Geophysical Prospecting, 1996, **44**, 525–549

Velocity sensitivity in transversely isotropic media[1]

C.H. Chapman[2] and D.E. Miller[3]

Abstract

We consider the problem of determining and predicting how the wave speeds in particular directions for a transversely isotropic (TI) medium depend on particular combinations of the density-normalized moduli A_{ij}. The expressions for the qP and qSV velocities are known to depend on four moduli. Normally, we can only determine three independent parameters from qP data, or two from qSV data, as the others have much lower sensitivity. The resolvable parameters are conveniently described by axial and off-axis parameters: for qP rays, $P_{0^\circ} = A_{11}$, $P_{90^\circ} = A_{33}$ and $P_{45^\circ} = (A_{11} + A_{33})/4 + (A_{13} + 2A_{55})/2$; and for qSV rays, $S_{0^\circ} = S_{90^\circ} = A_{55}$ and $S_{45^\circ} = (A_{11} + A_{33})/4 - A_{13}/2$. These parameters control the magnitude of the squared-velocities on the axes and at approximately 45°. For an arbitrary TI medium, if the medium is perturbed in a way that preserves a particular parameter, then slowness points in the associated direction and mode will be approximately preserved in the new medium. We refer to these parameters as 'push-pins', i.e. if a parameter is fixed, the associated part of the slowness surface is pinned in place.

Because, these five push-pins only contain four independent moduli, we can only fix at most three push-pins. Perturbing one of the other parameters inevitably perturbs the other. Numerical results illustrating the linkage between two push-pins, when three are fixed, are presented.

So-called anomalous TI media occur when the roles of the qP and qSV waves are reversed: in some directions the faster ray has transverse polarization. That, in turn, requires anomalous velocities at the push-pins, i.e. $S_{0^\circ} > P_{0^\circ}$, $S_{45^\circ} > P_{45^\circ}$ and/or $S_{90^\circ} > P_{90^\circ}$ (equivalent to the usual anomalous conditions $A_{11} < A_{55}$, $A_{13} + A_{55} < 0$ and/or $A_{33} < A_{55}$). In the Appendix, we confirm that anomalous sensitivities of the velocities at the five push-pins only occur in such media, although the push-pins still apply if interpreted appropriately. Truly anomalous sensitivities, in which push-pins play no role, only occur in media near the boundary between normal and anomalous.

[1] Received February 1995, revision accepted October 1995.
[2] Schlumberger Cambridge Research Ltd, High Cross, Madingley Road, Cambridge CB3 0EL, UK.
[3] Schlumberger-Doll Research, Old Quarry Road, Ridgefield CT 06877, USA.

1. Introduction

A transversely isotropic (TI) medium is described by five independent elastic moduli. It is well known that the exact phase velocities for both qP and qSV rays depend on four moduli. However, simple numerical experiments indicate that the qP velocity is only sensitive to three parameters, and the qSV velocity to two. We investigate the sensitivity of the velocity in TI media to the elastic moduli. We find that certain combinations conveniently describe the squared-velocities on the symmetry axis, perpendicular to the axis and at approximately 45° to the axis. For qP rays this leads to three significant parameters; for qSV rays to only two, as the axial parameters are equal. Using these parameters, we can readily determine what parameters can be found in a particular experiment, e.g. a cross-well tomography experiment, and which are unresolvable. If some parameters are known, uncertainties in the others are linked. The simple rules for sensitivities apply in all normal TI media. We also investigate in what circumstances they break down, e.g. when is the qP velocity sensitive to a fourth parameter. In anomalous TI media, when the roles of the qP and qSV waves are reversed, the sensitivity rules still apply if the parameters are assigned correctly. Only in intermediate media, when the roles of the qP and qSV waves overlap, will the simple rules break down.

In this introduction we review the theory for the exact results in TI media. In the second section we introduce the new parameters by considering the velocities at 45° to the axis. The third section investigates the sensitivities using perturbation theory, and this is followed by a collection of numerical examples. Results for anomalous TI media are discussed in the Appendix.

Ray theory in anisotropic media is well known (Červený 1972) and we follow a fairly standard notation. We define the density-normalized elastic (velocity-squared) parameters as

$$a_{ijkl} = c_{ijkl}/\rho, \tag{1}$$

where ρ is the density and c_{ijkl} are the anisotropic stiffness parameters, the phase slowness vector is defined as

$$\mathbf{p} = \nabla T, \tag{2}$$

where T is the traveltime, and the Christoffel matrix is defined as

$$\Gamma_{jk} = p_i p_l a_{ijkl}. \tag{3}$$

The slowness components must satisfy the eigen-equation

$$(\Gamma_{jk} - \delta_{jk})\hat{g}_k = 0, \tag{4}$$

where $\hat{\mathbf{g}}$ is the normalized, i.e. unit, polarization vector. Given $\mathbf{p}$, existence of a solution to (4) is equivalent to the condition that $\mathbf{p}$ satisfy the dispersion relationship

$$F(\mathbf{p}) = 0, \tag{5}$$

where

$$F(\mathbf{p}) = |\Gamma_{jk} - \delta_{jk}|. \tag{6}$$

Given a phase slowness vector $\mathbf{p}$, the associated group velocity vector (or ray-vector) $\mathbf{V}$ is parallel to ∇F and satisfies

$$\mathbf{p} \cdot \mathbf{V} = 1 \tag{7}$$

(Musgrave 1970, eq. (7.4.1a)). (This result follows from the geometry of the ray and the wavefronts). It follows (Musgrave 1970, eq. (7.4.9)) that

$$\mathbf{V} = \nabla F \frac{1}{\mathbf{p} \cdot \nabla F}. \tag{8}$$

The set of (endpoints of the) phase slowness vectors satisfying (5) form the *phase slowness surface* of the medium. The associated group velocity vectors (8) form the *wavefront surface* (*impulse response*) of the medium. They are the points lying exactly at a unit traveltime from the origin. If $\mathbf{p}$ and $\mathbf{V}$ are associated points, then $\mathbf{p}$ is normal to the wavefront surface at $\mathbf{V}$ (equation (2)), $\mathbf{V}$ is normal to the phase slowness surface at $\mathbf{p}$, (equations (5) and (8)). As (7) holds, the phase slowness surface and the wavefront surface are polar reciprocals.

Throughout this article, we restrict ourselves to TI media. In such media, the 6×6 matrix representation of the tensor of velocity-squared parameters reduces to

$$\begin{pmatrix} A_{11} & A_{11} - 2A_{66} & A_{13} & 0 & 0 & 0 \\ A_{11} - 2A_{66} & A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{13} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0A_{66} \end{pmatrix}, \tag{9}$$

(note the A_{ij} are the density-normalized elastic parameters in the two-index (Voigt) notation, and we have $A_{11} = A_{22}$ and $A_{44} = A_{55}$, and use $A_{11} = a_{1111}$ and $A_{55} = a_{3131}$ preferentially, as we will be interested mainly in displacements in the x_1–x_3 plane). As the system is azimuthally symmetric (about the x_3-axis), we can take $p_2 = 0$ without loss of generality, and restrict the rays to the x_1–x_3 plane.

As is well known (Musgrave 1970, p. 95), the solution for the slowness surface in TI media is straightforward. The eigenvector equation (4) reduces to

$$\begin{pmatrix} A_{11}p_1^2 + A_{55}p_3^2 - 1 & 0 & ap_1p_3 \\ 0 & A_{66}p_1^2 + A_{55}p_3^2 - 1 & 0 \\ ap_1p_3 & 0 & A_{55}p_1^2 + A_{33}p_3^2 - 1 \end{pmatrix} \hat{\mathbf{g}} = \mathbf{0}, \tag{10}$$

where

$$a = A_{13} + A_{55}. \tag{11}$$

One solution, a qSH wave, has the polarization in the transverse, horizontal direction, e.g. $\hat{\mathbf{g}}^{\mathrm{T}} = (0, 1, 0)$, and does not concern us here. The other solution requires (Musgrave 1970, eqs (8.2.7) and (8.2.1))

$$A_{11}A_{55}p_1^4 + A_{33}A_{55}p_3^4 + Ap_1^2p_3^2 - (A_{11} + A_{55})p_1^2 - (A_{33} + A_{55})p_3^2 + 1 = 0, \tag{12}$$

where

$$A = A_{11}A_{33} + A_{55}^2 - a^2. \tag{13}$$

For a given p_1 we can solve this for p_3, i.e.

$$p_3^2 = \frac{B \mp [B^2 - 4A_{33}A_{55}(A_{11}p_1^2 - 1)(A_{55}p_1^2 - 1)]^{1/2}}{2A_{33}A_{55}}, \tag{14}$$

where

$$B = A_{33} + A_{55} - Ap_1^2, \tag{15}$$

or for a given slowness direction $\hat{\mathbf{p}} = v\mathbf{p}$, the exact phase velocity v is given by

$$v^2 = \tfrac{1}{2}\left(A_{55} + A_{33}\hat{p}_3^2 + A_{11}\hat{p}_1^2 \pm \sqrt{((A_{33} - A_{55})\hat{p}_3^2 - (A_{11} - A_{55})\hat{p}_1^2)^2 + 4\hat{p}_1^2\hat{p}_3^2a^2}\right), \tag{16}$$

where the two signs correspond to qP and qSV. The corresponding eigenvectors (not normalized) are

$$\mathbf{g} = \begin{pmatrix} 2a\hat{p}_1\hat{p}_3 \\ 0 \\ (A_{33} - A_{55})\hat{p}_3^2 - (A_{11} - A_{55})\hat{p}_1^2 \pm \\ \{((A_{33} - A_{55})\hat{p}_3^2 - (A_{11} - A_{55})\hat{p}_1^2)^2 + 4\hat{p}_1^2\hat{p}_3^2a^2\}^{1/2} \end{pmatrix}. \tag{17}$$

With a bit of algebra, the components of the group velocity vector can be derived from (8) and (12) (Musgrave 1970, eqs (8.2.8) and (8.2.9)) and are given by

$$V_1 = \hat{p}_1[2A_{11}A_{55}\hat{p}_1^2 + A\hat{p}_3^2 - A_{11} - A_{55})]/D, \tag{18}$$

$$V_2 = \hat{p}_3[A\hat{p}_1^2 + 2A_{33}A_{55}\hat{p}_3^2 - A_{33} - A_{55})]/D, \tag{19}$$

where (Musgrave 1970, eq. (8.2.10))

$$D = (A_{11} + A_{55})\hat{p}_1^2 + (A_{33} + A_{55})\hat{p}_3^2 - 2. \tag{20}$$

Note that these results depend on four of the five parameters, as A_{66} only appears in the results for the qSH wave. Although these results are complete and exact, they are not very illuminating.

