Compressive strength and elastic properties of a transversely isotropic calcareous mudstone

Douglas E. Miller¹, Richard Plumb², Gregory Boitnott³

Abstract

This paper reports measurements of compressive strength, and static and dynamic elastic properties on a block of calcareous mudstone retrieved from an exploration well. Measurements of mechanical properties indicate that the mudstone is anisotropic with respect to all three properties and that the magnitude of anisotropy decreases with increasing confining stress. A detailed analysis of the elastic moduli computed using small unload reload cycles and simultaneous ultrasonic wave velocities shows both strong anisotropy and strong anelasticity. Surprisingly, the measurements are consistent with a mathematical description as a special type of anisotropic linear viscoelastic medium that is obtained by adding a set of cracks (or "joints", or "fractures") to an isotropic standard linear solid and is determined by density plus four parameters defining the viscoelastic solid and the excess normal compliance associated with the cracks. The model predicts a full set of TIV parameters for both low and high strain rate behavior.

Introduction

Wellbore instability in shale and mudstone is not news. But, it is unusual that large samples of the problem formation are retrieved from an exploration well. When a driller is faced with wellbore instability, the first thing he wants to know is what mud weight should be used to stop it. Answering this question requires information about the mechanical properties and the state of stress in the problem formations, or at least a reasonable approximation of them. For the case in question, there were no logs or cores from anywhere on the structure. So our rock sample provided the

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first opportunity to characterize the mechanical properties of the problem formation.

It is widely accepted that fine-grained argillaceous rocks are anisotropic with respect to elastic moduli and compressive strength. Moreover, elastic moduli measured in standard triaxial compression tests are commonly found to be significantly less than those computed from bulk density and elastic wave velocities measured on the same rock either in the lab or in the field. Whereas there is a considerable literature treating these issues separately, very little has been published considering them together for a single sample of rock.

Following is a description of a rock, laboratory tests performed on it, and an analysis of the measurements. Results of the mechanical testing are presented in two parts. The first part discusses the test protocol and basic observations concerning the static moduli and strengths as a function of confining pressure. Observations made within this context illustrate the interrelationships and stress dependence of key mechanical properties needed to build a geomechanical model for this material. The second part combines measurements from unload-reload cycles and ultrasonic elastic wave velocities to show that the mudstone’s mechanical properties are well described by a TIV linear viscoelastic model with a significant rate dependent component leading to a strong shift between static and dynamic moduli.

**Rock Characterization**

![Sample of mudstone](image)

**Figure 1.** Sample of mudstone recovered from an exploration well. Notice the marks indicating the location of and size of plug samples. The circular markings on the rock are 20mm in diameter.

The rock described in this paper is a calcareous mudstone retrieved from the top-hole section of an exploration well during a hole cleaning operation (Figure 1).
Several blocks of rock, like the one shown in Figure 1 comprise two textures: a calcareous mudstone and fine-grained calcareous grainstone. The focus of this paper is the mudstone as it was responsible for the majority of the wellbore instability problems.

Figure 2 shows backscattered electron microscope images of the mudstone taken at three different magnifications. Bedding, visible in all three images, is defined by pyrite nodules (bright components).
Figure 2. Backscattered SEM images of the mudstone sample: (a) 50x magnification showing homogeneous texture and traces of bedding; (b) 312x magnification showing bedding defined by organic matter (dark) and pyrite nodules (bright) but no cementation; (c) 1250x magnification showing high volume fraction of clay minerals. Some of the porosity, visible at the boundaries between different mineral phases, may be artifacts of unloading.

Modal mineralogy was measured using a scanning energy dispersive X-ray system. Mineral assemblages obtained for mudstone samples cut parallel and perpendicular to bedding were similar to each other whereas those of the mudstone and the grainstone were distinctly different (Table 1). Figure 3 shows a mineral map of a 4 cm x 9 cm section of mudstone. The map demonstrates that smectite clay minerals form the continuous, load-bearing solid phase in the mudstone (olive green). Floating in the clay are grains of serpentine (brown), calcite, and dolomite (blues).

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4 http://www.fei-natural-resources.com/uploadedFiles/Documents/NaturalResources/Products/BR0037-QEMSCAN650-web.pdf
Figure 3. Mineral composition map of the mudstone illustrating the homogeneous nature of the rock at the cm scale.

Table 1. Mineralogy determined by scanning energy dispersive X-ray

<table>
<thead>
<tr>
<th>Mineral</th>
<th>Calcareous Mudstone</th>
<th>Carbonate Grainstone</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>parallel perpendicular</td>
<td>parallel perpendicular</td>
</tr>
<tr>
<td>Quartz</td>
<td>2.6</td>
<td>8.5</td>
</tr>
<tr>
<td>Calcite</td>
<td>12.8</td>
<td>36.2</td>
</tr>
<tr>
<td>Dolomite</td>
<td>1.2</td>
<td>4.1</td>
</tr>
<tr>
<td>Albite</td>
<td>0.7</td>
<td>1.6</td>
</tr>
<tr>
<td>K-feldspar</td>
<td>2.9</td>
<td>5.0</td>
</tr>
<tr>
<td>Illite</td>
<td>1.8</td>
<td>0.8</td>
</tr>
<tr>
<td>Smectite</td>
<td>60.0</td>
<td>22.2</td>
</tr>
<tr>
<td>Serpentine</td>
<td>14.7</td>
<td>17.3</td>
</tr>
<tr>
<td>Chlorite</td>
<td>1.7</td>
<td>2.5</td>
</tr>
<tr>
<td>Pyrite</td>
<td>1.0</td>
<td>0.6</td>
</tr>
<tr>
<td>Rutile</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>Chromite</td>
<td>0.0</td>
<td>0.4</td>
</tr>
<tr>
<td>Fe-oxides</td>
<td>0.0</td>
<td>0.4</td>
</tr>
<tr>
<td>Others</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Porosity estimated for the mudstone is approximately 9%. Porosity was computed from measurements of grain density and bulk density. Plug ends were used to obtain a measure of grain density. They were crushed, pulverized and then vacuum oven dried at 110 °C prior to testing in a helium pycnometer. These density porosity may be slightly underestimated because they neglect effects of adsorbed water in
the plug samples (room dry condition).

