Chordal structure in computer algebra: Permanents

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Algebraic tools in discrete math

Several problems from discrete mathematics can be approached with tools from computer algebra.

**Sensor network localization:**
Find positions, given a few known fixed anchors and pairwise distances.

\[
\|x_i - x_j\|^2 = d_{ij}^2 \quad ij \in A
\]
\[
\|x_i - a_k\|^2 = e_{ij}^2 \quad ik \in B
\]

This is a system of quadratic polynomial equations. Can be solved using *Gröbner bases.*
Algebraic tools in discrete math

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**Lattice walks**
Let $\mathcal{A} \subset \mathbb{Z}^n$ consist of integer vectors. Consider a graph with vertex set $\mathbb{N}^n$ in which $\alpha, \beta$ are adjacent if $\alpha - \beta \in \mathcal{A}$. Describe the connected components.

This can be solved using a primary decomposition.
Several problems from discrete mathematics can be approached with tools from computer algebra.

**Permanents (this talk)**
Given a $n \times n$ matrix $M$, compute

$$\text{Perm}(M) := \sum_{\pi} \prod_{i} M_{i,\pi(i)}$$

sum over permutations $\pi$.
Important case: sparse matrices.
Algebraic tools in discrete math

- Many more problems can be naturally approached with algebraic tools: graph colorings, independent sets, Hamiltonian cycles, crypto problems, etc.
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Idea: exploit the graphical structure (chordality) of the problem!

Complexity aspects? Identify families of tractable instances.
Permanent computation

General facts:

- (Ryser’63) Best general and exact method; complexity $O(n 2^n)$.
- (Valiant’79) Computing the permanent of a matrix is $\#P$-hard.
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Tractable under special structure:
- (Fisher, Kasteleyn, Temperley’67) Bipartite graph is planar.
- (Barvinok’96) Rank is bounded.
- (Courcelle et al’01) Treewidth is bounded.
Graph abstractions of a matrix

The sparsity structure of a matrix $M$ can be represented with a graph. Let $a_1, \ldots, a_n$ denote the rows and $x_1, \ldots, x_n$ denote the columns. Consider these abstractions:

- $G$, bipartite adjacency graph: $(a_i, x_j) \in E$ iff $M_{i,j} \neq 0$
- $G^s$, (symmetrized) adjacency graph: $(i,j) \in E$ iff $|M_{i,j}| + |M_{j,i}| \neq 0$
- $G^X$, column graph: $(x_j, x_k) \in E$ iff exists $a_i$ such that $M_{i,j}M_{i,k} \neq 0$.

Equivalently, the adjacency graph of $M^T M$. 

Graph abstractions of a matrix

\[
\begin{pmatrix}
    a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\
    x_1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
    x_2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\
    x_3 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
    x_4 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
    x_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
    x_6 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\
    x_7 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\
    x_8 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\
    x_9 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
**Definition:** A tree decomposition of a graph $G = (X, E)$ is a pair $(T, \chi)$, where $T$ is a tree and $\chi : T \rightarrow \{0, 1\}^X$ assigns some $\chi(t) \subset X$ to each node $t$ of $T$, that satisfies the following conditions.

i. The union of $\{\chi(t)\}_{t \in T}$ is the whole vertex set $X$.

ii. For every $(x_i, x_j) \in E$, there is some node $t$ of $T$ with $x_i, x_j \in \chi(t)$.

iii. For every $x_i \in X$ the set $\{t : x_i \in \chi(t)\}$ forms a subtree of $T$.

The width $\omega$ of the decomposition is the largest $|\chi(t)|$. 
The *treewidth* of $G$ is the minimum width among all possible tree decompositions. The treewidth $\omega(G)$ of a graph $G$ can be though of as a measure of complexity: the smaller $\omega(G)$, the simpler the graph (closer to a tree).

Meta-theorem: NP-complete problems are “easy” on graphs of small treewidth.
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Meta-theorem: NP-complete problems are “easy” on graphs of small treewidth.

