quad E\{U(e'x)\} = \int_{R^n} U(e'x)\, dP \tag{1}$$
$$\text{subject to } x \in X.$$

In the above problem $e = (e_1, \ldots, e_n)$ is an n-dimensional (column) random vector of uncertain parameters. One may view e_i as rate of return of the ith risky asset or as price of a corresponding input or output of a firm. We denote by e' the transpose of e. The vector $x = (x_1, \ldots, x_n)$ is a (column) vector with elements x_i denoting the amount of ith asset held. The (perhaps subjective) probability measure P characterizes probabilistically the rate of return of the risky assets. The feasible set X expresses budget, technological and other constraints. A similar problem is considered in Leland [1] and we refer to that paper for further discussion and interpretation.

We shall make the following assumptions concerning U, X, and P.

A1: U is a concave monotonically nondecreasing function.

A2: X is a nonempty, closed, convex set.

A3: For every $x \in R^n$ we have $E\{|e'x|\} < \infty$ and $E\{|U(e'x)|\} < \infty$. This is true in particular if e is bounded with probability one.

Now by defining the function $V: R^n \to R$

$$V(x) = E\{U(e'x)\}$$

problem (1) is rewritten as

$$\max_{x \in X} V(x). \tag{2}$$

It is clear from our assumptions that V is a well defined concave function. Thus the problem of existence of a solution to problem (2) can be analyzed by means of the powerful corresponding results of convex function theory. Central to these results is the notion of a direction of recession of a convex set and a convex function.

A vector $y \neq 0$ is said to be a direction of recession [2] of a convex set $X \subset R^n$ if $x_0 + ay \in X$ for every $x_0 \in X$ and $a \geqslant 0$. It is to be noted that if $x_0 + ay \in X$ for some $x_0 \in X$ and all $a \geqslant 0$ the same is true for every $x_0 \in X$. Thus a direction of recession of a convex set is a direction along which this set is unbounded.

Now given a concave function $V: R^n \to R$ we shall call $y \neq 0$ a direction of recession of V if $V(x_0 + ay)$ is a monotonically nondecreasing function of a for every $x_0 \in R^n$. Again the function $V(x_0 + ay)$ is nondecreasing in a for every $x_0 \in R^n$ if it is nondecreasing for some $x_0 \in R^n$. We note that Rockafellar [2] defines a direction of recession somewhat differently (in terms of the recession function of a convex function). For the case that we consider the definition we gave is equivalent ([2, Theorem 8.6]).

We shall make use of the following result ([2, Theorems 27.1, 27.3]).

Proposition 1. *Problem* (2) *and hence also Problem* (1) *has a nonempty and compact solution set if and only if the convex set X and the concave function V have no common direction of recession.*

The propositions which follow are obtained simply by deriving the necessary and sufficient conditions for a given direction to be a direction of recession of the function V. Due to the particular structure of V these conditions take a simple form.

Let s^+ and s^- be the asymptotic slopes of U

$$s^+ = \lim_{c \to +\infty} \frac{dU(c)}{dc}, \qquad s^- = \lim_{c \to -\infty} \frac{dU(c)}{dc}, \qquad +\infty \geqslant s^- \geqslant s^+ \geqslant 0.$$

We note that the limits in the above relations are defined since U being a concave function is differentiable almost everywhere (with respect to Lebesgue measure) even though it is perhaps not differentiable everywhere. We denote by s^* the ratio of the asymptotic slopes s^- and s^+

$$s^* = s^-/s^+.$$

In order to resolve indeterminacies in the above equation, we define

$$\begin{aligned} s^* &= +\infty \qquad \text{if} \quad s^- = +\infty \\ s^* &= +\infty \qquad \text{if} \quad s^- > 0, \quad s^+ = 0 \\ s^* &= 0 \qquad \text{if} \quad s^- = s^+ = 0. \end{aligned}$$

Notice that s^* is invariant with respect to positive linear transformations of the utility function U. Furthermore s^* can be viewed as a measure of risk aversion of the decision maker. Indeed let U be a twice continuously differentiable function with first and second derivatives denoted by U' and U'' and index of absolute risk aversion [3] denoted by

$$r(c) = -(U''(c)/U'(c)).$$

Then it can be easily seen that

$$\ln s^* = \lim_{a \to \infty} \int_{-a}^{a} r(c)\, dc$$

thus showing that s^* can be meaningfully viewed as a measure of risk aversion associated with the utility function U.

