A Series of Lectures on Approximate Dynamic Programming Lecture 1

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Our Aim

Discuss optimization by Dynamic Programming (DP)

and the use of approximations

Purpose: Computational tractability in a broad variety of practical contexts

The Scope of these Lectures

After an intoduction to exact DP, we will focus on approximate DP for optimal control under stochastic uncertainty

- The subject is broad with rich variety of theory/math, algorithms, and applications
- Applications come from a vast array of areas: control/robotics/planning, operations research, economics, artificial intelligence, and beyond ...
- We will concentrate on control of discrete-time systems mostly for a finite number of stages (a finite horizon), and the expected value criterion
- We will focus mostly on algorithms ... less on theory and modeling

We will not cover:

- Infinite horizon problems (except briefly)
- Imperfect state information and minimax/game problems
- Simulation-based methods: reinforcement learning, neuro-dynamic programming
- A series of video lectures on the latter can be found at the author's web site

Reference: The lectures will follow Chapters 1 and 6 of the author's book

"Dynamic Programming and Optimal Control," Vol. I, Athena Scientific, 2017

Lectures Plan

Overview of Exact DP

- The basic problem formulation; some examples
- The DP algorithm for finite horizon problems with perfect state information
- Computational limitations; motivation for approximate DP

Approximate DP - I (limited lookahead with cost-to-go approximation)

- Approximation in value space
- Parametric cost approximation, feature-based architectures, neural networks

Approximate DP - II (model-free and simulation-based ideas)

- Q-factor approximation and model-free approximate DP
- Simulation-based on-line approximation; rollout and Monte Carlo tree search
- Applications in backgammon and AlphaGo

Approximate DP - III (some alternative approaches)

- Approximation in policy space
- Limited lookahead with problem approximation

EXACT DYNAMINC PROGRAMMING

Outline

Basic Problem Structure for DP

Discrete-time system

$$x_{k+1} = f_k(x_k, u_k, w_k), \qquad k = 0, 1, \ldots, N-1$$

- x_k: State; summarizes past information that is relevant for future optimization at time k
- u_k: Control; decision to be selected at time k from a given set $U_k(x_k)$
- w_k: Disturbance; random parameter with distribution $P(w_k \mid x_k, u_k)$
- For deterministic problems there is no w_k

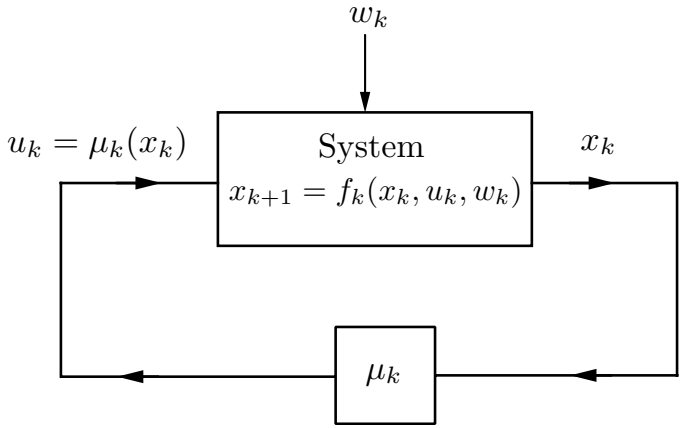
Cost function that is additive over time

$$E\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \right\}$$

Perfect state information

The control u_k is applied with (exact) knowledge of the state x_k

Optimization over Feedback Policies



- Feedback policies: Rules that specify the control to apply at each possible state x_k that can occur
- Major distinction: We minimize over sequences of functions of state $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$, with $u_k = \mu_k(x_k) \in U_k(x_k)$ - not sequences of controls $\{u_0, u_1, \ldots, u_{N-1}\}$

Cost of a policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$ starting at initial state x_0

$$J_\pi(x_0) = E\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k\left(x_k, \mu_k(x_k), w_k\right) \right\}$$

Optimal cost function:

$$J^*(x_0) = \min_\pi J_\pi(x_0)$$

Scope of DP

Any optimization (deterministic, stochastic, minimax, etc) involving a sequence of decisions fits the framework