**Mechanical testing**

The primary objective of the laboratory testing was to measure parameters required by standard geomechanics software: Young’s modulus, Poisson’s ratio, unconfined compressive strength, and friction angle. In addition we wanted to evaluate the magnitude of rock anisotropy, and we needed laboratory measurements to calibrate mechanical properties derived from geophysical logs. Toward this end, triaxial compression tests to failure were run at confining pressures of 0 MPa, 20 MPa and 40 MPa. Large pieces of rock retrieved from the well were subsampled to obtain 6 plug samples of the mudstone: three perpendicular and three parallel to bedding. In order to orient the plugs properly, pieces were trimmed from the hand samples, hand polished, and examined by SEM to identify grain scale fabric. Hand polishing with a small amount of water was found to enhance the exposure of the grain scale fabric of the mudstone.

Sample deformation was measured using strain gauges bonded directly to the plug sample. In the case of the unconfined compression tests, samples were instrumented with two axial and two radial strain gauges, each diametrically opposite the other similarly oriented gauge (Figure 4).

For the tests at elevated confining pressure, two axial gauges and one radial gauge were used. Due to the unusually small size of the plugs (19mm in diameter), each gauge averaged over a significant fraction of the plug’s circumference, so the orientation dependence of radial strain could not be resolved. For the horizontal plugs, radial gauges were oriented at 45° to bedding to provide an “average” radial response.
Compression tests were performed on room dry samples. Axial loading at fixed confining pressure was controlled in displacement feedback. Periodic small amplitude unload-reload cycles were performed to obtain a measure of unloading moduli as a function of deformation. Ultrasonic velocities were measured prior to each unloading cycle, providing a comparison of dynamic moduli with the static loading and static unloading moduli. Figure 5 shows an example of the test protocol.

For each test, a static Young's modulus and Poisson's ratio was computed from slopes of the stress vs. strain data fit to an early stage loading segment for each test at a differential stress level of approximately 15 MPa. Corresponding reloading moduli were computed based on fits to the data from the subsequent unload-reload cycle. The amplitude of the stress cycles is approximately 5 MPa. Elastic properties were computed from measurements of bulk density and ultrasonic compressional and shear velocities. Velocities were measured by time of flight method on axially propagating ultrasonic waves. A compressional and 2 orthogonally polarized shear waves were acquired at each measurement point. Shear wave polarizations were oriented with respect to bedding for the horizontal plugs. The different methods of computing elastic moduli are designated by subscripts: "t", "r" and "u" refer to tangent loading, unload-reload cycle and ultrasonic measurements respectively.

Compressive strengths were measured by manually identifying the maximum differential axial stress attained during the test. Samples were taken to various stages of post failure depending on particular circumstances of the test.
Figure 5. Example of Stress vs. strain results for a typical test. Young’s moduli and Poisson’s ratio’s are computed for each test by measuring the slopes for the stress vs. strain curves over early stage segments of the loading where deformation was linear. Unloading moduli are computed by fitting slopes to the subsequent unloading cycle. Dynamic moduli are computed from the velocity measurements corresponding to the same loading increments.

Results from compression tests

A summary of test results is shown in Table 2. Results revealed the expected anisotropy with respect to loading direction, with the horizontal plugs yielding systematically greater Young’s moduli and strengths than the vertical plugs. Note that the tabulated Young’s moduli and Poisson ratios were calculated directly from slopes of the load or reload crossplots and, while they need to be properly interpreted, no assumption of isotropy was used in this calculation.
Table 2: Summary of test results

<table>
<thead>
<tr>
<th>Plug</th>
<th>Orientation</th>
<th>Confining Strength</th>
<th>Load</th>
<th>E_t</th>
<th>ν_t</th>
<th>Reload</th>
<th>E_r</th>
<th>ν_r</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>MPa</td>
<td>MPa</td>
<td>GPa</td>
<td></td>
<td>GPa</td>
<td></td>
</tr>
<tr>
<td>1bv1</td>
<td>Vertical</td>
<td>0</td>
<td>31</td>
<td>4.6</td>
<td>0.06</td>
<td>9.1</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td>1av1</td>
<td></td>
<td>20</td>
<td>76</td>
<td>4.9</td>
<td>0.12</td>
<td>13.4</td>
<td>0.18</td>
<td></td>
</tr>
<tr>
<td>1av2</td>
<td></td>
<td>40</td>
<td>119</td>
<td>4.9</td>
<td>0.11</td>
<td>14.3</td>
<td>0.22</td>
<td></td>
</tr>
<tr>
<td>1bh1a</td>
<td>Horizontal</td>
<td>0</td>
<td>53</td>
<td>9.1</td>
<td>0.19</td>
<td>15.3</td>
<td>0.17</td>
<td></td>
</tr>
<tr>
<td>1ah1</td>
<td></td>
<td>20</td>
<td>97</td>
<td>8.9</td>
<td>0.06</td>
<td>16.7</td>
<td>0.17</td>
<td></td>
</tr>
<tr>
<td>1bh1b</td>
<td></td>
<td>40</td>
<td>124</td>
<td>7.6</td>
<td>0.07</td>
<td>15.4</td>
<td>0.15</td>
<td></td>
</tr>
</tbody>
</table>