Červený (1982) and Červený and Jech (1982) have developed a theory for linearized perturbations to the traveltime in anisotropic media. This has been extended to cover the case of degenerate qS rays by Jech and Pšenčík (1989). To the first order, perturbations to the slowness surface of a general anisotropic medium are given by

$$\delta p = -\tfrac{1}{2}p^3\delta a_{ijkl}\hat{p}_i\hat{p}_l\hat{g}_j\hat{g}_k \tag{21}$$

for fixed phase direction, and for the phase velocity (Chapman and Pratt 1992, eq. (17); Every and Sachse 1992, eq. (11))

$$\delta v = \frac{1}{2v}\delta a_{ijkl}\hat{p}_i\hat{p}_l\hat{g}_j\hat{g}_k. \tag{22}$$

Let us consider the general perturbation results, (21) and (22), restricted to TI media and a fixed slowness direction. We can take $\hat{g}_2 = \hat{p}_2 = 0$. Further let us consider the perturbation from an isotropic medium and for simplicity consider first the qP solution (so $\hat{\mathbf{g}} = \hat{\mathbf{p}}$). From (21), the slowness perturbation is (Chapman and Pratt 1992)

$$\delta p = -\tfrac{1}{2}p^3\{\hat{p}_1^4\delta A_{11} + 2\hat{p}_1^2\hat{p}_3^2\delta(A_{13} + 2A_{55}) + \hat{p}_3^4\delta A_{33}\}. \tag{23}$$

Making the approximations

$$v^2 \simeq v_0^2 + 2v\delta v \simeq v_0^2 - \frac{2}{p^3}\delta p, \tag{24}$$

and noting that

$$\hat{p}_1^4 + 2\hat{p}_1^2\hat{p}_3^2 + \hat{p}_3^4 = 1, \tag{25}$$

we can combine (23) and (24) into a form analogous to (16),

$$\begin{aligned} v^2 &\simeq v_0^2 + \hat{p}_1^4\delta A_{11} + 2\hat{p}_1^2\hat{p}_3^2\delta(A_{13} + 2A_{55}) + \hat{p}_3^4\delta A_{33} \\ &= \hat{p}_1^4 A_{11} + 2\hat{p}_1^2\hat{p}_3^2(A_{13} + 2A_{55}) + \hat{p}_3^4 A_{33}, \end{aligned} \tag{26}$$

where we have used $v_0^2 = A_{11} = A_{13} + 2A_{55} = A_{33}$ in an isotropic medium.

Comparing this result with the exact result (16), several questions arise.

- Why does $A_{13} + 2A_{55}$ appear as a combination in (26) but apparently not in (16)? Or what is the physical significance of $A_{13} + 2A_{55}$?
- Why are there only three independent parameters in (26) and not four, as in (16)? Or to what extent does the qP slowness depend on A_{55} independently?

Similar questions arise for the qSV perturbation results. Because the polarization is known *a priori*, i.e. $\hat{g}_1 = \hat{p}_3$ and $\hat{g}_3 = -\hat{p}_1$, the degenerate perturbation theory of Jech and Pšenčík (1989) is easy to apply. From (21), the slowness perturbation is (Chapman and Pratt 1992)

$$\delta p = -\tfrac{1}{2}p^3\{\hat{p}_1^2\hat{p}_3^2\delta(A_{11} + A_{33} - 2A_{13} - 4A_{55}) + \delta A_{55}\}, \tag{27}$$

and the corresponding velocity expression is

$$v^2 = \hat{p}_1^2\hat{p}_3^2(A_{11} + A_{33} - 2A_{13} - 4A_{55}) + A_{55}. \tag{28}$$

Again, the questions arise:

- Why does $A_{11} + A_{33} - 2A_{13} - 4A_{55}$ appear as a combination in (28) but apparently not in (16)? Or what is the physical significance of $A_{11} + A_{33} - 2A_{13} - 4A_{55}$?
- Why are there only two independent parameters in (28) and not four, as in (16)? Or to what extent does the qSV slowness depend on the other parameters, e.g. A_{11} and A_{33}, independently?

We attempt to answer these and related questions. Note that we have not always pursued the analysis in complete detail as simple, exact numerical solutions of (16) are always available. Rather we have attempted to find explanations which can aid our physical intuition (rather than our numerical solutions).

A discussion of these same issues from a slightly different viewpoint has been given by Costa (1993) and Every and Sachse (1992).

2. Velocity at 45° in TI media

For qP, the velocity on the axis of symmetry is $\sqrt{A_{33}}$ and in the perpendicular direction $\sqrt{A_{11}}$. For qSV waves, the velocities in both directions are $\sqrt{A_{55}}$. It is therefore to be expected that the other parameters in the perturbation results (26) and (28) control the velocity off-axis. Let us consider the results at 45° to the axis of symmetry.

The general result for the rotation of the 4th-order tensor of elastic parameters is well known, i.e. $a'_{i'j'k'l'} = g_{i'i}g_{j'j}g_{k'k}g_{l'l}a_{ijkl}$, where $g_{i'i}$ are the direction cosines of the rotation. Chapman and Pratt (1992, eqns (D4) and (D5)) showed that for rotation about the x_2-axis, the parameters

$$\mathbf{q}^{\mathrm{T}} = (A_{11}, 4A_{15}, 2A_{13} + 4A_{55}, 4A_{35}, A_{33}) \tag{29}$$

are rotated by a 5×5 matrix (similar results have been given by Neighbours and Schacher 1967), i.e. these parameters, with the combination $2A_{13} + 4A_{55}$, transform independently of other parameters (for general anisotropy). They are important here because they include the important parameters for qP rays in TIV media (26) with $A_{15} = A_{35} = 0$. Applying a rotation of 45° to a TIV medium, we obtain a TI medium

$$\begin{pmatrix} A'_{11} \\ 4A'_{15} \\ 2A'_{13} + 4A'_{55} \\ 4A'_{35} \\ A'_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 1 & -\frac{1}{2} & 0 & \frac{1}{2} & -1 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} A_{11} \\ 0 \\ 2A_{13} + 4A_{55} \\ 0 \\ A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}(A_{11} + A_{33} + 2A_{13} + 4A_{55}) \\ A_{11} - A_{33} \\ \frac{1}{2}(3A_{11} - 2A_{13} - 4A_{55} + 3A_{33}) \\ A_{11} - A_{33} \\ \frac{1}{4}(A_{11} + A_{33} + 2A_{13} + 4A_{55}) \end{pmatrix}. \tag{30}$$

Notice that on the new axes, i.e. at 45° to the old axes, the longitudinal parameter is $(A_{11} + A_{33} + 2A_{13} + 4A_{55})/4$, i.e. it contains the combination $A_{13} + 2A_{55}$. Similarly the parameters

$$\mathbf{r}^{\mathrm{T}} = (A_{11} + A_{33} - 2A_{13}, 4A_{35} - 4A_{15}, 4A_{55}) \tag{31}$$

are rotated by a 3×3 matrix (Chapman and Pratt 1992, eq. (D6)). These are the important parameters for qSV rays (28). Applying a rotation of 45° to a TIV medium with $A_{35} = A_{15} = 0$), we obtain a TI medium

$$\begin{aligned} \begin{pmatrix} A'_{11} + A'_{33} - 2A'_{13} \\ 4(A'_{35} - A'_{15}) \\ 4A'_{55} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} + A_{33} - 2A_{13} \\ 0 \\ 4A_{55} \end{pmatrix} \\ &= \begin{pmatrix} 4A_{55} \\ 0 \\ A_{11} + A_{33} - 2A_{13} \end{pmatrix}. \end{aligned} \tag{32}$$

The axial parameter A_{55} has been replaced by $(A_{11} + A_{33} - 2A_{13})/4$.

Alternatively, we can use (16) to evaluate the phase velocity at 45°. We obtain

$$v^2 = \tfrac{1}{4}\left\{A_{11} + A_{33} + 2A_{55} \pm \sqrt{4(A_{13} + A_{55})^2 + (A_{11} - A_{33})^2}\right\} \tag{33}$$

$$\simeq \tfrac{1}{4}\left\{A_{11} + A_{33} + 2A_{55} \pm 2(A_{13} + A_{55})\left(1 + \frac{(A_{11} - A_{33})^2}{8(A_{13} + A_{55})^2} + \cdots\right)\right\}$$

$$\simeq \tfrac{1}{4}(A_{11} + A_{33} + 2A_{13} + 4A_{55}) + O(A_{11} - A_{33})^2 \tag{34}$$

$$\text{or} \simeq \tfrac{1}{4}(A_{11} + A_{33} - 2A_{13}) + O(A_{11} - A_{33})^2, \tag{35}$$

where (34) is obtained by a Taylor expansion of (33) with the plus sign, and (35) is obtained with the minus sign.