Given any vector $y \in R^n$, $y \neq 0$ let us also consider the ratio

$$p(y) = \frac{E\{e'y \mid e'y > 0\}\, P(e'y > 0)}{E\{-e'y \mid e'y < 0\}\, P(e'y < 0)}$$

where in the above relation we define

$$p(y) = +\infty \qquad \text{if} \quad P(e'y < 0) = 0.$$

The ratio $p(y)$ can be viewed as a measure of expected profitability of holding assets in fixed ratios specified by the direction y. It is the ratio of expected rate of positive return provided a positive return occurs, multiplied by the probability of a positive return, over the corresponding expected rate of negative return multiplied by the probability of a negative return. Notice that $p(y)$ depends on the probability distribution of the random vector e alone and is not dependent upon the form of the utility function U.

We have the following proposition the proof of which is given in the Appendix.

PROPOSITION 2. *A vector $y \neq 0$ is such that*

$$V(x_0 + ay) = E\{U[e'(x_0 + ay)]\}$$

is a monotonically nondecreasing function of $a \in R$ for every $x_0 \in R^n$ (and hence y is a direction of recession of V) if and only if

$$p(y) \geqslant s^*. \tag{3}$$

The interpretation of the above proposition is quite clear. Whenever y is such that $V(x_0 + ay)$ is monotonically nondecreasing in a the decision maker will tend to invest (in addition to x_0) arbitrarily large amounts in fixed ratios specified by the direction y. Such behavior will occur if and only if the scalar measure $p(y)$ of expected profitability in the direction y exceeds the scalar measure s^* of the decision maker's risk aversion.

Naturally, in order for an investment in the direction y to be infinitely reproducible, y must be a direction of recession of the feasible set X. By combining Propositions 1 and 2 we have:

PROPOSITION 3. *In order that Problem* (1) *has a nonempty and compact solution set it is necessary and sufficient that*

$$p(y) < s^* \tag{4}$$

for all directions of recession y of the convex set X.

If we denote by p^* the maximum of $p(y)$ over all directions of recession of X (assuming that this maximum is attained)

$$p^* = \max\{p(y) \mid y \text{ is a direction of recession of } X\}$$

an equivalent statement of the above proposition is the following:

Problem (1) *has a nonempty and compact solution set if and only if*

$$p^* < s^*$$

where p^* is a measure of profitability of the gambling opportunities available and s^* is a measure of risk aversion of the decision maker.

We note that the above proposition contains as a special case Leland's sufficient condition which states that if $s^+ = 0$ and $P(e'y < 0) > 0$ for all directions of recession y of X then an optimal solution to Problem (1) exists. The conditions of Proposition 3 are however much more powerful.

For example, existence is assured if $s^- = +\infty$ and $P(e'y < 0) > 0$ for all directions of recession y of X. We also note that if X is bounded, and hence compact, it has no directions of recession and (4) is trivially satisfied. In this case, of course, the solution set is known to be nonempty and compact by Weierstrass' Theorem and the continuity of V.

We mention that a partial converse of Proposition 3 can also be proved, namely that Problem (1) does not have any optimal solution whenever

$$s^+ E\{e'y \mid e'y > 0\}\, P(e'y > 0) + s^- E\{e'y \mid e'y < 0\}\, P(e'y < 0) > 0$$

for some direction of recession y of the set X. Excluding the degenerate situations $s^- = 0$ and $P(e'y > 0) = P(e'y < 0) = 0$ the inequality above is equivalent to $p(y) > s^*$. A typical example of this situation is when the decision maker is risk neutral ($s^* = 1$) and $E\{e'y\} > 0$ for some direction of recession y of the set X. In the case where $p^* = s^*$ there are two possibilities. Either the problem has no solution or the problem has a nonempty but unbounded solution set. One can easily construct examples to show that both situations are possible.