Results are summarized graphically in figures 6-8. Strengths plotted in Mohr Coulomb space yield two linear and parallel failure envelopes, with a suggestion of downward curvature in the horizontal plugs at the highest stresses (Figure 5). Friction angle is 22 degrees.

![Strength Mohr Coulomb](image)

**Figure 6.** Measured strengths ($\sigma_1$) as a function of confining pressure ($\sigma_2$) plotted in Mohr Coulomb presentation.

Figures 7 and 8 share a common symbology and color-coding. Triangles represent measurements on vertical plugs, squares depict horizontal plugs. Parameters measured on static loading are shown in blue, those measured using unload-reload data are plotted in red and green represents dynamic data. Figure 7 summarizes measurements of Young’s modulus. No systematic pressure dependence is observed for $E_t$, $E_r$ or $E_u$. The rock is stiffer parallel to bedding than perpendicular to bedding and $E_u > E_r > E_t$, as expected.
**Figure 7.** Summary of measurements of Young’s modulus as a function of confining pressure. Note the rock is stiffer parallel to bedding and \( E_u > E_r > E_t \).

**Figure 8.** Summary of measurements of Poisson’s ratio as a function of confining pressure. Note, there is no systematic pressure dependence on Poisson’s ratio and generally \( \nu_r > \nu_t \).

Poisson’s ratios are plotted as a function of confining pressure in **Figure 8**. In general, we find no systematic dependence of Poisson’s ratio on confining pressure, but \( \nu_r \) is systematically greater than \( \nu_t \). Considering that the rock is a mudstone it is interesting that all measures of Poisson’s ratio are relatively low. In the next section
we find a clear relationship between Poisson’s ratio obtained from reloading cycles and moduli derived from ultrasonic measurements.

Mechanical anisotropy decreases significantly as confining pressure increases, as seen in Figure 9. Strength anisotropy is observed to be nearly identical to the anisotropy in the unloading modulus, with the material becoming nearly isotropic at the highest confining pressures. The anisotropy in the loading Young’s moduli decreases, but maintains measureable anisotropy (The ratio of Young moduli parallel and normal to bedding, $E/E'$ is greater than 1) even at the highest confining pressures.

![Strength and Modulus Anisotropy](image)

**Figure 9.** Strength and Young’s modulus anisotropy as a function of confining pressure. Note the strong correspondence between the anisotropy in unloading modulus and the anisotropy in strength, while the anisotropy in loading modulus follows a higher trend.

In summary we find for the mudstone:

- Compressive strength is greater measured parallel to bedding than perpendicular to it.
- All measures of Young’s moduli are greater measured parallel to bedding than perpendicular to it.
- $E_r > E$, and $\nu_r > \nu_t$.
- There is a strong dependence of compressive strength on confining pressure while the dependence of elastic moduli on confining pressure is relatively weak.
- The magnitude of elastic anisotropy decreases with increasing confining pressure.

In the next section, a more nuanced view of these data demonstrates that the mudstone is characterized by a TIV elastic symmetry and a strong viscoelastic component.
Calculation of Anisotropic Elastic Moduli

A wide range of notation has been used in the literature describing anisotropic media. For this section, we have adopted Voigt notation for compliances and moduli. For velocities we use the convention that the first subscript indicates the direction of propagation and the second subscript indicates the direction of polarization. We treat 3 as the vertical axis and refer to media that have rotational symmetry around this axis as TIV ("Transversely Isotropic with a Vertical symmetry axis"). A discussion of alternate notations is included in the Appendix.

![Calculation of Anisotropic Elastic Moduli](image)

**Figure 10.** *Velocities estimated from ultrasonic measurements.*

Figure 10 shows axial ultrasonic measurements recorded during the early part of the test in Figure 2. The raw measurements, indicated by colored symbols, were interpolated to give values at 15 MPa axial stress (where reloading cycles were recorded). Those values are indicated by the black dots and labeled on the left of the plot.

The clear mismatch of both compressional ($V_{11} = 1.25 \times V_{33}$) and shear ($V_{12} = 1.17 \times V_{13}$) wavespeeds is an indication of strong anisotropy. The lack of shear splitting when measured on the vertical plug is an indication of transverse isotropy. The small (less than 2%) mismatch between crossplane shear measured on the horizontal plug ($V_{13}$) and on the vertical plug ($V_{31}, V_{32}$) is an indication of the small but non-zero sample heterogeneity at the scale of the plug size.

Table 4 summarizes the moduli (squared velocity times density) derived from the ultrasonic velocities. These are four of the five moduli needed to characterize a
general TIV medium. The fifth ($C_{13}$) cannot be calculated directly from ultrasonic data without additional information about the medium or an off-axis measurement.