These results, the first and fifth components of (30) and the final component of (32), or (34) and (35), suggest introducing the new parameters,

$$P_{0^\circ} = A_{11}, \tag{36}$$

$$P_{45^\circ} = \tfrac{1}{4}(A_{11} + A_{33} + 2A_{13} + 4A_{55}), \tag{37}$$

$$P_{90^\circ} = A_{33}, \tag{38}$$

$$S_{0^\circ} = S_{90^\circ} = A_{55}, \tag{39}$$

$$S_{45^\circ} = \tfrac{1}{4}(A_{11} + A_{33} - 2A_{13}), \tag{40}$$

for the squared-velocities. The notation is suggestive of the direction in which these are the squared-velocities, but remember that although the results (30) and (32) are correct for the rotated elastic parameters, they do not yield the exact ray results at 45° unless $A_{11} = A_{33}$. The subscript angle is measured from the x_1-axis, i.e. the angle ψ

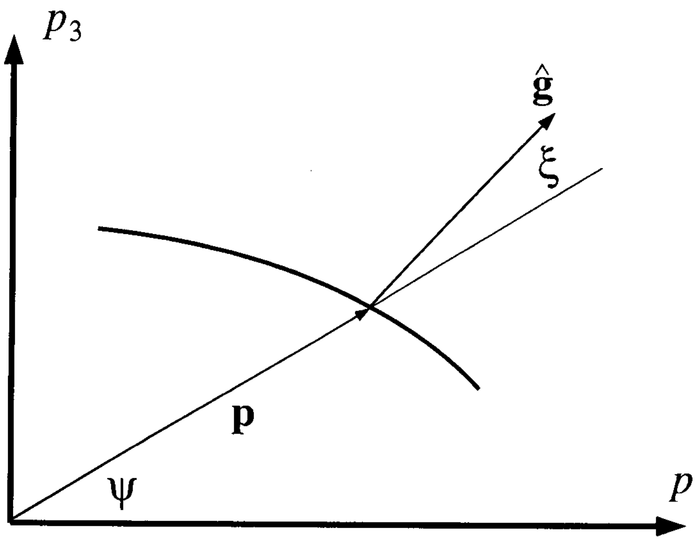

Figure 1. The slowness and polarization vectors, $\mathbf{p}$ and $\hat{\mathbf{g}}$. The angle between the slowness vector and the x_1-axis is ψ, and between the slowness and polarization vectors, ξ.

(Fig. 1). With this change of variables, the perturbation result (23) becomes

$$\delta p = -\tfrac{1}{2}p^3\{\hat{p}_1^2(\hat{p}_1^2 - \hat{p}_3^2)\delta P_{0^\circ} + 4\hat{p}_1^2\hat{p}_3^2\delta P_{45^\circ} + \hat{p}_3^2(\hat{p}_3^2 - \hat{p}_1^2)\delta P_{90^\circ}\}. \tag{41}$$

Notice that the coefficients of the δPs are unity in the appropriate direction and zero in the other directions (Fig. 2). Similarly (27) becomes

$$\delta p = -\tfrac{1}{2}p^3\{(1 - 4\hat{p}_1^2\hat{p}_3^2)\delta S_{0^\circ} + 4\hat{p}_1^2\hat{p}_3^2\delta S_{45^\circ}\}. \tag{42}$$

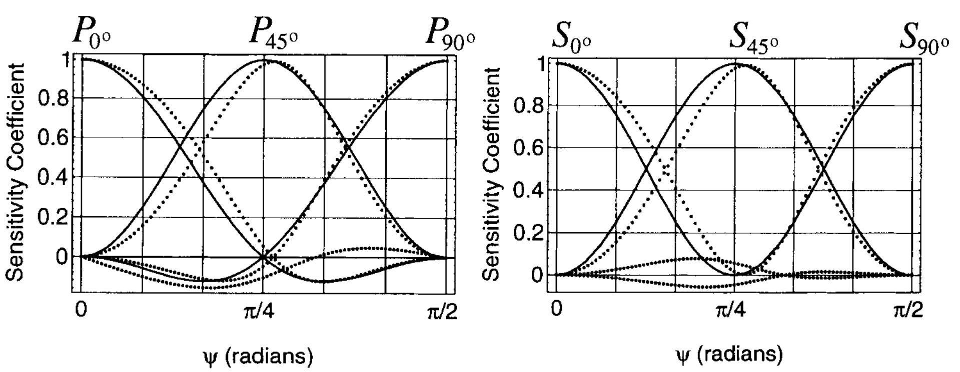

Figure 2. The exact (dotted line) and approximate (solid line) sensitivity functions for a representative shale medium. In units of km^2/s^2, the medium has density-normalized moduli $\{A_{11}^0, A_{13}^0, A_{33}^0, A_{55}^0\} = \{7.0, 2.5, 5.5, 1.0\}$. In the qP case, they are the coefficients in (41) and (47), respectively. In the qSV case, they are the coefficients in (42) and (48).

Again the coefficient of $\delta S_{0°} = \delta S_{90°}$ is unity on the axes and zero at 45°, and the coefficient of $\delta S_{45°}$ is the opposite (Fig. 2). We are now in a position to answer partly the questions posed earlier: what is the physical significance of the combinations $A_{13} + 2A_{55}$ and $A_{11} + A_{33} - 2A_{13} - 4A_{55}$ appearing in (26) and (28)? They are related to the velocities at 45° and can be rewritten

$$A_{13} + 2A_{55} = 2P_{45°} - \tfrac{1}{2}(P_{0°} + P_{90°}), \tag{43}$$

$$A_{11} + A_{33} - 2A_{13} - 4A_{55} = 4(S_{45°} - S_{0°}). \tag{44}$$

As we have seen, (23) and (27) have been rewritten as (41) and (42) in terms of only the new squared-velocity parameters (36)–(40). It remains to answer why the other independent parameters in (16) do not appear, or alternatively when are they significant. In the next section, we show that the same five parameters, (36)–(40), apply in so-called anomalous TI media provided they are assigned correctly. The 'missing' parameters, A_{55} for qP rays and A_{11} and A_{33} independently for qSV rays, are only significant in intermediate media, i.e. neither normal nor anomalous.

It may also be helpful to note that in an isotropic medium, ρA_{55} and ρA_{13} are the shear modulus μ and the Lamé constant λ, respectively, while ρA_{11} and ρA_{33} are both $\lambda + 2\mu$. Thus, $P_{45°}$ is an average of the 'natural' ways to form $\alpha^2 = (\lambda + 2\mu)/\rho$ and $S_{45°}$ is the average of the two 'natural' ways to obtain $\beta^2 = \mu/\rho$ from the axial compressional moduli.

3. Perturbation of TI media

We now investigate the perturbation of a TI medium in more detail. In the general formula (21), we can still set $\hat{g}_2 = \hat{p}_2 = 0$, and the perturbation reduces to

$$\begin{aligned}\delta p &= -\tfrac{1}{2}p^3\{\hat{p}_1^2\hat{g}_1^2\delta A_{11} + 2\hat{p}_1\hat{p}_3\hat{g}_1\hat{g}_3\delta(A_{13} + 2A_{55}) + \hat{p}_3^2\hat{g}_3^2\delta A_{33} \\ &\quad + (\hat{p}_1\hat{g}_3 - \hat{p}_3\hat{g}_1)^2\delta A_{55}\} & (45)\\ &= -\tfrac{1}{2}p^3\{\cos^2\psi\cos^2(\psi+\xi)\delta A_{11} + 2\cos\psi\sin\psi\cos(\psi+\xi)\sin(\psi+\xi)\delta(A_{13} + 2A_{55}) \\ &\quad + \sin^2\psi\sin^2(\psi+\xi)\delta A_{33} + \sin^2\xi\delta A_{55}\}, & (46)\end{aligned}$$

where ψ is the angle between the slowness direction and the x_1-axis (Fig. 1), i.e. $\hat{\mathbf{p}} = (\cos\psi, 0, \sin\psi)$, and ξ is the angle between the polarization vector and slowness direction (measured in the same direction as ψ, i.e. $\hat{\mathbf{g}} = (\cos(\psi+\xi), 0, \sin(\psi+\xi))$.

Eliminating $\delta(A_{13} + 2A_{55})$ in favour of $P_{45°}$ or $S_{45°}$, we can rewrite (46) in the equivalent forms

$$\begin{aligned}\delta p &= -\tfrac{1}{2}p^3\{\cos\psi\cos(\psi+\xi)\cos(2\psi+\xi)\delta P_{0°} + \sin 2\psi\sin 2(\psi+\xi)\delta P_{45°} \\ &\quad - \sin\psi\sin(\psi+\xi)\cos(2\psi+\xi)\delta P_{90°} + \sin^2\xi\delta A_{55}\} & (47)\\ &= -\tfrac{1}{2}p^3\{(\sin^2\xi + \sin 2\psi\sin 2(\psi+\xi))\delta S_{0°} - \sin 2\psi\sin 2(\psi+\xi)\delta S_{45°} \\ &\quad + \cos\xi(\cos\psi\cos(\psi+\xi)\delta A_{11} + \sin\psi\sin(\psi+\xi)\delta A_{33})\}. & (48)\end{aligned}$$

In a normal TI medium, at every propagation angle ψ, the faster solution of the dispersion relationship, (16), satisfies the conditions $\xi \simeq 0$, while the slower solution satisfies the condition $\xi \simeq \pi/2$. In physically anomalous TI media (as defined by Helbig and Schoenberg 1987, which always implies the reversal of the qP and qSV roles, see Appendix), this order may be reversed in some slowness directions, and there may be directions where ξ is neither close to zero nor to $\pi/2$ (see Appendix). In such media, it still makes sense to classify points as qP when $\xi \simeq 0$ and qSV when $\xi \simeq \pi/2$. If we set $\xi = 0$ in (47), the equation simplifies to the weak (near-isotropic) form (41). Similarly, the substitution $\xi = \pi/2$ transforms (48) to (42). We may conclude that the sensitivity relationship (41) and (42) are valid for qP and qSV points in arbitrary TI media, insofar as those terms are meaningful. It follows that, for an arbitrary TI medium:

3.1 If the medium is perturbed to a new medium in a way that preserves $P_{0^\circ}, P_{90^\circ}$ and P_{45°, then all qP phase slownesses will be approximately preserved.

