A Series of Lectures on Approximate Dynamic Programming Lecture 4

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APPROXIMATE DYNAMIC PROGRAMMING III

Outline

1. Approximation in Policy Space

2. Tail Problem Approximation

Using a Parametric Approximation Architecture for Policies

- Parametrize policies with a parameter vector $r = (r_0, \ldots, r_{N-1})$:

$$\pi(r) = \{\tilde{\mu}_0(x_0, r_0), \ldots, \tilde{\mu}_{N-1}(x_{N-1}, r_{N-1})\}$$

- Compute/train off-line the parameters using some optimization
- Great advantage: After off-line training, the on-line calculation of the controls is very fast

An important use: Implement policies obtained by approximation in value space

- Train off-line a cost function approximation and compute many state-control pairs (x_k^s, u_k^s), $s = 1, \ldots, q$
- Train a policy approximation architecture on these pairs. For example by solving for each k the least squares problem

$$\min_{r_k} \sum_{s=1}^{q} \left\| u_k^s - \tilde{\mu}_k(x_k^s, r_k) \right\|^2 + \text{(Regularization term)}$$

- This idea applies more generally. Generate many “good" state-control pairs (x_k^s, u_k^s), using a software or human “expert" and train in policy space

Cost Optimization Approach for Training a Policy Architecture

- Minimize the cost $J_{\pi(r)}(x_0)$ over r
- Aim directly for an optimal policy within the parametric class
- Gradient-based optimization may be a possibility
- Random search in the space of r is another possibility (e.g., cross entropy method)

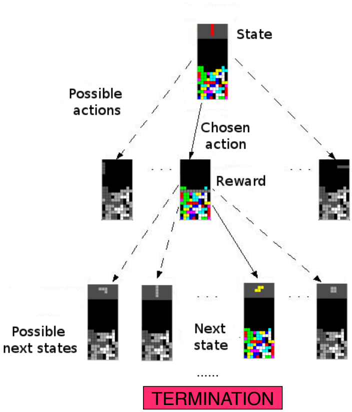
An important special case: Policy parametrization through cost function parametrization

- For a given value space parametrization $r = (r_0, \ldots, r_{N-1})$, we define

$$\tilde{\mu}_k(x_k, r_k) = \arg \min_{u_k \in U_k(x_k)} E\Big\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}\big(f_k(x_k, u_k, w_k), r_k\big) \Big\}$$

- Has achieved success in a number of test problems (e.g., tetris)

An Example: Tetris (Often Used as Testbed in Competitions)



- Number of states $> 2^{200}$ (for 10×20 board)
- $J^*(x)$: optimal score starting from board position x
- Common choice: 22 features, readily recognized by tetris players as capturing important aspects of the board position (heights of columns, etc)
- Long history of successes and failures

Lookahead Minimization

Cost-to-go Approximation

First ℓ Steps

"Future"

$$\min_{u_k,\mu_{k+1},\ldots,\mu_{k+\ell-1}} E\left\{ g_k(x_k,u_k,w_k) + \sum_{m=k+1}^{k+\ell-1} g_k\big(x_m,\mu_m(x_m),w_m\big) + \tilde{J}_{k+\ell}(x_{k+\ell}) \right\}$$

Tail problem approximation

Tail Problem Approximation Ideas

Obtain $\tilde{J}_{k+\ell}$ as the cost-to-go of a simplified problem
which is solved exactly or approximately

Enforced decomposition of interconnected subsystems

Applies to problems involving a collection I of interconnected subsystems, with each subsystem $i \in I$ applying control u_k^i at time k

- One-at-a time optimization: Obtain $\tilde{J}_{k+\ell}$ by optimizing one subsystem at a time, with controls of other subsystems fixed at nominal values
- Constraint relaxation: Artificially decouple subsystems by modifying the constraint set
- Lagrangean relaxation: Artificially decouple subsystems by using Lagrange multipliers (we will not cover)

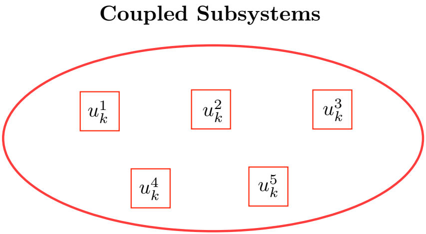
Probabilistic approximation

Simplify the probabilistic structure (e.g., replace random variables with deterministic)

Aggregation

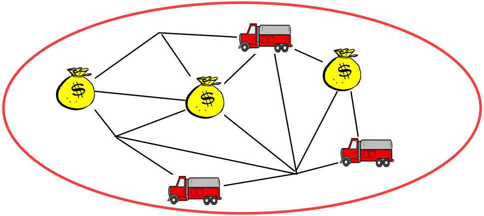
Reduce the size of the problem; e.g., by “combining" states into aggregate states