Table 4

<table>
<thead>
<tr>
<th>Modulus</th>
<th>$C_{11}$</th>
<th>$C_{33}$</th>
<th>$C_{66}$</th>
<th>$C_{55}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(GPa)</td>
<td>19.5</td>
<td>12.4</td>
<td>8.12</td>
<td>5.91</td>
</tr>
</tbody>
</table>

Figure 11. Reloading cycles at 15 MPa for six strain gauges. VA1 and VA2 are axial gauges on the vertical plug. VR is the radial gauge on the vertical plug, etc. Slopes of the superimposed lines correspond to the compliances listed in Table 4.

Figure 11 shows a set of reloading cycles from the test like the one shown in Figure 2. Straight lines that fit the data are superimposed. The slopes of these lines are direct measures of the compliances listed in Table 5. Table 5 also lists moduli derived from these compliances using the TIV versions of equations (A3) to (A8). The fifth modulus ($C_{55}$) cannot be calculated directly from axial load cycles without additional information about the medium or shear measurements.
Table 5

<table>
<thead>
<tr>
<th>Compliance (1/GPa)</th>
<th>$s_{11}$</th>
<th>$s_{33}$</th>
<th>$s_{12}$</th>
<th>$s_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0714</td>
<td>0.114</td>
<td>-0.012</td>
<td>-0.012</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Modulus (GPa)</th>
<th>$c_{11}$</th>
<th>$c_{33}$</th>
<th>$c_{66}$</th>
<th>$c_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.2</td>
<td>9.03</td>
<td>5.91</td>
<td>1.57</td>
<td></td>
</tr>
</tbody>
</table>

Three moduli, $c_{11}$, $c_{33}$, and $c_{66}$, are estimated by both the ultrasonic and quasi-static methods. They disagree in a very systematic way. Each ultrasonic modulus is 38% larger than the corresponding quasi-static modulus. Several other Algebraic Relations can be observed:

1. $s_{12} = s_{13}$. This is clearly seen in Figure 11 where radial load cycle data from the two plugs are seen to match each other and match the overlain line so well as to be nearly indistinguishable. Considering the vertical plug, the slope of the line gives $s_{13}$ directly. Since the radial gauge was placed on the horizontal plug to be sensitive to an average of $S12$ and $S13$, and since that average is equal to $s_{13}$, it follows that $s_{12} = s_{13}$.

2. Dimensionless ratios of moduli are the same whether measured by ultrasonics (Table 5) or reload cycles (Table 6).

3. Load Cycle $c_{66} = $ Ultrasonic $C_{55}$

Ockham’s principle of parsimony suggests that entities are not to be multiplied beyond necessity.

The assumption that Algebraic Relation (2) applies to all moduli is a parsimonious solution to the problem of finding a matched pair of elastic descriptions that fit all the data.

As above and for the remainder of this section, we use uppercase letters for ultrasonically derived moduli, lowercase letters for reloading moduli.

Since $C_{11}/c_{11} = 1.38$, and $c_{13} = 1.57$, $C_{13} = C_{11} * c_{13}/C_{11} = C_{11} * c_{13}/c_{11} = C_{11}/c_{11} * c_{13} = 1.38 * 1.57 = 2.16$ GPa.

Similarly, $c_{55} = C_{55}/1.38 = 4.30$ GPa.
Interpretation of Anisotropic Elastic Moduli

A simple physical model that agrees with these measurements can be found in the domain of linear viscoelasticity.

The linearity of the stress/strain cycles seen in Figure 11 strongly indicates that the mismatch between ultrasonic and reloading moduli is not a strain-amplitude effect. While it is true that the strain amplitudes in the load cycles (1 to 4 \(10^{-4}\)) are three orders of magnitude larger than the strain amplitudes in the ultrasonic signals (roughly \(3 \times 10^{-7}\)), each cycle can be regarded as a sequence of many small-amplitude subsequences, all with the same slope. It would violate Ockham’s principle to suppose that the slope at invisibly small scales does not agree with the consistent slope seen at measurable scales.

The load cycle time was 120 seconds. Thus, the strain rate for the load cycles is about \(3 \times 10^{-2}\) sec\(^{-1}\). The ultrasonic frequency was 500 kHz, hence the ultrasonic strain rate was about \(2 \times 10^{-4} \times 5 \times 10^5 = 1 \times 10^2\) sec\(^{-1}\). Thus, while the strain amplitude for the ultrasonics is three orders of magnitude lower than the strain amplitude in the load cycles, the strain rate is four orders of magnitude higher.

The theory of linear viscoelasticity is very well established (e.g. Biot 1954, Lakes 1998, Carcione, 2001, Shearer, 2009). A linear viscoelastic medium satisfies a rate-dependent form of Hook’s law that can be thought of as family of rate-dependent, complex-valued elastic tensors. A key observable for all linearly viscoelastic materials is that stiffness is an increasing function of strain rate.

While the theory was developed in full anisotropic generality, most of the literature has been directed at man-made isotropic materials. We do not know a reference to a study of shales aimed at an anisotropic viscoelastic description.

A mathematically simple form of viscoelasticity that has been studied in connection with solid earth seismology (e.g. Shearer, 2009 section 6.6.4) is the standard linear solid which adds rate-dependent terms to Hook’s law and is characterized by a pair of relaxation time constants whose ratio (the “relaxation ratio”) is equal to the ratio between “unrelaxed” and “relaxed” moduli. In its simplest form, a single relaxation ratio, which we will denote as \(\varphi\), suffices for all moduli (shear, Young, bulk, etc). An isotropic standard linear solid (ISLS) can be parameterized for example, by a density \(\rho\), an unrelaxed Young modulus \(E^U\), a Poisson ratio \(\eta\), and a relaxation ratio \(\varphi\) which is the ratio between unrelaxed and relaxed moduli.