Finally let us consider the case where assumption A3 is relaxed and replaced by the following assumption:

A3': For every $x \in R^n$ we have $E\{|U(e'x)|\} < \infty$.

Then we have the following proposition the proof of which is given in the Appendix:

PROPOSITION 4. *Assume that* A1, A2, A3' *hold and furthermore* $s^+ = 0$, $s^- > 0$, *and* $P(e'y < 0) > 0$ *for all directions of recession* y *of the set* X. *Then Problem* (1) *has a nonempty and compact solution set.*

The result above is somewhat stronger than a similar result shown by Leland [1].

The results developed above apply to both portfolio selection problems and to the problem of a firm choosing inputs and outputs to maximize expected utility of revenue. A specific case of a portfolio problem which will also be considered in a dynamic framework in the next section is the following:

There are n risky assets each offering a random net return e_i per unit invested in the ith asset ($i = 1,..., n$). There is also a riskless asset offering a riskless net return r per unit invested. The budget available for allocation among the $(n + 1)$ assets is B_0 and the problem is to find the allocation $(x_0, x_1,..., x_n)$ which maximizes

$$E\left\{U\left[(1 + r)\, x_0 + \sum_{i=1}^{n} (1 + e_i)\, x_i\right]\right\}$$

subject to

$$x_0 + \sum_{i=1}^{n} x_i = B_0, \qquad x_i \geqslant 0, \qquad i = 1,..., n.$$

Notice that x_0 may be negative thus allowing for the possibility of borrowing.

The problem above is equivalent to the problem

$$\max_{x_i \geqslant 0} E\left\{U\left[(1 + r)\, B_0 + \sum_{i=1}^{n} (e_i - r)\, x_i\right]\right\} \qquad i = 1,..., n. \tag{5}$$

For any $y \in R^n$, $y \neq 0$ let

$$p(y) = \frac{E\{(e - \bar{r})'\, y \mid (e - \bar{r})'\, y > 0\}\, P[(e - \bar{r})'\, y > 0]}{E\{-(e - \bar{r})'\, y \mid (e - \bar{r})'\, y < 0\}\, P[(e - \bar{r})'\, y < 0]}$$

where $\bar{r}$ denotes the vector $(r, r,..., r)'$.

Let also s^* denote the asymptotic slope ratio s^-/s^+ corresponding to the utility function U.

By applying the result of Proposition 3 we have that Problem (5) has a nonempty and compact solution set if and only if

$$p(y) < s^*$$

for all $y = (y_1,..., y_n) \neq 0$, $y_i \geqslant 0$, $i = 1,..., n$. This result essentially means that whenever the condition above is violated the decision maker will tend to invest in arbitrarily large amounts of some of the risky assets by borrowing a correspondingly large amount of money. Conversely when the condition above is satisfied such behavior will not occur.

III. The Dynamic Portfolio Problem

The dynamic portfolio problem that we shall initially consider is an N-stage version of problem (5) of the previous section. The decision-maker's wealth B_k at period k evolves according to the equation

$$B_{k+1} = (1 + r_k)\, B_k + \sum_{i=1}^{n} (e_i^k - r_k)\, x_i \qquad k = 0, 1,..., N - 1 \tag{6}$$

where e_i^k is the random net return of the ith asset at the kth period and $r_k \geqslant 0$ is the riskless rate of return at the kth period. We assume that returns are uncorrelated over time. The objective is to select strategies

$x_i^k(B_k)$, $k = 0,\dots, N-1$, $i = 1,\dots, n$ which maximize the terminal utility function

$$E\{U(B_N)\}. \tag{7}$$

Furthermore, we must require that the functions $x_i^k(\cdot)$ are Borel measurable in order for the expectation (7) to be well defined. The solution of the above problem by dynamic programming is well known. The optimal strategy is obtained from the recursive algorithm

$$I_k(B_k) = \sup_{x_i^k \geqslant 0} E\left\{I_{k+1}\left[(1+r_k)\, B_k + \sum_{i=1}^{n} (e_i^k - r_k)\, x_i^k\right]\right\}$$

$$i = 1,\dots, n, \quad k = 0,\dots, N-1 \tag{8}$$

$$I_N(B_N) = U(B_N). \tag{9}$$

We use the notation

$$e_k = (e_1^k,\dots, e_n^k)' \in R^n$$
$$\bar{r}_k = (r_k,\dots, r_k)' \in R^n.$$