A continuous-state example: Linear-quadratic optimal control

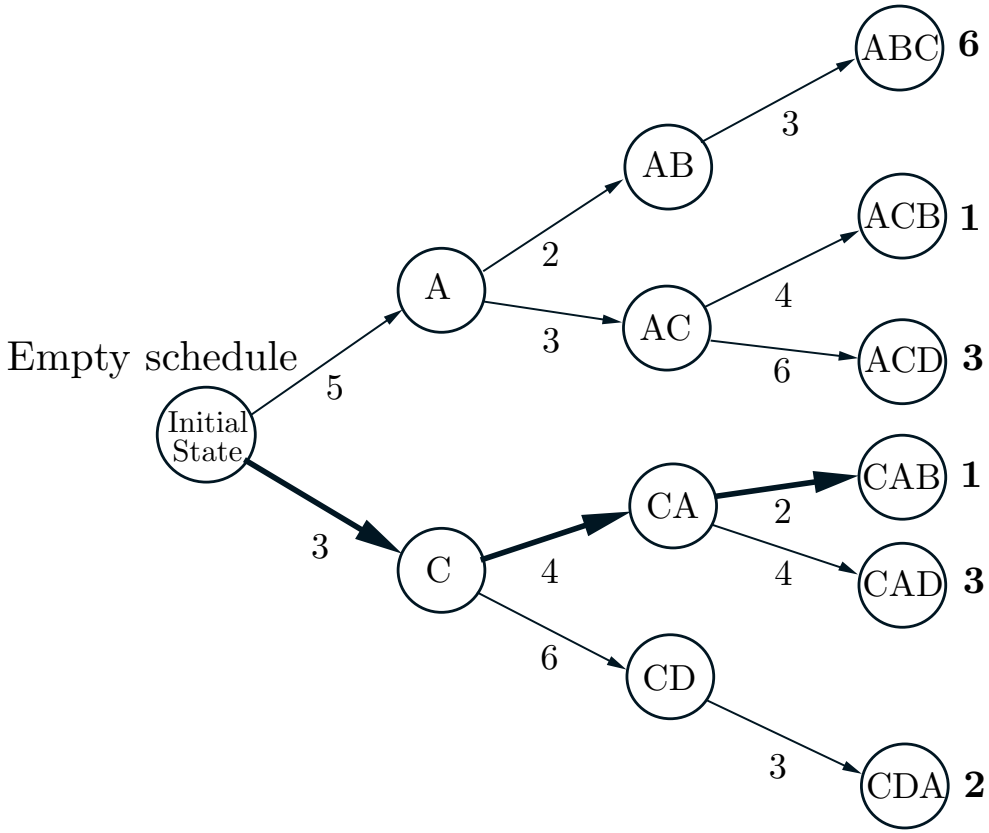
- Linear discrete-time system: $x_{k+1} = Ax_k + Bu_k + w_k, \quad k = 0, \ldots, N-1$
- $x_k \in \Re^n$: The state at time k
- $u_k \in \Re^m$: The control at time k (no constraints in the classical version)
- $w_k \in \Re^n$: The disturbance at time k ($w_0, \ldots, w_{N-1}$ are independent random variables with given distribution)

Quadratic Cost Function

$$E\left\{x_N' Q x_N + \sum_{k=0}^{N-1} \left(x_k' Q x_k + u_k' R u_k\right)\right\}$$