Strikingly and somewhat surprisingly, our measurements admit a representation by a special type of anisotropic linear viscoelastic medium that is obtained by adding a set of cracks (or “joints”, or “fractures”) to an ISLS.

As shown in the Appendix, our Algebraic Relation (1) is a necessary and sufficient
condition for a TIV medium to be represented as a fractured isotropic model in the sense of Schoenberg and Douma (1988). Such a medium is determined by a background isotropic medium plus dimensionless normal and tangential fracture excess compliances. Sayers and Kachanov (1995) describe a mathematically equivalent formalism based on assumptions of a transversely isotropic distribution of penny-shaped cracks. Equations connecting the two formalisms are given in the Appendix. To be consistent with our use of Poisson ratio and relaxation ratio, we will use dimensionless compliance ratios $\chi$ and $\eta$ defined in the Appendix.

Because both the fracture theory and the viscoelastic theory are linear in the stresses and strains, they can be combined without significant modification. The resulting fractured isotropic standard linear solid (FISLS) has anisotropic relaxed and unrelaxed moduli which are obtained from the background isotropic relaxed and unrelaxed moduli exactly as in the elastic case. Details are given in the Appendix. As with the relaxation parameter with respect to moduli, the use of the same dimensionless fracture parameters for relaxed and unrelaxed states is not required, but can serve to constrain and simplify the model.

As shown in the Appendix, a FISLS satisfies our Algebraic Condition (3) if and only if it satisfies $\varphi = \chi$. We will call such a medium a "Special Fractured Isotropic Standard Linear Solid" (SFISLS). That’s a six-letter acronym for a medium which is determined by five parameters: density $\rho$, unrelaxed Young modulus $E^U$, a Poisson ratio $\nu$, a relaxation ratio $\varphi$, and a dimensionless compliance ratio $\eta$.

Given any SFISLS, equations described in the Appendix yield values for all the directly measured quantities listed in the first rows of Tables 1 and 2 (four quasi-static compliances plus five ultrasonic velocities). A simple optimization minimizing the total squared error, was used to obtain the SFISLS listed in Table 6.

While the condition that a single relaxation constant condition applies to all moduli is not generally assumed for manufactured viscoelastic materials, it is not unheard of. Koppelmann (1958) shows measurements indicating that plexiglass (M33) has this property. We do not (yet) know a good physical account for the special relation $\varphi = \chi$. 

Table 6. Parameters defining the best-fit Special Fractured Isotropic Standard Linear Solid.

<table>
<thead>
<tr>
<th>density (g/ml)</th>
<th>relaxed Young modulus (GPa)</th>
<th>Poisson ratio (nd)</th>
<th>relaxation ratio (nd)</th>
<th>compliance ratio (nd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.38</td>
<td>13.6</td>
<td>0.148</td>
<td>1.38</td>
<td>.645</td>
</tr>
</tbody>
</table>

Table 7 lists values associated with the best-fit Special Fractured Isotropic Standard Linear Solid from Table 6. Values associated with the unrelaxed (ultrasonic, high strain-rate) behavior are in uppercase ($C_{ij}$, $S_{ij}$, $V_{ij}$ etc). Values associated with the unrelaxed (ultrasonic, high strain-rate) behavior are in lowercase ($c_{ij}$, $s_{ij}$, $v_{ij}$ etc). Equations used to calculate these values are in the Appendix. Values compared to measured values in the optimization calculation are highlighted in red.
Table 7. Derived properties of the best-fit Special Fractured Isotropic Standard Linear Solid from Table 6. Model parameters are in blue. Derived values fit to measured data are in red.

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>$\rho$</th>
<th>$E^R$</th>
<th>$\nu$</th>
<th>$\varphi$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.38</td>
<td>13.6</td>
<td>0.148</td>
<td>1.38</td>
<td>0.645</td>
</tr>
</tbody>
</table>

Unrelaxed Modulus | $C_{11}$ | $C_{33}$ | $C_{66}$ | $C_{55}$ | $C_{13}$ | $C_{12}$ |
| (GPa)            | 19.5    | 12.4   | 8.12    | 5.91     | 2.16    | 3.21     |

relaxed modulus | $c_{11}$ | $c_{33}$ | $c_{66}$ | $c_{55}$ | $c_{13}$ | $c_{12}$ |
| (GPa)            | 14.2    | 9.03    | 5.91    | 4.29     | 1.57    | 2.33     |

Unrelaxed Velocity | $V_{11}$ | $V_{33}$ | $V_{12}$ | $V_{13}$ |
| (km/sec)          | 2.86    | 2.28    | 1.85    | 1.58     |

relaxed velocity | $v_{11}$ | $v_{33}$ | $v_{12}$ | $v_{13}$ |
| (km/sec)          | 2.44    | 1.95    | 1.58    | 1.34     |

Unrelaxed Compliance | $S_{11}$ | $S_{33}$ | $S_{66}$ | $S_{55}$ | $S_{13}$ | $S_{12}$ |
| (1/GPa)           | 0.0536  | 0.0833  | 0.123   | 0.169    | -0.00794 | -0.00794 |

relaxed compliance | $s_{11}$ | $s_{33}$ | $s_{66}$ | $s_{55}$ | $s_{13}$ | $s_{12}$ |
| (1/GPa)           | 0.0737  | 0.115   | 0.169   | 0.233    | -0.0109  | -0.0109  |