The following proposition gives a sufficient condition for existence of a solution to the problem described above:

PROPOSITION 5. *Assume that for all* k, $B \in R$ *and* $x \in R^n$ *with* $x_i \geqslant 0$, $i = 1,\dots, n$

$$E\{|(e_k - \bar{r}_k)'x|\} < \infty, \qquad P[(e_k - \bar{r}_k)'x < 0] > 0,$$
$$E\{|U[B + (e_k - \bar{r}_k)'x]|\} < \infty$$

and furthermore that

$$\lim_{c\to\infty} (dU(c)/dc) = 0, \qquad \lim_{c\to-\infty} (dU(c)/dc) > 0. \tag{10}$$

Then the functions I_k *of* (8), (9) *are real valued, concave, monotonically nondecreasing, and the problem indicated in Eq.* (8) *has a nonempty and compact solution set for all* $B_k \in R$ *and all* $k = 0, 1,\dots, N-1$. *Furthermore there exists a Borel measurable strategy* $x_i^k(\cdot)$, $k = 0,\dots, N-1$, $i = 1,\dots, n$ *maximizing* $E\{U(B_N)\}$.

The sufficient condition of the above proposition follows from the results of the previous section once it is proved that the functions I_k of (8), (9), satisfy

$$\lim_{c\to\infty} (dI_k(c)/dc) = 0, \qquad \lim_{c\to-\infty} (dI_k(c)/dc) > 0$$

for every k. This fact together with the existence of a measurable strategy is shown in the Appendix. Notice that the proof of this fact would be almost immediate if the function U were bounded above since in that case the functions I_k would also be bounded above.

We can further generalize the dynamic model of this section by introducing intermediate consumption. Denoting by c_k the consumption at period k we have

$$B_{k+1} = (1+r_k)\, B_k + \sum_{i=1}^{n} (e_i^k - r_k)\, x_i^k - (1+r_k)\, c_k$$

and the objective is to find an investment and consumption strategy maximizing

$$E\left\{U(B_N) + \sum_{k=0}^{N-1} U_k(c_k)\right\}$$

where U_k are upper-semicontinuous concave, monotonically nondecreasing functions on $[0, \infty)$. The consumption c_k is constrained to be nonnegative for all k. The question of existence of an optimal solution to this problem can be analyzed by using similar methods as for the previous problems. Rather than presenting the complete analysis we state without proof that if $\lim_{c\to\infty}(dU_k(c)/dc) = 0$ for all k then the assumptions of Proposition 5 constitute a sufficient condition for existence of a solution.

APPENDIX

In this appendix we give the proofs of Propositions 2, 4, and 5.

Proof of Proposition 2. By [2, Theorem 8.5] we have that $y \neq 0$ is a direction of recession of V if and only if

$$\lim_{n\to\infty} \frac{V(\lambda_n y) - V(0)}{\lambda_n} = \lim_{n\to\infty} \frac{E\{U(\lambda_n e'y)\} - U(0)}{\lambda_n} \geqslant 0$$

for every increasing sequence λ_n with $\lambda_n > 0$, $\lim_{n\to\infty} \lambda_n = +\infty$. We shall assume that $P(e'y > 0) > 0$ and $P(e'y < 0) > 0$. The proof can be trivially modified if either $P(e'y > 0) = 0$ or $P(e'y < 0) = 0$. The above inequality is equivalent to

$$\lim_{n\to\infty} \left[\frac{E\{U(\lambda_n e'y) \mid e'y > 0\} - U(0)}{\lambda_n} P(e'y > 0) + \frac{E\{U(\lambda_n e'y) \mid e'y < 0\} - U(0)}{\lambda_n} P(e'y < 0)\right] \geqslant 0. \tag{11}$$