where Q and R are positive definite symmetric matrices

Discrete-State Deterministic Scheduling Example

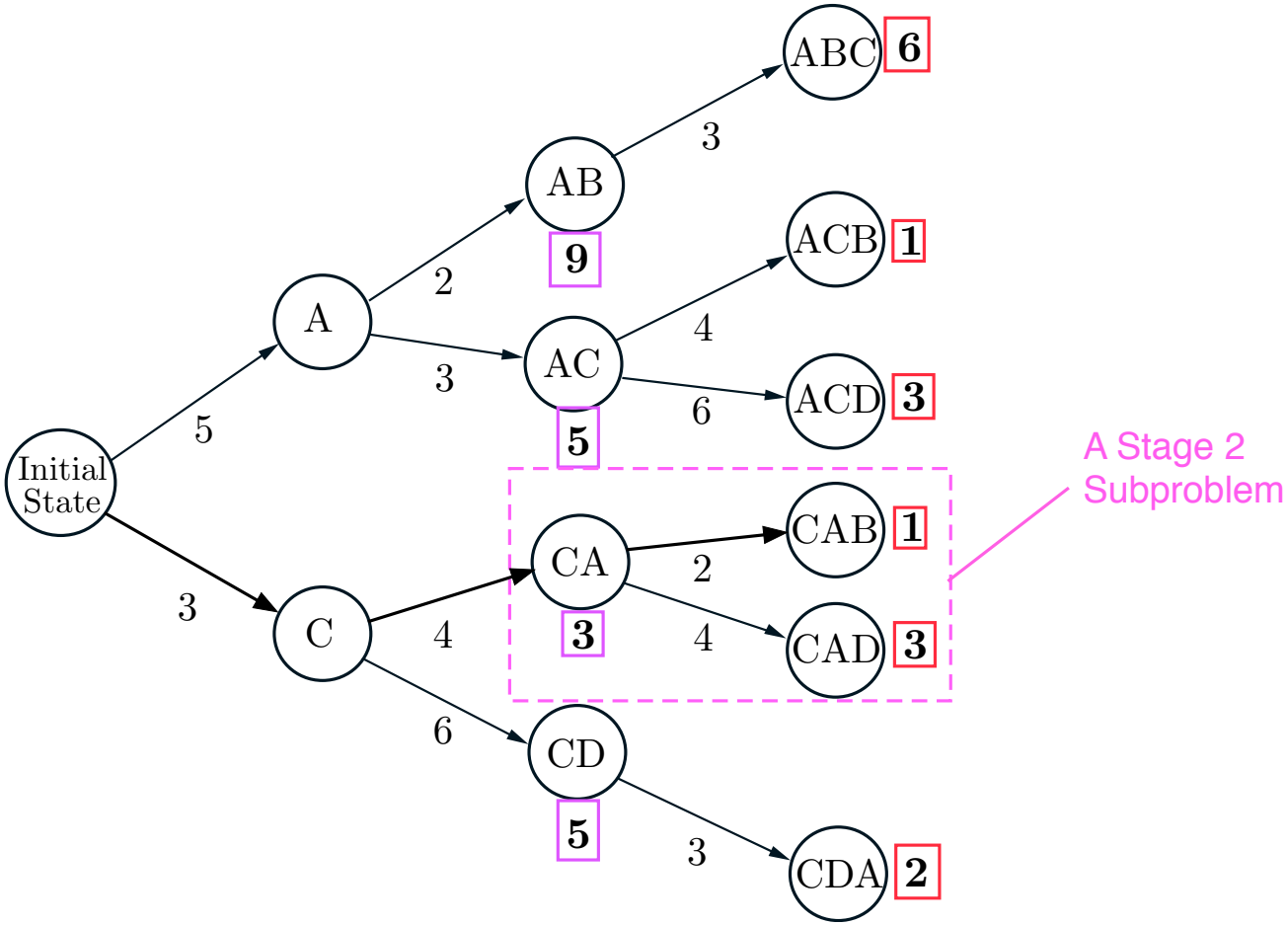


Find optimal sequence of operations A, B, C, D (A must precede B and C must precede D)

DP Problem Formulation

- States: Partial schedules; Controls: Stage 0, 1, and 2 decisions
- DP idea: Break down the problem into smaller pieces (tail subproblems)
- Start from the last decision and go backwards