Unrelaxed Engineering | $E$ | $E'$ | $G'$ | $G$ | $\nu$ | $\nu'$ |
| (GPa) or (nd)     | 18.7   | 12.0   | 8.13  | 5.92 | 0.148 | 0.0954 |

relaxed engineering | $e$ | $e'$ | $g'$ | $g$ | $\nu$ | $\nu'$ |
| (GPa) or (nd)     | 13.6   | 8.70   | 5.92  | 4.29 | 0.148 | 0.0954 |

Thomsen | $\varepsilon$ | $\gamma$ | $\delta$ |
|         | 0.284   | 0.188   | 0.142   |

Sayers/Kachanov Schoenberg/Douma | $B_T$ | $B_N$ | $B_N/B_T$ | $E_T$ | $E_N$ |
| (1/GPa) or (nd) | 0.0462 | 0.0298 | 0.645   | 0.375 | 0.586 |

$b_T$ | $b_N$ |
| (1/GPa) | 0.0635 | 0.0409 |
Discussion

Analysis of laboratory data indicate that our mudstone is a transversely isotropic rock which responds in a linear viscoelastic manner to small load cycles but is inelastic when subjected to high differential stress. A conceptual model of mudstone and its response to stress is illustrated in Figure 12. Figure 12a shows a macroscopic view of the rock including mechanical properties of the layered TIV medium; 12b is a schematic of the microstructure; 12c is a schematic stress strain curve illustrating the mudstone’s response to three loading protocols. Macroscopic layering is primarily responsible for the directional dependence on compressive strength and elastic properties. At the microscopic scale, deformation is viscoelastic so long as induced shear stresses do not exceed the inter-particle frictional strength. Since smectite clay minerals are the continuous load-bearing mineral, plastic deformation should occur when the frictional strength of smectite is exceeded. When this happens the particles move past one another resulting in a low Young’s modulus and non recoverable plastic strain.

![Figure 12. Conceptual geomechanical model of mudstone.](image)

Systematic errors in geomechanical models of mudstone are to be expected when the TIV and viscoelastic nature of the rock is not honored. Predictions of earth stress, rock strength and wellbore stability will be affected. Errors in wellbore stability are compounded since earth stress and rock strength are inputs to wellbore stability models. Finally the viscoelastic nature of the mudstone influences the appropriate use of geophysical logs to derive elastic parameters for use in the stress and wellbore stability models. We have seen here that the moduli for the mudstone depend on strain rate and stress level. Today these complexities are handled by calibrating geophysical logs to core measurements. A better understanding of the intrinsic rock properties that govern the mechanical properties of mudrocks will improve detection of anisotropic strata and will improve the inversion of
geophysical data for rock mechanical properties and the geomechanical calculations that depend upon them.

Summary

This paper has characterized the mechanical properties of a calcareous mudstone. We presented measurements of compressive strength, and static and dynamic elastic properties on a block rock retrieved from an exploration well. Measurements of mechanical properties indicate that the mudstone is anisotropic with respect to all three properties and that the magnitude of anisotropy decreases with increasing confining stress. A detailed analysis of the elastic moduli computed using small unload reload cycles and simultaneous ultrasonic wave velocities shows both strong anisotropy and strong anelasticity. Surprisingly, the measurements are consistent with a mathematical description as a special type of anisotropic linear viscoelastic medium that is obtained by adding a set of cracks (or "joints", or "fractures") to an isotropic standard linear solid and is determined by density plus four parameters defining the viscoelastic solid and the excess normal compliance associated with the cracks. The model predicts a full set of TIV parameters for both low and high strain rate behavior.

Acknowledgments

The authors are grateful to Neil Judge and Don Schultz of Hunt Oil Co. for providing the rock sample and for permission to publish the rock mechanics data. They also want to thank Herman Lemmens and Gerda Gloy of FEI Corporation for providing the mineralogical analysis.
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Appendix

Orthorhombic Media

Hook’s law for an orthotropic medium can be written in terms of moduli

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{12} & C_{22} & C_{23} \\
C_{13} & C_{23} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{23} \\
2\epsilon_{13} \\
2\epsilon_{12}
\end{bmatrix}
\]  

(A1)

or in terms of compliances

\[
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{23} \\
2\epsilon_{13} \\
2\epsilon_{12}
\end{bmatrix}
= \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{12} & S_{22} & S_{23} \\
S_{13} & S_{23} & S_{33}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
2\sigma_{23} \\
2\sigma_{13} \\
2\sigma_{12}
\end{bmatrix}
\]  

(A2)

When written as above in Voigt notation, the modulus and compliance tensors are matrix inverses to one another:

\[
[S_{ij}] = [C_{ij}]^{-1}
\]  

(A3)

Thus,

\[
S_{44} = C_{44}^{-1}, \quad S_{55} = C_{55}^{-1}, \quad S_{66} = C_{66}^{-1}.
\]  

(A4)

\[
S_{11} = (C_{22}C_{33} - C_{23}^2)/D, \\
S_{22} = (C_{11}C_{33} - C_{13}^2)/D, \\
S_{33} = (C_{11}C_{22} - C_{12}^2)/D.
\]  

(A5)
and

\[
S_{12} = \frac{(C_{23}C_{13} - C_{12}C_{33})}{D}, \\
S_{13} = \frac{(C_{12}C_{23} - C_{13}C_{22})}{D}, \\
S_{23} = \frac{(C_{12}C_{13} - C_{11}C_{23})}{D},
\]

where

\[
D = \det \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{12} & C_{22} & C_{23} \\
C_{13} & C_{23} & C_{33}
\end{bmatrix}
= C_{11}C_{22}C_{33} - C_{11}C_{23}C_{23} \\
+ C_{12}C_{23}C_{13} - C_{12}C_{12}C_{33} \\
+ C_{13}C_{12}C_{23} - C_{13}C_{22}C_{13}.
\]

\[(A6)\]

The above equations are for calculating compliances from given moduli. To reverse the calculation, one simply interchanges the letters ‘C’ and ‘S’ in all expressions.