3.2 If the medium is perturbed to a new medium in a way that preserves S_{0° and S_{45°, then all qSV phase slownesses will be approximately preserved.

As the expression (48) reduces to either (41) or (42), and the sensitivity functions (Fig. 2) are localized, we obtain local results such as:

3.3 If an arbitrary TI medium is perturbed to a new medium in a way that preserves P_{45°, then all qP phase slownesses with phase angle near 45° will be approximately preserved.

Thus, the five parameters, (36)–(40), act as 'push-pins' controlling the magnitude of the squared-velocities on the axes and at approximately 45°. By a push-pin we mean that if a parameter is fixed, the associated part of the slowness surface is pinned in place.

3.4 If an arbitrary TI medium is perturbed in a way that preserves a given push-pin, then slowness points in the associated direction and mode will be approximately preserved in the new medium.

It is clear that the five new parameters, (36)–(40), compared with four parameters, A_{11}, A_{33}, A_{55} and A_{13}, in the original exact equations (16), cannot be independent. If one parameter is perturbed, at least one other must also be altered. For instance, if S_{0° is perturbed, then we can keep the qP slowness at all angles fixed (by adjusting A_{13} to compensate for the change in A_{55}) but then S_{45° must be perturbed by an equal but opposite amount to S_0. Alternatively, if we keep S_{45° fixed, then P_{45° must be perturbed by an equal amount to S_{0° if P_{0° and P_{90° are fixed. Similar results exist connecting other points on the slowness surfaces. These are illustrated in the next section using numerical examples.

4. Numerical examples

We illustrate some of the ideas in the preceding section by showing the phase slowness and wavefront surfaces for various media obtained as perturbations of a reference TI medium. The medium is based on the measured shale medium

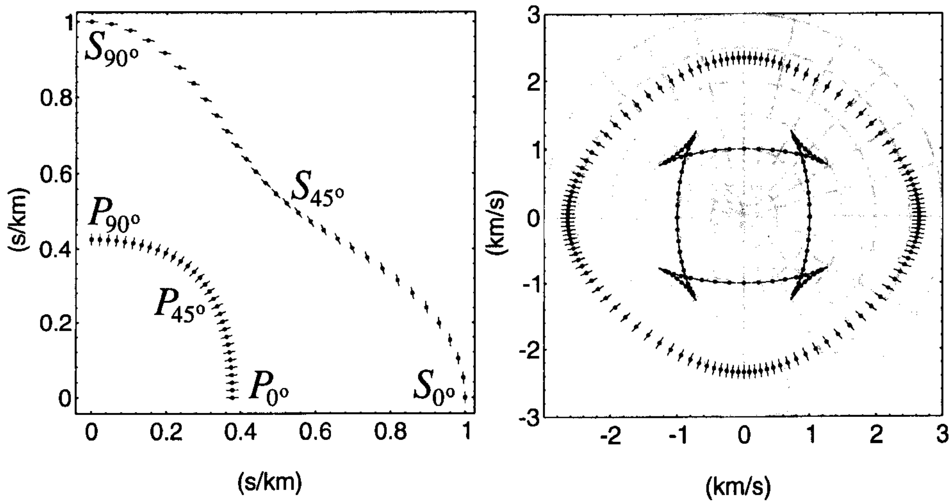

Figure 3. The phase slowness and group velocity surfaces for the representative shale medium in Fig. 2. The push-pin points on the slowness surfaces are labelled with the symbols $P_{0°}, P_{45°}$, etc.

described by Miller, Leaney and Borland (1994), which is similar to other shales that have been measured *in situ* (Costa 1993; Miller and Chapman 1991) and in the laboratory (Jones and Wang 1981; Hornby 1995). The four density-normalized moduli that determine qP and qSV propagation for this medium are

$$\{A_{11}^0, A_{13}^0, A_{33}^0, A_{55}^0\} = \{7.0, 2.5, 5.5, 1.0\}, \tag{49}$$

all in units km^2/s^2. From (37) and (40) we can calculate

$$\{P_{45°}^0, S_{45°}^0\} = \{5.375, 1.875\}, \tag{50}$$

so the qP velocity near 45° is approximately equal to the vertical qP velocity, and the qSV velocity near 45° is about 35% faster than the axial qSV velocity. Figure 3 shows the phase slowness and group velocity surfaces for this medium. In each case the function is sampled in 3° increments of phase angle and the associated polarization vectors are illustrated as tic-marks. Expressions (16)–(20) were used to calculate these results, and similarly for Fig. 5–17.

Figure 4 plots the difference angle ξ between phase and polarization directions for qP waves and Fig. 2 shows the exact and approximate sensitivity functions for this medium. In the qP case, the sensitivity functions are the coefficients in (41) and (47) respectively. In the qSV case, they are the coefficients in (42) and (48).

Figures 5–9 compare the phase slowness and group velocity surfaces in our reference medium with those in perturbed media obtained by fixing some of the push-pin parameters. In each case, the reference surfaces are plotted in light grey and the surfaces associated with the perturbed medium are plotted in black. The heavy dots in each plot indicate the push-pin parameters that have been held fixed. The

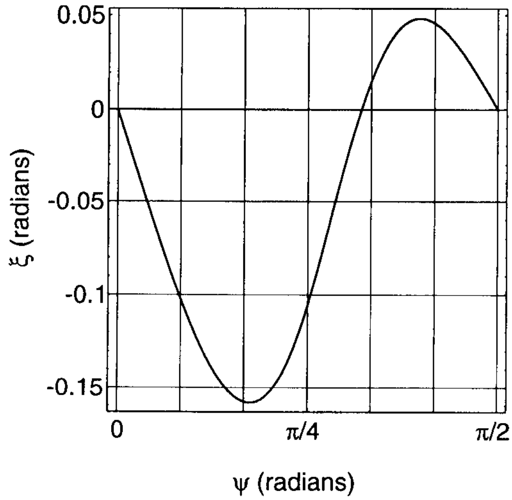

Figure 4. The angle ξ between the slowness vector and polarization vector for the representative medium in Fig. 3.

dots are plotted at the velocity or slowness derived from the push-pin parameter, e.g. $\sqrt{P_{0^\circ}}$ and $1/\sqrt{P_{0^\circ}}$ for the group velocity and phase slowness plots, respectively. Off-axis, they are plotted at 45°. Note that these points are not exactly on the slowness or velocity surfaces (as already mentioned, definitions (37) and (39) are not exactly the

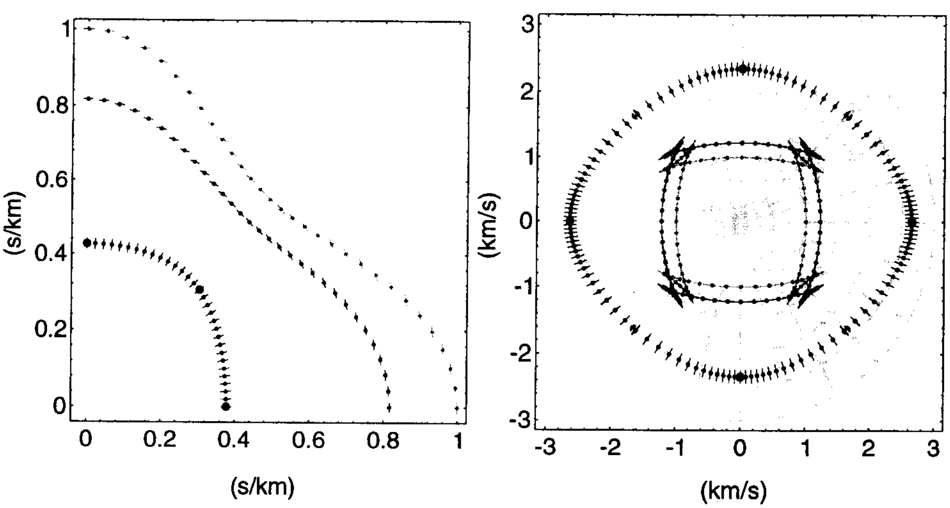

Figure 5. The result of perturbing the reference shale medium with a 50% increase in S_{0°, keeping the three qP push-pin parameters, $P_{0^\circ}, P_{45^\circ}$ and P_{90° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, 1.5, 5.5, 1.5\}$.

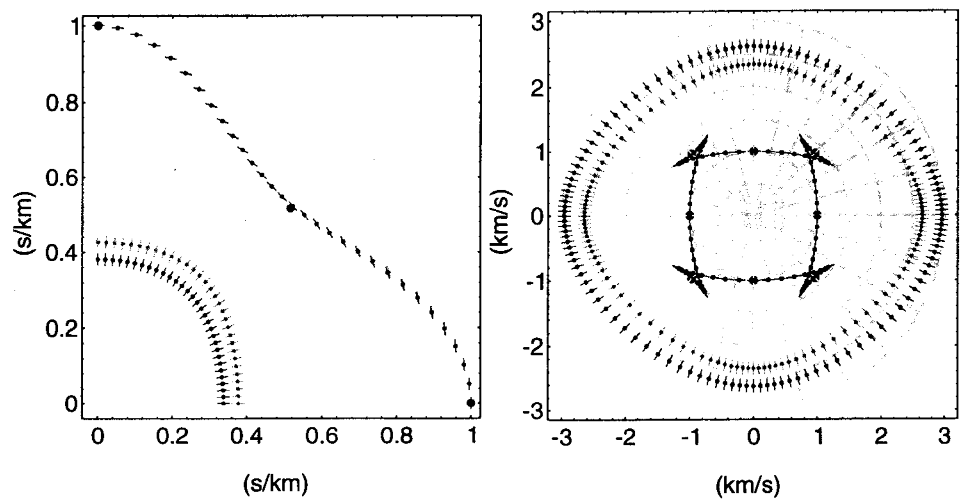

Figure 6. The result of perturbing the reference shale medium with a 25% increase in P_{0°, keeping the two qS push-pin parameters S_{0° and S_{45°, and P_{0°/P_{90° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{8.75, 4.0625, 6.875, 1.0\}$.

squared-velocity at 45°), but the difference is only conspicuous in Fig. 16, when A_{11} and A_{33} differ significantly.