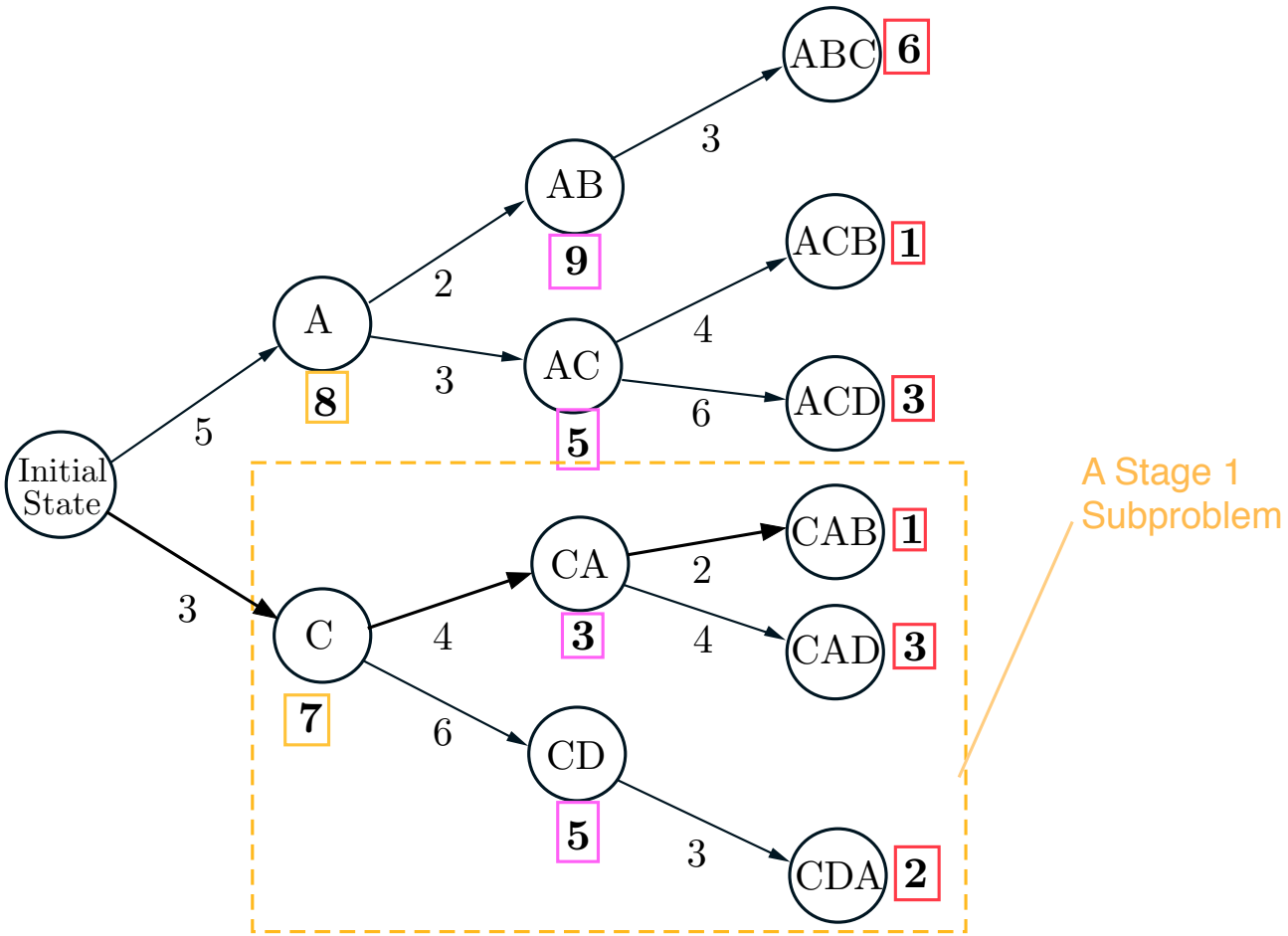
Scheduling Example Algorithm I



Solve the stage 2 subproblems (using the terminal costs)

At each state of stage 2, we record the optimal cost-to-go and the optimal decision