Axial velocities in orthorhombic media have a simple relation to moduli. \(\rho\) will denote density. Adopting a convention that \(V_{ij}\) is the velocity for propagation parallel to the i-axis with polarization parallel to the j-axis, we find in full tensor notation \(\rho V_{ij}^2 = C_{ijij}\). Thus, using Voigt notation for the moduli,

\[
\rho V_{11}^2 = C_{11}, \quad \rho V_{12}^2 = C_{66}, \quad \rho V_{13}^2 = C_{55},
\]

et cetera.

\[\text{TIV Media}\]

When the medium is Transversely Isotropic with respect to the 3-axis (TIV) the additional relations

\[
C_{44} = C_{55}, \quad C_{11} = C_{22}, \quad C_{13} = C_{23}, \\
C_{11} = C_{12} + 2C_{66}, \\
S_{44} = S_{55}, \quad S_{11} = S_{22}, \quad S_{13} = S_{23}, \\
S_{11} = S_{12} + S_{66}/2
\]

\[(A9)\]
are satisfied. For the remainder of the Appendix, we assume TIV symmetry. In this case

\[ D = C_{33}(C_{11}^2 - C_{12}^2) - 2C_{13}^2(C_{11} - C_{12}) \]

\[ = (C_{11} - C_{12})(C_{33}(C_{11} + C_{12}) - 2C_{13}^2) \]  \hspace{1cm} (A10)

Hence, for example

\[ S_{11} = \frac{(C_{11}C_{33} - C_{13}^2)}{(C_{11} - C_{12})(C_{33}(C_{11} + C_{12}) - 2C_{13}^2)} \]

\[ = \frac{(C_{11}C_{33} - C_{13}^2)}{4C_{66}(C_{33}(C_{11} - C_{66}) - C_{13}^2)} \]  \hspace{1cm} (A11)

The reciprocal of \( S_{11} \) is known in the engineering literature (e.g. Amadei (1996), Jaeger and Cook (2007)) as \( E \), the *Young’s modulus in the plane of isotropy*.

\[ S_{33} = \frac{(C_{11}^2 - C_{12}^2)}{(C_{11} - C_{12})(C_{33}(C_{11} + C_{12}) - 2C_{13}^2)} \]

\[ = \frac{C_{11} - C_{66}}{C_{33}(C_{11} - C_{66}) - C_{13}^2} \]  \hspace{1cm} (A12)

The reciprocal of \( S_{33} \) is known in the engineering literature (e.g. Amadei (1996)) as \( E' \), the *Young’s modulus perpendicular to the plane of isotropy*.

The ratio \( S_{13}/S_{33} \) is known in the engineering literature as \( \nu' \), the *Poisson ratio perpendicular to the plane of isotropy*.

\[ \frac{S_{13}}{S_{33}} = \frac{(C_{12}C_{13} - C_{13}C_{11})}{(C_{11}^2 - C_{12}^2)} \]

\[ = \frac{-C_{13}}{2(C_{11} - C_{66})} \]  \hspace{1cm} (A13)
Fractured Isotropic Media

The ratio $S_{13}/S_{33}$ is observed to be 1 in our load cycle data.

$$\frac{S_{12}}{S_{13}} = \frac{(C_{12}C_{33} - C_{13}C_{13})}{(C_{13}C_{11} - C_{12}C_{13})}$$

$$= \frac{(C_{12}C_{33} - C_{13})}{(C_{11} - C_{12})}$$ \hfill (A14)

Thus $S_{12} = S_{13}$ if and only if

$$C_{11} - C_{12} = \frac{C_{12}}{C_{13}}C_{33} - C_{13}$$ \hfill (A15)

In view of (8), an equivalent condition is

$$C_{13}(C_{13} + 2C_{66}) = (C_{11} - 2C_{66})C_{33}$$ \hfill (A16)

When the condition $S_{12} = S_{13}$ is satisfied in a TIV medium, the compliance tensor $S$ can be written

$$S = S_b + \begin{bmatrix} 0 & 0 & B_N \\ 0 & B_T & B_T \\ 0 & B_T & 0 \end{bmatrix}$$ \hfill (A17)

where

$$S_b = \begin{bmatrix} E^{-1} & \nu E^{-1} & \nu E^{-1} \\ \nu E^{-1} & E^{-1} & \nu E^{-1} \\ \nu E^{-1} & \nu E^{-1} & \nu E^{-1} \end{bmatrix}$$ \hfill (A18)

is the compliance matrix of a background isotropic medium with parameters
and $B_N$ and $B_T$ are excess normal and tangential compliances defined by

$$B_N = S_{33} - S_{11}, \quad B_T = S_{55} - S_{66}.$$  \hspace{1cm} (A20)

Sayers and Kachonov (1995) and Sayers (2008) describe a physical theory which associates these excess compliances with tensorial averages of excess compliances from a distribution of small, low aspect-ratio cracks.