Now for every e such that $e'y > 0$ the functions

$$f_n(e) = [U(\lambda_n e'y) - U(0)]/\lambda_n \tag{12}$$

satisfy

$$f_1(e) \geqslant \cdots \geqslant f_n(e) \geqslant f_{n+1}(e) \geqslant \cdots \tag{13}$$

by the concavity of U. Furthermore the integrals $E\{f_n(e) \mid e'y > 0\}$ exist and are finite. In addition we have by the definition of s^+ (see also [2, Theorem 8.5]).

$$\lim_{n\to\infty} f_n(e) = s^+e'y. \tag{14}$$

Hence (13) and (14) imply, by the Monotone Convergence Theorem [4], that

$$\lim_{n\to\infty} E\{f_n(e) \mid e'y > 0\} = s^+E\{e'y \mid e'y > 0\}$$

or equivalently

$$\lim_{n\to\infty} \frac{E\{U(\lambda_n e'y) \mid e'y > 0\} - U(0)}{\lambda_n} = s^+E\{e'y \mid e'y > 0\}.$$

Similarly we have

$$\lim_{n\to\infty} \frac{E\{U(\lambda_n e'y) \mid e'y < 0\} - U(0)}{\lambda_n} = s^-E\{e'y \mid e'y < 0\}$$

and (11) is equivalent to

$$s^+E\{e'y \mid e'y > 0\}\, P(e'y > 0) + s^-E\{e'y \mid e'y < 0\}\, P(e'y < 0) \geqslant 0.$$

The above relation is in turn equivalent to (3) which was to be proved. Q.E.D.

Proof of Proposition 4. Let y be an arbitrary direction of recession of the set X. We shall again give the proof under the assumption $P(e'y > 0) > 0$. The case $P(e'y > 0) = 0$ is trivial. Proceeding as in the proof of Proposition 2 we have that y is a direction of recession of V if and only if (11) holds. Furthermore, for every e such that $e'y > 0$ (13), (14), still hold. By the assumption $s^+ = 0$ we have for any e such that $e'y > 0$ by (14)

$$\lim_{n\to\infty} f_n(e) = s^+e'y = 0. \tag{15}$$

From (13), (15) and the Monotone Convergence Theorem

$$\lim_{n\to\infty} E\{f_n(e) \mid e'y > 0\} = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{E\{U(\lambda_n e'y) \mid e'y > 0\} - U(0)}{\lambda_n} = 0. \tag{16}$$

Since $s^- > 0$ we may assume without loss of generality that U has a positive slope s_0 at the origin (otherwise we consider the functions $[U(\lambda_n e'y + c) - U(c)]/\lambda_n$ in place of $f_n(e)$, where c is a point such that $dU(c)/dc > 0$). Let e be any point such that $e'y < 0$. We have by the concavity of U

$$0 > s_0e'y \geqslant f_1(e) \geqslant \cdots \geqslant f_n(e) \geqslant f_{n+1}(e) \geqslant \cdots.$$

Hence

$$E\{f_n(e) \mid e'y < 0\} \leqslant s_0E\{e'y \mid e'y < 0\} < 0$$

and

$$\limsup_{n\to\infty} \frac{E\{U(\lambda_n e'y) \mid e'y < 0\} - U(0)}{\lambda_n} \leqslant s_0E\{e'y \mid e'y < 0\} < 0. \tag{17}$$

From (16) and (17) we have that the limit in (11) is strictly negative. Hence y is not a direction of recession of V and, by Proposition 1, Problem (1) has a nonempty and compact solution set. Q.E.D.

Proof of Proposition 5. If it could be proved that the maximization problem in (8) has a solution for every $B \in R$ then it would be easy to show recursively that the functions I_k are well defined as real valued, concave, and monotonically nondecreasing functions. We thus concentrate on proving the existence of a solution to the problem in (8).