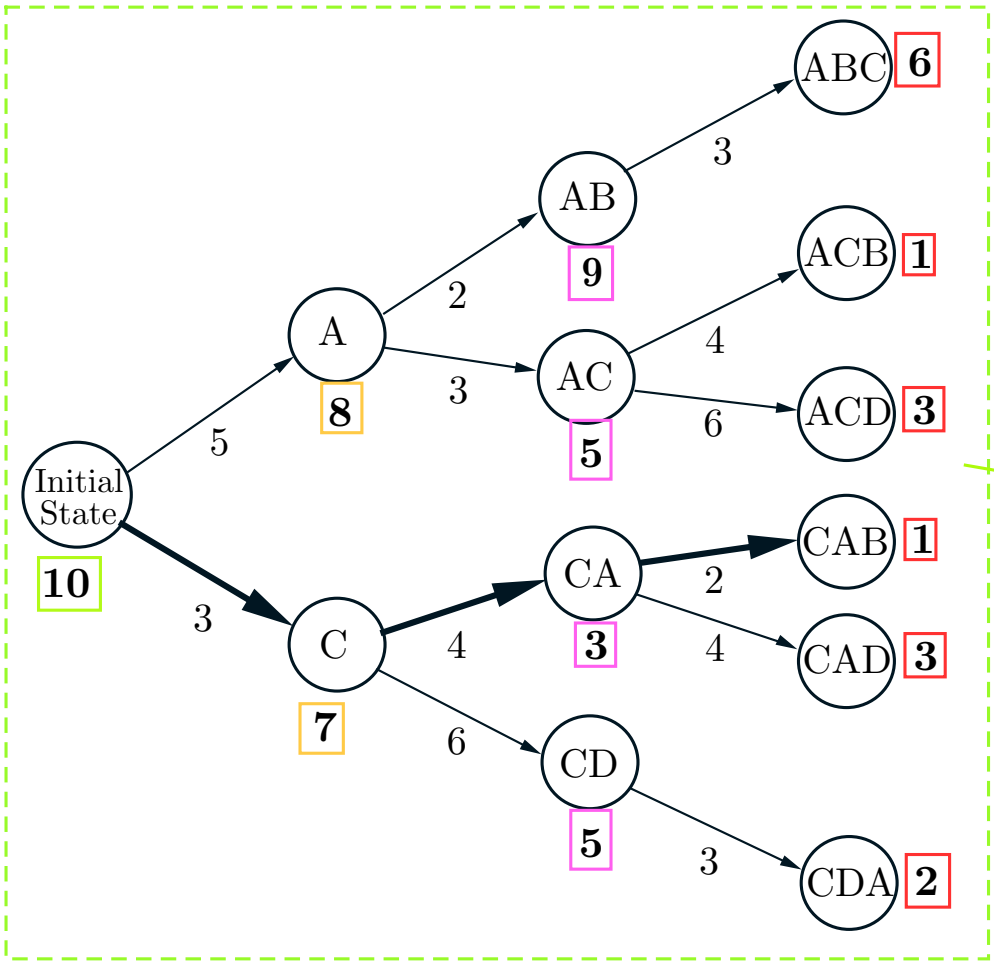
Scheduling Example Algorithm II



Solve the stage 1 subproblems (using the solution of stage 2 subproblems)

At each state of stage 1, we record the optimal cost-to-go and the optimal decision

Scheduling Example Algorithm III



Solve the stage 0 subproblem (using the solution of stage 1 subproblems)

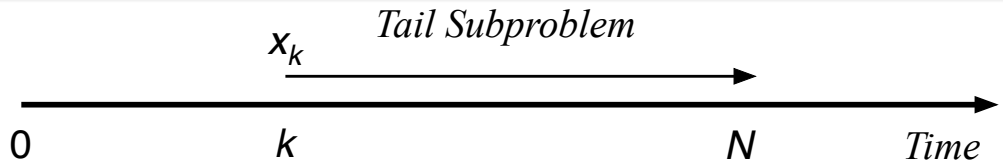
- The stage 0 subproblem is the entire problem
- The optimal value of the stage 0 subproblem is the optimal cost J^*(initial state)
- Construct the optimal sequence going forward

Principle of Optimality

- Let $\pi^* = \{\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*\}$ be an optimal policy
- Consider the "tail subproblem" whereby we are at x_k at time k and wish to minimize the "cost-to-go" from time k to time N

$$E\left\{ g_N(x_N) + \sum_{m=k}^{N-1} g_m\big(x_m, \mu_m(x_m), w_m\big) \right\}$$

Consider the "tail" $\{\mu_k^*, \mu_{k+1}^*, \ldots, \mu_{N-1}^*\}$ of the optimal policy



THE TAIL OF AN OPTIMAL POLICY IS OPTIMAL FOR THE TAIL SUBPROBLEM

DP Algorithm

- Start with the last tail (stage $N-1$) subproblems
- Solve the stage k tail subproblems, using the optimal costs-to-go of the stage $(k+1)$ tail subproblems
- The optimal value of the stage 0 subproblem is the optimal cost J^*(initial state)
- In the process construct the optimal policy

Formal Statement of the DP Algorithm

Computes for all k and states x_k: $J_k(x_k)$: opt. cost of tail problem that starts at x_k

Go backwards, $k = N-1, \ldots, 0$, using

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} E_{w_k}\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}\big(f_k(x_k, u_k, w_k)\big) \Big\}$$

Interpretation: To solve a tail problem that starts at state x_k

Min over u_k the (kth-stage cost + Opt. cost of the tail problem that starts at state x_{k+1})

Notes:

- $J_0(x_0) = J^*(x_0)$: Cost generated at the last step, is equal to the optimal cost
- Let $\mu_k^*(x_k)$ minimize in the right side above for each x_k and k. Then the policy $\pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\}$ is optimal
- Proof by induction

Approximate DP

Approximate DP is primarily motivated by the often ENORMOUS computational demands of exact DP

Some perspectives

- The connection of theory and algorithms (convergence, rate of convergence, complexity, etc) is solid for exact DP and most of optimization
- By contrast, for approximate DP, the connection of theory and algorithms is fragile
- There is a great variety of approximate DP approaches
- Some approximate DP algorithms have been able to solve impressively difficult problems. We often do not fully understand why - a lot of research is ongoing
- There are success stories without theory
- There is theory without success stories
- The theory available is interesting but may involve some assumptions not always satisfied in practice
- Implementation is often an art
- For a given problem: The challenge is how to bring to bear the right mix from a broad array of methods and theoretical ideas