Figures 5 and 6 illustrate our observations 3.1 and 3.2. Notice that despite the large perturbation to A_{55} in Fig. 5, no change in the qP surface is visible (see Miller and

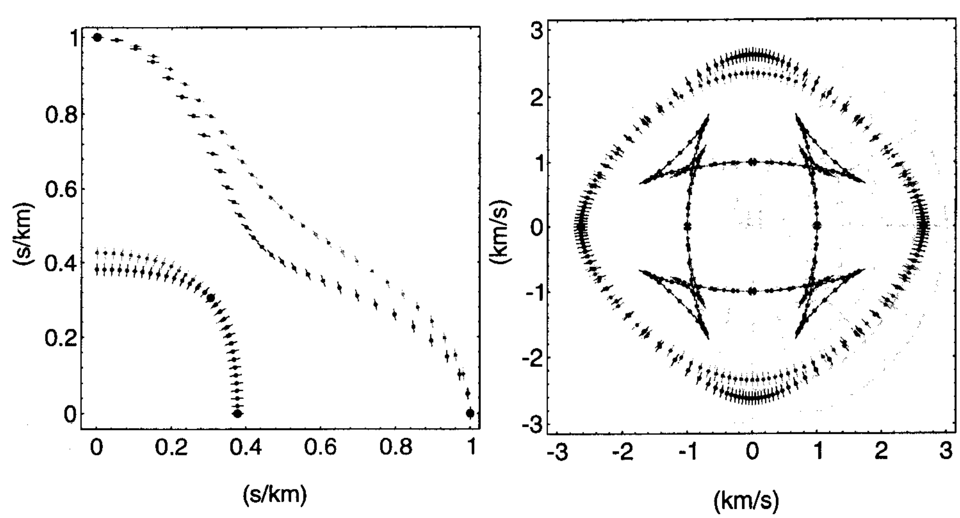

Figure 7. The result of perturbing the reference shale medium with a 25% increase in P_{90°, keeping P_{0°, S_{0° and P_{45° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, 1.8125, 6.875, 1.0\}$.

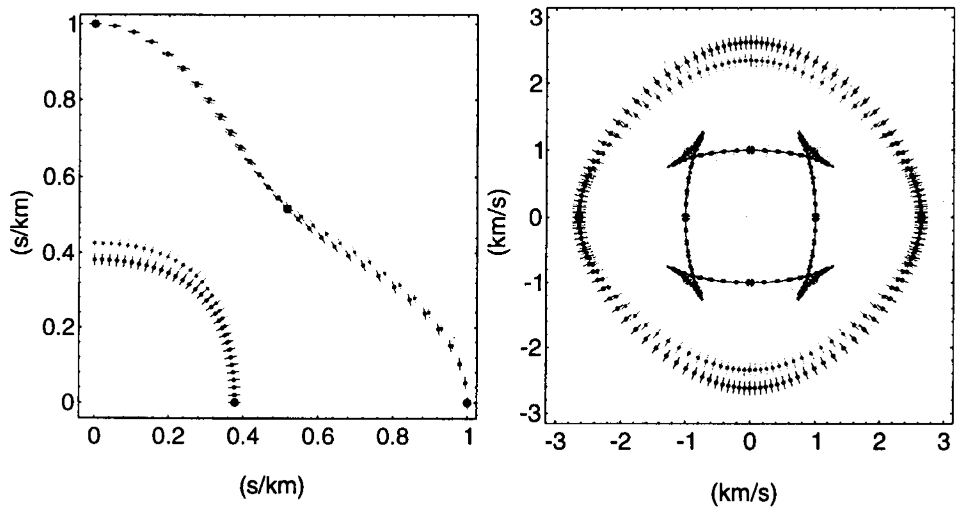

Figure 8. The result of perturbing the reference shale medium with a 25% increase in P_{90°, keeping P_{0°, S_{0° and S_{45° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, 3.1875, 6.875, 1.0\}$.

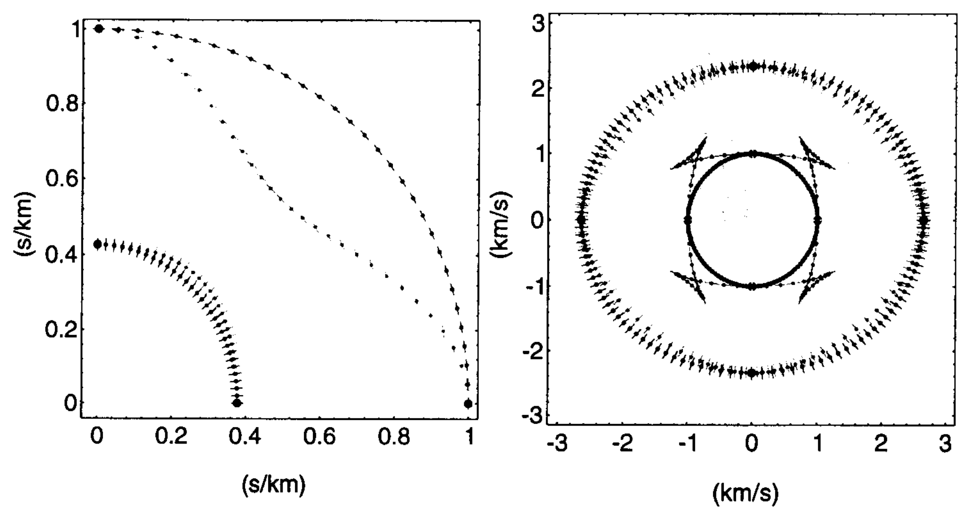

Figure 9. The result of perturbing the reference shale medium to one with equal qSV push-pin parameters, i.e. $S_{0^\circ} = S_{45^\circ}$, keeping $P_{0^\circ}, P_{45^\circ}$ and S_{0° fixed. The new medium has an isotropic qSV surface and an elliptical qP surface. The new moduli are $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, 4.25, 5.5, 1.0\}$.

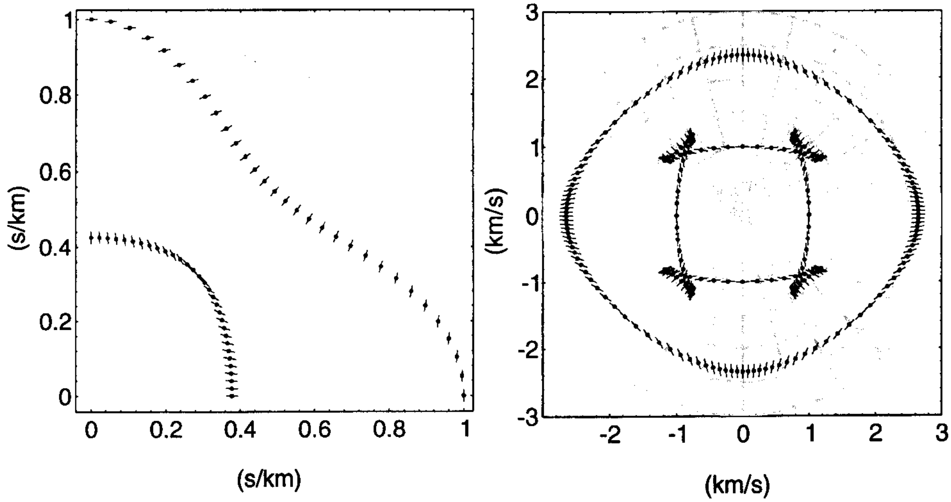

Figure 10. A new reference medium obtained by interchanging the off-axis push-pin parameters of the original medium, i.e. $P_{45^\circ} \leftrightarrow S_{45^\circ}$. The new medium has moduli $\{A^1_{11}, A^1_{13}, A^1_{33}, A^1_{55}\} = \{7.0, -4.5, 5.5, 1.0\}$. This figure is identical to Fig. 3 except for the polarizations.

Spencer 1994, Fig. 3; Every and Sachse 1992, Fig. 2). The perturbations to S_{0° and S_{45° are equal, i.e. $\delta S_{0^\circ} = \delta A_{55} = -\delta A_{13}/2 = \delta S_{45^\circ}$. In Fig. 6, S_{0° and S_{45° are fixed, and the perturbation to the qSV surface is not visible. We have arranged the perturbation so that A_{11}/A_{33} remains fixed and the qP surface scales proportionally.