Enforced Decomposition: One Subsystem at a Time

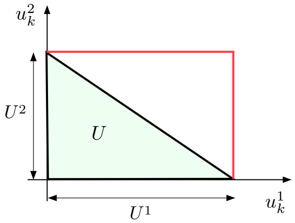


- Let $u_k = (u_k^1, \ldots, u_k^n)$, with u_k^i corresponding to the ith subsystem
- To compute cost-to-go approximation $\tilde{J}_k(x_k)$:
  - Start with subsystem 1, optimize over $(u_k^1, \ldots, u_{N-1}^1)$, with all future controls of other subsystems $i \neq 1$ held at nominal values $(\tilde{u}_k^i, \ldots, \tilde{u}_{N-1}^i)$
  - Fix the nominal values of subsystem 1 to the optimal sequence thus obtained
  - Repeat for all subsystems $i = 2, \ldots, n$ (with intermediate adjustment of the nominal control values)

Example: Optimize the Routes of n Vehicles Through a Road Network



- Aim: Execute a number of tasks with given values
- The value of a task is collected only once; a finite horizon is assumed
- This is a very complex combinatorial problem
- The single vehicle problem is typically much simpler (e.g., can be solved exactly or with a high-quality heuristic)
- Solve (suboptimally) the tail subproblem one-vehicle-at-a-time. The nominal decisions of the other vehicles can be determined using some heuristic



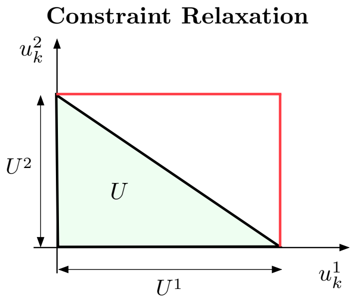
- Let $x_k = (x_k^1, \dots, x_k^n)$, $u_k = (u_k^1, \dots, u_k^n)$, $w_k = (w_k^1, \dots, w_k^n)$, with (x_k^i, u_k^i, w_k^i) corresponding to the ith subsystem
- Assume that the only coupling between subsystems is the control constraint

$$(u_k^1, \dots, u_k^n) \in U, \qquad \text{e.g., } u_k^i \in U^i,\ u_k^1 + \cdots + u_k^n \le b_k$$

- Approximate U with a decomposed constraint $U^1 \times \dots \times U^n$
- The problem decomposes into n decoupled subproblems. Let $\tilde{J}_k^i$ be the optimal cost to go functions for the ith decoupled subproblem (obtained by DP off-line)
- Use one-step lookahead with cost-to-go approximation

$$\tilde{J}_{k+1}(x_{k+1}) = \tilde{J}_{k+1}^1(x_{k+1}^1) + \cdots + \tilde{J}_{k+1}^n(x_{k+1}^n)$$

Example: Production Planning



A work center producing n product types

- x_k^i, u_k^i, w_k^i: the amounts stored, produced, and demanded of product i at time k
- State is the stock vector $x_k = (x_k^1, \ldots, x_k^n)$, where $x_{k+1}^i = x_k^i + u_k^i - w_k^i$
- U represents the (shared) production capacity of the work center
- In a more complex version (involving equipment failures), U depends on a random parameter α_k that changes according to a Markov chain

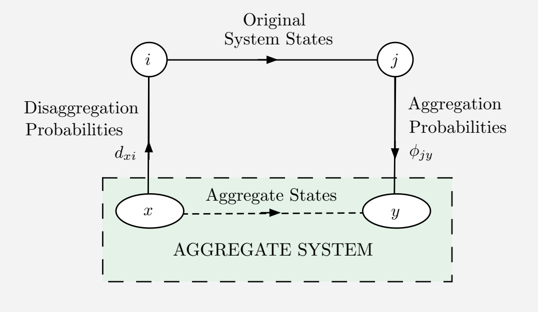
Probabilistic Approximation

Modify the probability distributions $P(w_k \mid x_k, w_k)$ to simplify the calculation of $\tilde{J}_{k+\ell}$ and/or the lookahead minimization

Certainty equivalent control (inspired by linear-quadratic control problems)

- Replace uncertain quantities with deterministic nominal values
- The lookahead and tail problems are deterministic so they can be solved by DP or by special deterministic methods
- Use expected values or forecasts to determine nominal values; update policy when forecasts change (on-line replanning)
- A variant: Partial certainty equivalence. Fix only some uncertain quantities to nominal values
- A generalization: Approximate $E\{\cdot\}$ by limited simulation

Tail Problem Approximation by Aggregation



- Construct a “smaller" aggregate tail problem by introducing aggregate states
- Use the exact costs-to-go of the aggregate tail problem as approximate costs-to-go for the original

Aggregation examples:

- State discretization-interpolation schemes
- Grouping of states into subsets, which serve as aggregate states
- Feature-based discretization; aggregate problem operates in the space of features

Concluding Remarks

What we covered

- Approximate DP for finite horizon problems with perfect state information
- Approximation in value space
- Approximation in policy space; possibly in combination with approximation in value space

What we did not cover

- Approximate DP for infinite horizon problems
  - Lookahead and rollout ideas apply with essentially no change
  - Special training methods for approximation in value space
  - Temporal difference methods [e.g., TD(λ) and others]; TD(λ) is closely related with the proximal algorithm, but implemented by simulation (see internet videolecture)
- Imperfect state information problems can be converted to (much more complex) perfect state information problems. Approximate DP is essential for any kind of solution
- A variety of important lookahead/approximation in value space schemes: Model predictive control, open-loop feedback control, and others
- Alternative cost criteria: minimax/games, multiplicative/exponential cost, etc
- Approximation error bound analysis

Thank you!