Schoenberg and Douma (1988) described an equivalent formulation which associated the excess compliances with fractures interpreted as the zero-thickness limit of thin compliant layers in an effective medium theory. Schoenberg’s parameters are dimensionless quantities:

$$E_N = \frac{S_{33}}{S_{11}} - 1, \quad = \frac{B_N}{S_{11}},$$

$$E_T = \frac{S_{55}}{S_{66}} - 1, \quad = \frac{B_T}{S_{66}}.$$  \hspace{1cm} (A21)

We assign a symbol for the dimensionless compliance ratio:

$$\eta = \frac{B_N}{B_T}$$  \hspace{1cm} (A22)

Because $E_T$ is being used in the main text for one of the Young moduli, we assign an additional dimensionless crack tangential compliance ratio as

$$\chi = 1 + \frac{B_T}{S_{66}} = \frac{S_{55}}{S_{66}} = \frac{C_{66}}{C_{55}}$$  \hspace{1cm} (A23)

**Linear Viscoelastic Media**

A linear viscoelastic medium satisfies a rate-dependent form of Hooke’s law that can be thought of as family of rate-dependent, complex-valued elastic tensors. A simple model that exhibits viscoelastic behavior and is adequate for our measurements is the Standard Linear Solid (SLS) (e.g. Shearer (2009) section 6.6.4; Carcione (2001) chapter 2). Corresponding to a mechanical
model that combines two springs and a dashpot, the SLS satisfies a stress-strain relationship

\[ \sigma + \tau_\sigma \dot{\epsilon} = M^R (\epsilon + \tau_\epsilon \dot{\epsilon}) \]  \hspace{1cm} (A24)

where the dots indicate time derivatives, \( \tau_\epsilon \) is a characteristic "relaxation time" for strain in response to an applied step in stress, \( \tau_\sigma \) is a characteristic "relaxation time" for stress in response to an applied step in strain, and \( M^R \) is a "relaxed modulus" which gives the ratio of stress to strain at infinite time. Fourier transform of (A24) leads to a complex, frequency-dependent modulus

\[ M(\omega) = M^R \left( \frac{1 + i\omega \tau_\epsilon}{1 + i\omega \tau_\sigma} \right) \]  \hspace{1cm} (A25)

Taking the limit as \( \omega \to \infty \), one obtains the "unrelaxed modulus"

\[ M^U = M^R \left( \frac{\tau_\epsilon}{\tau_\sigma} \right) \]  \hspace{1cm} (A26)

Causality requires \( \tau_\epsilon \geq \tau_\sigma \), hence \( M^U \geq M^R \). Frequency-dependent attenuation in this model is associated with the imaginary part of \( M(\omega) \). It is maximum near the Debye peak, \( \omega = (\tau_\epsilon \tau_\sigma)^{-1/2} \), and zero at the relaxed and unrelaxed limits.

The formalism described here can be applied independently to anisotropic moduli which can be combined algebraically as in the case of pure elasticity. In particular, if all the moduli share the same time constants, the ratio between relaxed and unrelaxed combinations of moduli (and hence the shape of the anisotropy) will not vary. Such a restriction is consistent with our data. We assign a symbol for the common dimensionless relaxation ratio between unrelaxed and relaxed moduli:

\[ \phi = \left( \frac{\tau_\epsilon}{\tau_\sigma} \right) = \frac{M^U}{M^R} \]  \hspace{1cm} (A27)

For such a model, dimensionless parameters such as Poisson’s ratio, \( v_p/v_s \) ratio, dimensionless Thomsen parameters, etc. are the same whether measured at the unrelaxed or relaxed limits.
Fractured Linear Viscoelastic Media

The relaxed and unrelaxed limits of an isotropic standard linear solid with a common relaxation ratio are described by a density $\rho$, a background relaxed Young modulus $E_R$, a background Poisson’s ratio $\nu$, and a relaxation ratio $\phi$.

If cracks are introduced to such a model with dimensionless parameters $\chi$ and $\eta$, anisotropic relaxed and unrelaxed compliances satisfying the constraint $S_{12} = S_{13}$ can be obtained directly from equations (A20) - (A23). These, in turn allow computation of the full set of relaxed and unrelaxed TIV moduli and axial velocities using the TIV versions of (A1)-(A8).

From (A27),

$$C^U_{66} = \phi C^R_{66} \quad \text{(A28)}$$

From (A23),

$$C^U_{66} = \chi C^U_{55} \quad \text{(A29)}$$

Thus, the constraint $C^U_{55} = C^R_{66}$, which was called Algebraic Relation (3) in the main text is equivalent to

$$\phi = \chi \quad \text{(A30)}$$

and the model used to fit the measurements is fully specified by the tuple $[\rho, E_R, \nu, \phi, \eta]$.

Given these values, relaxed and unrelaxed compliances can be computed using the following equations:

$$S^R_{11} = 1/E_R \quad \text{(A31)}$$

$$S^R_{12} + \nu S^R_{11} \quad \text{(A32)}$$

$$S^R_{66} = 2(S^R_{11} - S^R_{12}) \quad \text{(A33)}$$
Relaxed and unrelaxed moduli can be calculated from the compliances using the TIV versions of (A4) to (A7). Relaxed and unrelaxed axial velocities can then be calculated from the moduli using (A8).