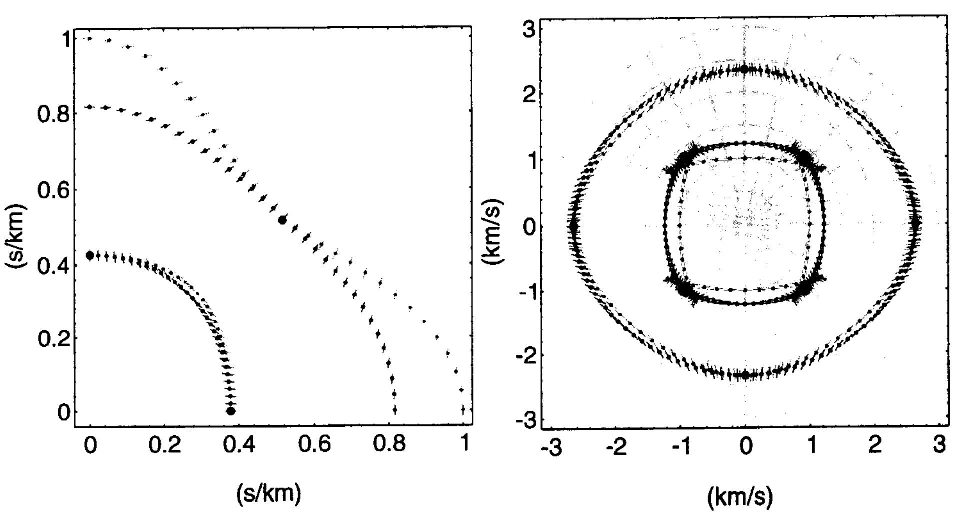

Figure 11. The result of perturbing the anomalous reference medium with a 50% increase in S_{0°, keeping the three qP push-pin parameters, $P_{0^\circ}, P_{45^\circ}$ and P_{90° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, -5.5, 5.5, 1.5\}$.

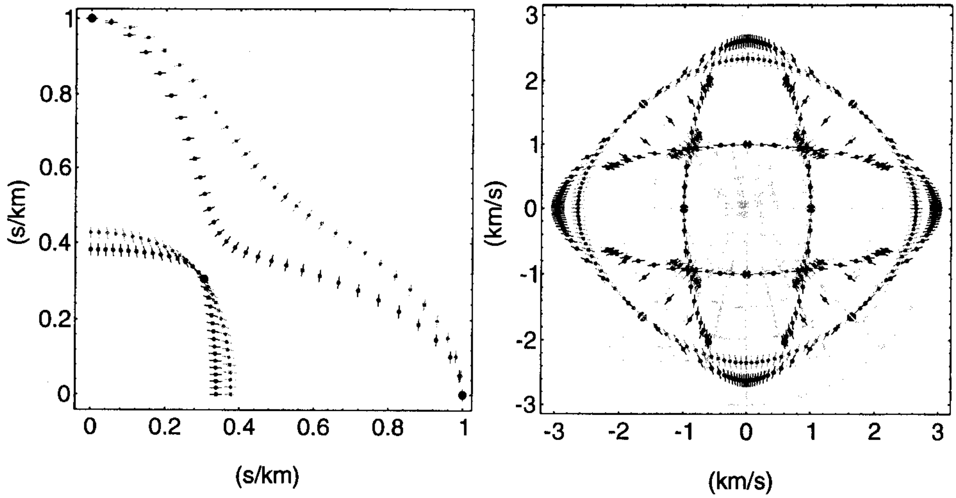

Figure 12. The result of perturbing the anomalous reference medium with a 25% increase in P_{0°, keeping the two qS push-pin parameters, S_{0° and S_{45°, and P_{0°/P_{90° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{8.75, -2.9375, 6.875, 1.0\}$.

Figure 7 illustrates observation 3.3. We have fixed P_{0°, P_{45° and S_{0°. It is of particular interest in anisotropic cross-well tomography, where observation angles beyond 45° may be unavailable. The qP surface is perturbed near the x_3-axis and the qSV surface near 45°. Figure 8 shows the effect of pinning the x_1-axis parameters P_{0° and S_{0°, and

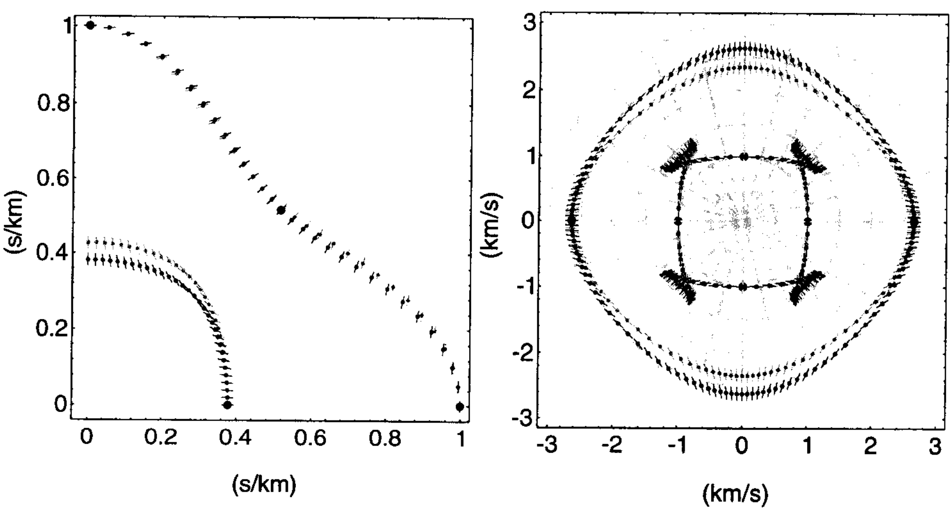

Figure 13. The result of perturbing the anomalous reference medium with a 25% increase in P_{90°, keeping P_{0°, S_{0° and P_{45° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, -5.1875, 6.875, 1.0\}$. This figure is identical to Fig. 8 except for the polarizations.

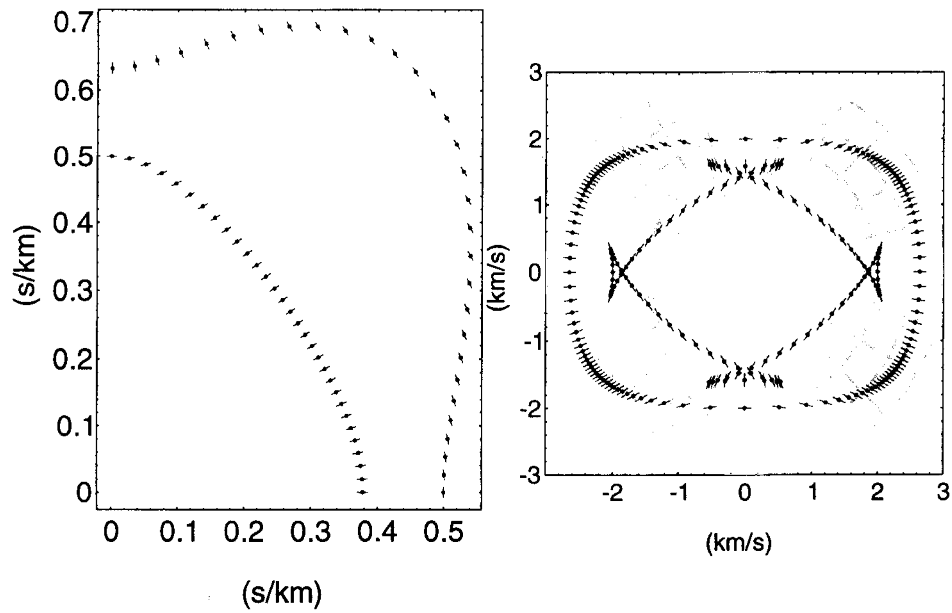

Figure 14. A new reference medium with anomalous polarization for vertical propagation. The new medium has moduli $\{A_{11}^2, A_{13}^2, A_{33}^2, A_{55}^2\} = \{7.0, 0.5, 2.5, 4.0\}$.

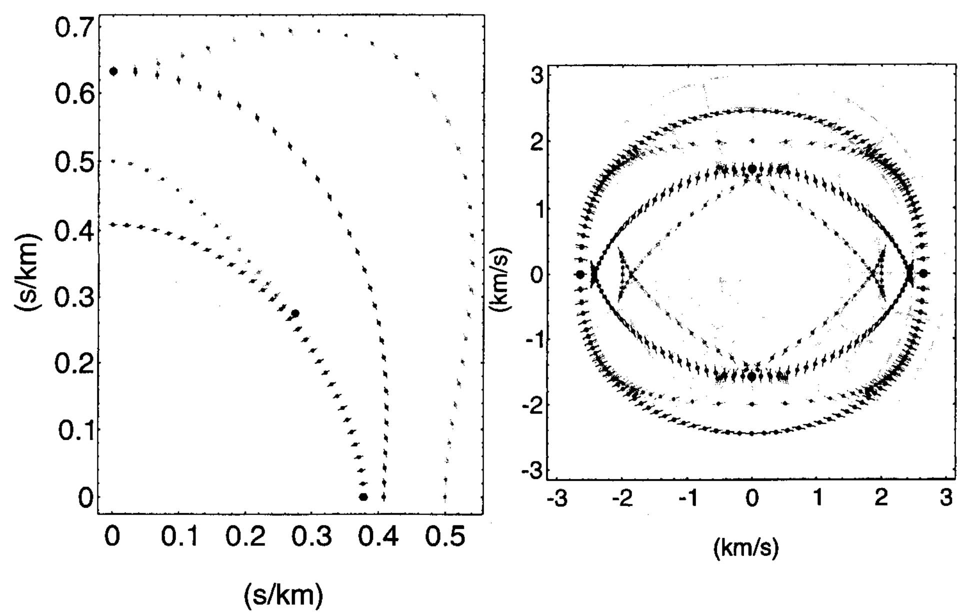

Figure 15. The result of perturbing the second anomalous reference medium with a 50% increase in S_{0°, keeping the three qP push-pin parameters, $P_{0^\circ}, P_{45^\circ}$ and P_{90° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, -3.5, 2.5, 6.0\}$.