We shall show that if $f: R \to R$ is a monotonically nondecreasing concave function with $\lim_{z\to\infty} df(z)/dz = 0$, $\lim_{z\to-\infty} df(z)/dz > 0$, $\alpha \geqslant 1$ is a given scalar, $\mu \in R^n$ is a random vector satisfying $E\{|\mu'x|\} < \infty$, $P(\mu'x < 0) > 0$, $E\{|f(\alpha B + \mu'x)|\} < \infty$ for all $B \in R$, $x \geqslant 0$ then the function

$$F(B) = \max_{x \geqslant 0} E\{f(\alpha B + \mu'x)\} \tag{18}$$

is concave monotonically nondecreasing and

$$\lim_{B\to\infty} dF(B)/dB = 0, \qquad 0 < \lim_{B\to-\infty} dF(B)/dB \leqslant \alpha \lim_{z\to-\infty} df(z)/dz.$$

Furthermore we will show that under these circumstances there exists a Borel measurable function $x(B)$ such that

$$F(B) = E\{f[\alpha B + \mu'x(B)]\} \tag{19}$$

and that in addition $E\{|F(B + \bar{\mu}'x)|\} < \infty$ for all $B \in R$, $x \geqslant 0$ and random vectors $\bar{\mu}$ for which $E\{|f(B + \bar{\mu}'x)|\} < \infty$, $E\{|\bar{\mu}'x|\} < \infty$. These facts in conjunction with the theory of Section 2 are sufficient to prove Proposition 4 by induction.

Now from Proposition 2 we have that the maximization problem in (18) has a nonempty and compact solution set. Furthermore it is clear that F is real valued, concave and monotonically nondecreasing. From the theory of recession functions [2] (Theorems 9.2, 9.3, 9.5, in particular) the right asymptotic slope of F can be calculated to be

$$\begin{aligned}\lim_{B\to\infty} dF(B)/dB &= \max_{x\geqslant 0} [\lim_{z\to\infty} (df(z)/dz)\, E\{\alpha + \mu'x \mid \alpha + \mu'x > 0\}\, P(\alpha + \mu'x > 0) \\ &\quad + \lim_{z\to-\infty} (df(z)/dz)\, E\{\alpha + \mu'x \mid \alpha + \mu'x < 0\}\, P(\alpha + \mu'x < 0)].\end{aligned}$$

Using the fact $\lim_{z\to\infty} df(z)/dz = 0$ we have

$$\begin{aligned}\lim_{B\to\infty} dF(B)/dB &= \max_{x\geqslant 0} [\lim_{z\to-\infty} (df(z)/dz)\, E\{\alpha + \mu'x \mid \alpha + \mu'x < 0\}\, P(\alpha + \mu'x < 0)] \\ &\leqslant 0.\end{aligned}$$

But F is monotonically nondecreasing implying $\lim_{B\to\infty} dF(B)/dB \geqslant 0$. Hence $\lim_{B\to\infty} dF(B)/dB = 0$.

The left asymptotic slope can be calculated to be

$$\begin{aligned}\lim_{B\to-\infty} dF(B)/dB &= -\max_{x\geqslant 0} [\lim_{z\to-\infty} (df(z)/dz)\, E\{-\alpha + \mu'x \mid -\alpha + \mu'x < 0\} \\ &\qquad \times P(-\alpha + \mu'x < 0)] \\ &\leqslant \lim_{z\to-\infty} (df(z)/dz)\, \alpha.\end{aligned}$$

If $\lim_{B\to-\infty} dF(B)/dB$ were zero that would imply that for some vector $\bar{x} \geqslant 0$ we have $P(-\alpha + \mu'\bar{x} < 0) = 0$. This is impossible by our assumptions. Hence $\lim_{B\to-\infty} dF(B)/dB > 0$. The existence of a Borel measurable function $x(B)$ satisfying (19) follows directly from Corollary 4.3 of [5]. The inequality $E\{|F(B + \bar{\mu}'x)|\} < \infty$ follows from the fact that $F(B)$ is bounded above by an increasing affine function and below by the function $f(\alpha B)$. Q.E.D.

Acknowledgment

The author thanks J. L. Sweeney for commenting on various aspects of this paper.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.