Alternatively, we can define treewidth in terms of chordal completions of the graph (chordal completions are in correspondance with tree decompositions).
Treewidth of matrix graphs

Simple fact:

- \( \text{tw}(G) \leq 2 \text{tw}(G^s) \).
- \( \text{tw}(G) \leq \text{tw}(G^x) + 1 \).
- For a fixed \( \text{tw}(G) \) both \( \text{tw}(G^s) \), \( \text{tw}(G^x) \) are unbounded.
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Previous tree decomposition methods are primarily based on the symmetrized graph \( G^s \) (Courcelle et al., Flarup et al., Meer), and they do not scale well with the treewidth \( O(n 2^{O(\omega^2)}) \).
Our results

- A tree decomposition method to compute permanents, based on the bipartite graph $G$, with complexity $\tilde{O}(n 2^\omega)$.
- The algorithm naturally extends to higher dimensional problems: mixed discriminants, hyperdeterminants. Complexity $\tilde{O}(n^2 + n 3^\omega)$.
- Hardness results for the case of mixed volumes.
Our results

- $\text{HDet}$: tensor
- $\text{MDisc}$: $n$ matrices
- $\text{MVol}$: $n$ polytopes
- $\text{Det}$: matrix
- $\text{Perm}$: matrix

- Hard (known)
- Easy

- Hard (small $\omega$)
- Easy (small $\omega$)

$(this \ paper)$
Basic idea

For block matrices

$$\text{Perm} \begin{pmatrix} A & A' \\ 0 & B \end{pmatrix} = \text{Perm}(A) \text{Perm}(B) = \text{Perm} \begin{pmatrix} 0 & A \\ B & B' \end{pmatrix}$$
Permanent expansion: column graph

Similarly, for the matrix

\[
M = \begin{pmatrix}
A_{1,1} & 0 & A_{1,3} & A_{1,4} & 0 \\
A_{2,1} & 0 & A_{2,3} & A_{2,4} & 0 \\
0 & C_{3,2} & C_{3,3} & C_{3,4} & 0 \\
0 & B_{4,2} & B_{4,3} & 0 & B_{4,5} \\
0 & B_{5,2} & B_{5,3} & 0 & B_{5,5}
\end{pmatrix}
\]

we have the following formula

\[
\text{Perm}(M) = \text{Perm} \begin{pmatrix} A_{1,1} & A_{1,3} \\ A_{2,1} & A_{2,3} \end{pmatrix} \text{Perm} \begin{pmatrix} C_{3,4} \end{pmatrix} \text{Perm} \begin{pmatrix} B_{4,2} & B_{4,5} \\ B_{5,2} & B_{5,5} \end{pmatrix} \\
+ \text{Perm} \begin{pmatrix} A_{1,1} & A_{1,4} \\ A_{2,1} & A_{2,4} \end{pmatrix} \text{Perm} \begin{pmatrix} C_{3,3} \end{pmatrix} \text{Perm} \begin{pmatrix} B_{4,2} & B_{4,5} \\ B_{5,2} & B_{5,5} \end{pmatrix} \\
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\]

To evaluate we need 14 multiplications: four $2 \times 2$ permanents.
Compare with: $4 \times 5! = 480$ multiplications using definition.
Permanent expansion: column graph

This simple formula exists because its column graph has a simple structure

\[
\begin{pmatrix}
M_{1,1} & 0 & M_{1,3} & M_{1,4} & 0 \\
M_{2,1} & 0 & M_{2,3} & M_{2,4} & 0 \\
0 & M_{3,2} & M_{3,3} & M_{3,4} & 0 \\
0 & M_{4,2} & M_{4,3} & 0 & M_{4,5} \\
0 & M_{5,2} & M_{5,3} & 0 & M_{5,5}
\end{pmatrix}
\]