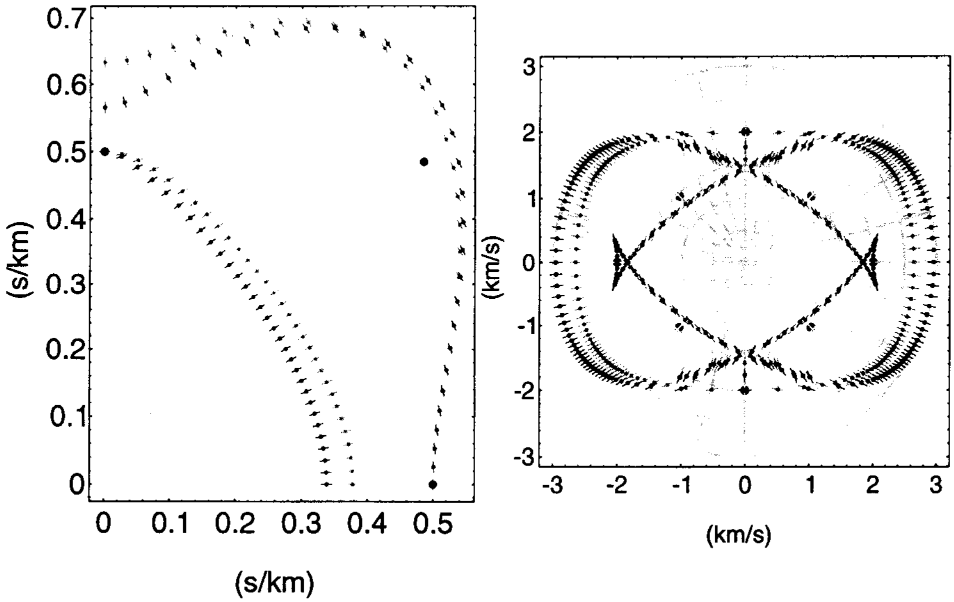

Figure 16. The result of perturbing the second anomalous reference medium with a 25% increase in P_{0°, keeping the two qS push-pin parameters, S_{0° and S_{45°, and P_{0°/P_{90° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{8.75, 1.6875, 3.125, 4.0\}$.

the off-axis qSV parameter S_{45°. The qSV surface is hardly perturbed, and the qP perturbation increases uniformly from 0° to 90°. Figure 9 shows the effect of pinning the three axial parameters and forcing $S_{45^\circ} = S_{0^\circ}$. This is a medium with elliptical anisotropy.

Figures 11–17 illustrate that our observations remain valid in anomalous media. It is important to emphasise that these examples are included only for mathematical interest. They should not be taken as representing any material that we are likely to encounter in physical studies of rocks. The reference medium for Figs. 11–13 is shown in Fig. 10. It is obtained from our original medium by pinning the axial parameters and interchanging the off-axis parameters. This is equivalent to changing the sign of $A_{13} + A_{55}$. Helbig and Schoenberg (1987) discuss this phenomenon. Notice that in Fig. 11, it is the slower surface that is fixed with P_{45° where the polarization is longitudinal. Similarly in Fig. 12, the faster surface is fixed with S_{45°. It is instructive to compare Fig. 13 with Figs 7 and 8. The shapes of the perturbed slowness and velocity surfaces in Fig. 13 are the same as Fig. 8, and yet the push-pins correspond to Fig. 7, i.e. P_{45° fixed. In this anomalous medium (Fig. 13), the slower surface has longitudinal polarization at 45° and so is fixed by P_{45°. The reference medium for Figs. 15–17, shown in Fig. 14, was chosen as a random example where anomalous polarization occurs on one of the axes. In Fig. 15, P_{90°

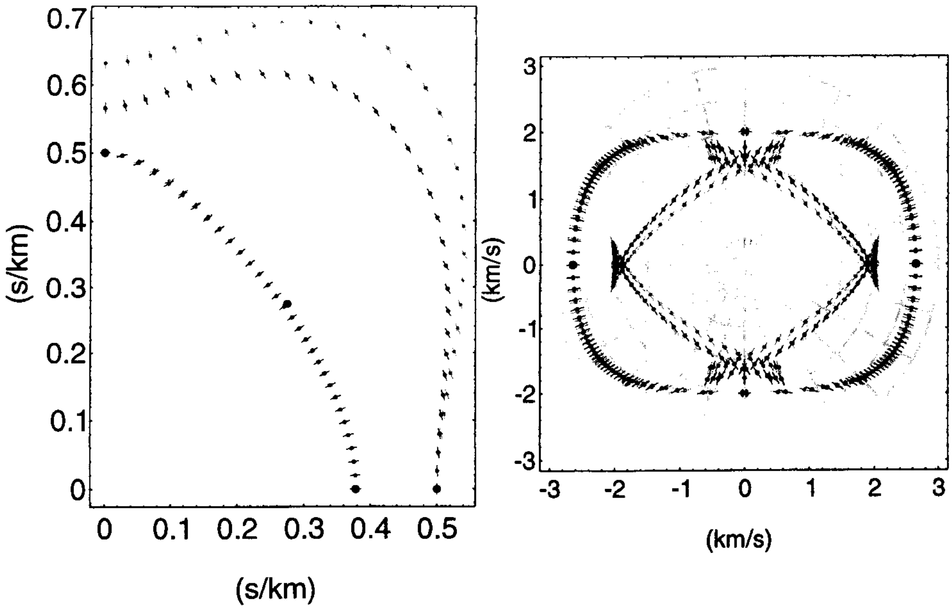

Figure 17. The result of perturbing the second anomalous reference medium with a 25% increase in P_{90°, keeping P_{0°, S_{0° and P_{45° fixed. The new medium has moduli $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, 0.1875, 3.125, 4.0\}$.

fixes the slower surface on the anomalous axis, and in Figs 16 and 17, S_{90° fixes the faster surface.

5. Conclusions

We have considered the problem of determining and predicting how the wave speeds in particular directions for a transversely isotropic medium depend on particular combinations of the density-normalized moduli A_{ij}. We have shown that the three axial parameters $P_{0^\circ} = A_{11}$, $P_{90^\circ} = A_{33}$, and $S_{0^\circ} = A_{55}$, together with two off-axis parameters, $P_{45^\circ} = (A_{11} + A_{33})/4 + (A_{13} + 2A_{55})/2$ and $S_{45^\circ} = (A_{11} + A_{33})/4 - A_{13}/2$, act as 'push-pins' controlling the magnitude of the squared-velocities on the axes and at approximately 45°. For an arbitrary TI medium, if the medium is perturbed in a way that preserves a given push-pin, then slowness points in the associated direction and mode will be approximately preserved in the new medium. The off-axis parameters, P_{45° and S_{45°, contain the combinations of parameters that appear in perturbation theory, i.e. (23) and (27), explaining their physical significance. We have also shown that in anomalous TI media the same 'push-pins' apply, provided they are appropriately assigned. Only in intermediate media, where the polarizations are neither dominantly longitudinal nor transverse, are the wave speeds sensitive to the 'missing' parameters: A_{55} for qP rays (47), or A_{11} and A_{33} for qSV rays (48).

Appendix

Anomalous polarizations

In normal TI media, the angle $\xi \simeq 0$ for qP waves and $|\xi| \simeq \pi/2$ for qSV waves. Expanding each term in (46) to leading order, we obtain

$$\delta p \simeq -\tfrac{1}{2}p^3\{\cos^2\psi\cos 2\psi\delta P_{0^\circ} + \sin^2 2\psi\delta P_{45^\circ} - \sin^2\psi\cos 2\psi\delta P_{90^\circ} + \sin^2\xi\delta A_{55}\} \tag{A1}$$

for the qP case, and

$$\delta p \simeq -\tfrac{1}{2}p^3\{\cos^2 2\psi\delta S_{0^\circ} + \sin^2 2\psi\delta S_{45^\circ} + \cos\xi\cos\psi\sin\psi\delta(A_{33} - A_{11})\} \tag{A2}$$

for the qSV case. Note these are not rigorous Taylor expansions of the complete expression (for instance in (47), the coefficient of δA_{11} contains terms $O(\xi)$), but indicate the order of magnitude of each term in its most significant range. Expression (A1) is equivalent to (41) but shows that the sensitivity to A_{55} of the qP slowness is $O(\xi^2)$. Expression (A2) is equivalent to (42) and shows the sensitivity to $(A_{33} - A_{11})$ is $O(\frac{\pi}{2} - \xi)$.

We can conclude that the slowness sensitivity is approximately (41) and (42), i.e. the weak TI expressions, unless the polarization is anomalous. Various authors have investigated anomalous polarizations, e.g. Helbig and Schoenberg (1987), Dellinger (1991). We review these results and confirm that the only circumstances in which the angle ξ differs significantly from 0 or $\pm\pi/2$ are when the velocities are anomalous.

Dellinger (1991) has given a compact expression for $\sin^2\xi$ (his equation (2.21)). For simplicity as the derivation requires much algebra, we just quote the result with minor notational changes

$$\cos 2\xi = \pm\frac{t_1\cos 2\psi - t_2}{(t_1^2 - t_2 a)^{1/2}}, \tag{A3}$$

where

$$t_1 = \cos^2\psi(A_{11} - A_{55}) - \sin^2\psi(A_{33} - A_{55}), \tag{A4}$$

$$t_2 = -4a\sin^2\psi\cos^2\psi, \tag{A5}$$

(Note that Dellinger's $s = \cos^2\psi$ as the angle is measured from the other axis, and a has been defined in (11)). We now investigate this expression.