\(G^x\) and \(T\)
Results

Lemma: Let \((T, \chi)\) be a tree decomposition of \(G^X\). Let \(c_1, \ldots, c_k\) be the children of a node \(t \in T\). Then

\[
\text{perm}(A_{T_t}, Y) = \sum_{Y} \text{perm}(A_t, Y_t) \prod_{j=1}^{k} \text{perm}(A_{T_{c_j}}, Y_{c_j})
\]

where the sum is over all \(Y = (Y_t, Y_{c_1}, \ldots, Y_{c_k})\) such that:

\[
Y = Y_t \cup (Y_{c_1} \cup \cdots \cup Y_{c_k})
\]

\[
\chi(T_{c_j}) \setminus \chi(t) \subset Y_{c_j} \subset \chi(T_{c_j}) \quad Y_t \subset \chi(t).
\]

Lemma: The above formula can be evaluated in \(\tilde{O}(k 2^\omega)\).
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\]

Lemma: The above formula can be evaluated in \(\tilde{O}(k 2^\omega)\).

Proof: Rewrite the equation as

\[
P_t(\bar{Y}) = \sum_{\bar{Y}_t \sqcup \bar{Y}_{c_1} \sqcup \cdots \sqcup \bar{Y}_{c_k} = \bar{Y}} Q_t(\bar{Y}_t) \prod_{j=1}^k Q_{c_j}(\bar{Y}_{c_j})
\]

and use the fast subset convolution (Björklund et al. 07).
Permanent expansion: bipartite graph

Perm$(M) = perm(\{a_1, a_2\}, \{x_1, x_4\}) \cdot perm(\{a_3, a_4, a_5\}, \{x_2, x_3, x_5\})$

$+ perm(\{a_1, a_2, a_3\}, \{x_1, x_3, x_4\}) \cdot perm(\{a_4, a_5\}, \{x_2, x_5\})$

$- M_{3,3} \cdot perm(\{a_1, a_2\}, \{x_1, x_4\}) \cdot perm(\{a_4, a_5\}, \{x_2, x_5\})$

 perm$(\{a_1, a_2, a_3\}, \{x_1, x_3, x_4\}) = M_{3,3} \cdot perm(\{a_1, a_2\}, \{x_1, x_4\}) + M_{3,4} \cdot perm(\{a_1, a_2\}, \{x_1, x_3\})$

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To evaluate we need 16 multiplications.
**Theorem:** Let $M$ be a matrix with associated bipartite graph $G$. Then we can compute $\text{Perm}(M)$ in $\tilde{O}(n2^\omega)$ where $\omega = \text{tw}(G)$. 

The mixed discriminant of $n$ matrices of size $n \times n$ is

$$\text{Disc}(A_1, \ldots, A_n) := \sum_{\pi, \rho} \text{sgn}(\pi) \text{sgn}(\rho) \prod_{i} A_{\pi(i)} \rho(i)$$

where the sum is over pairs of permutations $\pi, \rho$. 

**Theorem:** Let $M$ be a list of matrices with associated tripartite graph $G$. Then we can compute $\text{Disc}(M)$ in $\tilde{O}(n^2 + n^3 \omega)$, where $\omega = \text{tw}(G)$. 
Theorem: Let $M$ be a matrix with associated bipartite graph $G$. Then we can compute $\text{Perm}(M)$ in $\tilde{O}(n 2^\omega)$ where $\omega = \text{tw}(G)$.

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Theorem: Let $M$ be a list of matrices with associated tripartite graph $G$. Then we can compute $\text{Disc}(M)$ in $\tilde{O}(n^2 + n 3^\omega)$, where $\omega = \text{tw}(G)$. 
Further generalizations

The first Cayley hyperdeterminant is the simplest generalization of the determinant to multidimensional arrays.

**Theorem:** Let $M$ be a square $d$-dimensional tensor of length $n$, with $d$-partite graph $G$. We can compute its *hyperdeterminant* in $\tilde{O}(n^2 + n 3^\omega)$.

The mixed volume is a geometric generalization of determinants and permanents to arbitrary convex bodies on $\mathbb{R}^n$.

**Theorem:** Computing mixed volumes of $n$ zonotopes remains #P-hard when their associated graph has bounded treewidth.
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**Theorem:** Computing mixed volumes of $n$ zonotopes remains $\#P$-hard when their associated graph has bounded treewidth.
Summary


Thanks for your attention!