In the range $\psi = 0$ to $\pi/2$, t_1 is monotonic. For normal TI media, $A_{11} > A_{55}$ and $A_{33} > A_{55}$ and t_1 decreases from a positive value at $\psi = 0$ to a negative value at $\psi = \pi/2$. However, anomalous values $A_{11} - A_{55} < 0$ and/or $A_{33} - A_{55} < 0$ are possible and t_1 may increase and/or have a negative value at $\psi = 0$ and/or a positive value at $\psi = \pi/2$. The term $t_1\cos 2\psi$ equals $A_{11} - A_{55}$ at $\psi = 0$ and $A_{33} - A_{55}$ at $\psi = \pi/2$. For normal media, it is zero at $\psi = \pi/4$ and at a point in the range $\psi = \pi/4$

to $\pi/2$ (see Fig. 18a, although for this numerical example the zeros are at approximately $\psi = 0.79$ and 0.86, and the minimum of $t_1 \cos 2\psi$ is -0.026, barely resolvable on the scale of the figure). The other factor in the numerator, t_2, is zero at $\psi = 0$ and $\pi/2$ and peaks at $\pi/4$. The sign of the peak depends on $-a = -(A_{13} + A_{55})$, and for normal media will be negative, but anomalous positive values are possible.
The numerator is therefore (see Fig. 18a)

$$t_1 \cos 2\psi - t_2 = A_{11} - A_{55} \qquad \text{at } \psi = 0, \tag{A6}$$

$$= A_{13} + A_{55} \qquad \text{at } \psi = \frac{\pi}{4}, \tag{A7}$$

$$= A_{33} - A_{55} \qquad \text{at } \psi = \frac{\pi}{2}. \tag{A8}$$

The behaviour of the denominator is simpler as both terms are positive. Values are (see Fig. 18b)

$$(t_1^2 - t_2 a)^{1/2} = |A_{11} - A_{55}| \qquad \text{at } \psi = 0, \tag{A9}$$

$$= \sqrt{(A_{13} + A_{55})^2 + \left(\frac{A_{11} - A_{33}}{4}\right)^2} \qquad \text{at } \psi = \frac{\pi}{4}, \tag{A10}$$

$$= |A_{33} - A_{55}| \qquad \text{at } \psi = \frac{\pi}{2}. \tag{A11}$$

The behaviour of the numerator is significant. For normal media, the three values (A6)–(A8) are all positive and (Fig. 18(c))

$$\frac{t_1 \cos 2\psi - t_2}{(t_1^2 - t_2 a)^{1/2}} \lesssim 1. \tag{A12}$$

Equality always occurs at $\psi = 0$ and $\pi/2$ (the axes are always pure mode directions) and possibly at an intermediate angle. This intermediate pure mode direction requires (Dellinger 1991, eq. (2.22)) that

$$\cos^2 \psi = \frac{(A_{33} - A_{13} - 2A_{55})}{(A_{33} - A_{13} - 2A_{55}) + (A_{11} - A_{13} - 2A_{55})}, \tag{A13}$$

and only exists if

$$\frac{A_{11} - A_{13} - 2A_{55}}{A_{33} - A_{13} - 2A_{55}} = \frac{(P_{0^\circ} - S_{0^\circ}) - (P_{45^\circ} - S_{45^\circ})}{(P_{90^\circ} - S_{90^\circ}) - (P_{45^\circ} - S_{45^\circ})} > 0. \tag{A14}$$

For normal media, the angle ξ is either approximately 0 or $\pi/2$. Anomalous directions of polarization only occur when one or more of the factors $A_{11} - A_{55}, A_{33} - A_{55}$ or $A_{13} + A_{55}$ change sign. Thus the weak, TI approximations (41) and (42) will not be significantly wrong unless these factors are small or negative. There are no other circumstances in which ξ varies significantly, so for normal media the qP slowness is

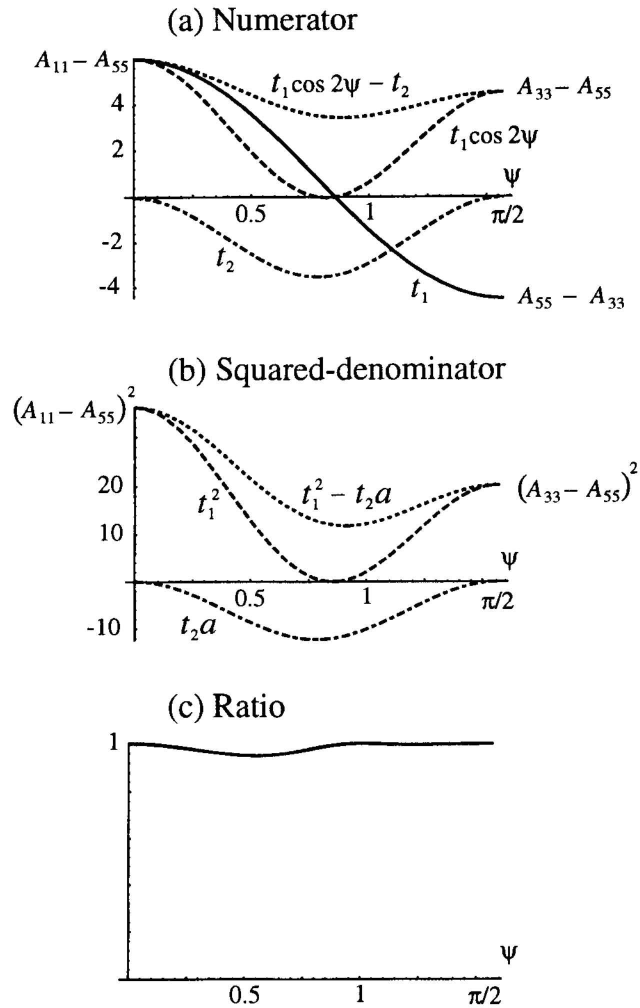

Figure 18. Expression (A3) for the TI medium in Fig. 3. (a). Factors in the numerator of (A3): t_1 (solid); $t_1 \cos 2\psi$ (dashed); t_2 (dot-dashed) and $t_1 \cos 2\psi - t_2$ (dotted). (b). Factors in the denominator of (A3): t_1^2 (dashed); $t_2 a$ (dot-dashed) and $t_1^2 - t_2 a$ (dotted). (c). The total expression (A3).

always insensitive to A_{55} (provided $A_{13} + 2A_{55}$ is fixed), and the qSV slowness is insensitive to A_{11} and A_{33} (provided $A_{11} + A_{33} - 2A_{13}$ is fixed). The only interesting behaviour is for anomalous media.

Anomalous TI media occur when at least one of the values (A6)–(A8) is negative. It is well known (from the positive-definite energy condition) that at most two of these values can be negative. If all values are negative then $-A_{13} > A_{55} > A_{11}$ and A_{33}, so $A_{13}^2 > A_{11}A_{33}$ which violates the energy condition. The exact result for the phase velocity (16) is symmetric with respect to an interchange of the indices 1 and 3,

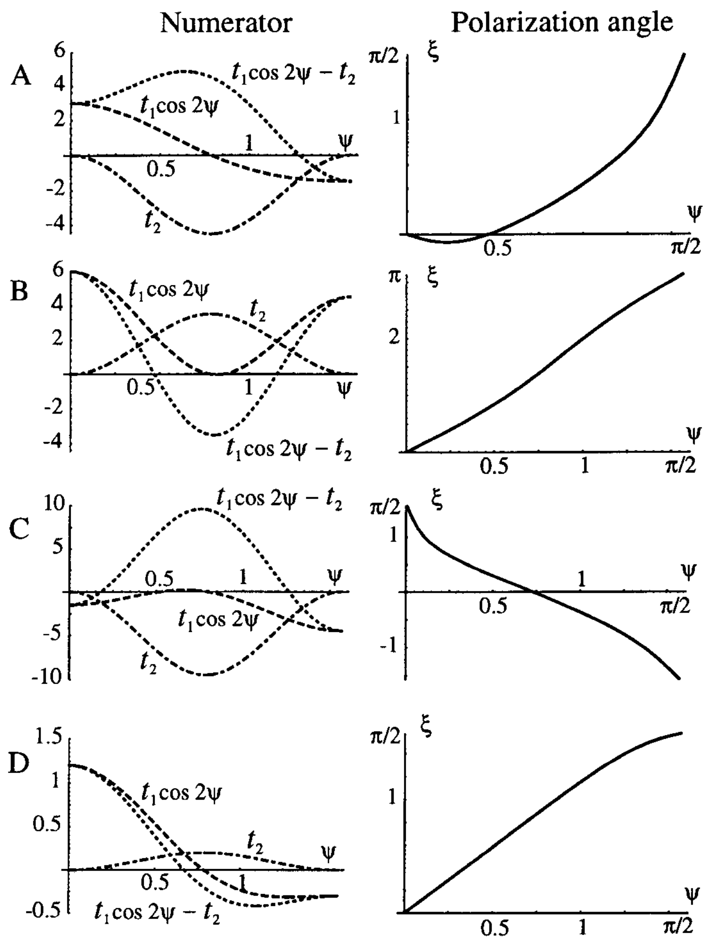

Figure 19. Results for anomalous media. For each case we have illustrated the terms in the numerator of (A3) and the polarization angle ξ for qP waves. Case A. $S_{90^\circ} > P_{90^\circ}$ with $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, 0.5, 2.5, 4.0\}$, i.e. Fig. 14; Case B. $S_{45^\circ} > P_{45}$ with $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, -4.5, 5.5, 1.0\}$, i.e. Fig. 10; Case C. $S_{0^\circ} > P_{0^\circ}$ and $S_{90^\circ} > P_{90^\circ}$ with $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{5.5, 2.5, 2.5, 7.0\}$; Case D. $S_{45^\circ} > P_{45^\circ}$ and $S_{90^\circ} > P_{90^\circ}$ with $\{A_{11}, A_{13}, A_{33}, A_{55}\} = \{7.0, -6.0, 5.5, 5.8\}$.

and so, without loss in generality, we can consider just $A_{11} > A_{33}$. First let us note that the anomalous conditions,

$$A_{11} < A_{55}, \tag{A15}$$

$$A_{13} + A_{55} < 0, \tag{A16}$$

$$A_{33} < A_{55}, \tag{A17}$$

are exactly equivalent to

$$P_{0^\circ} < S_{0^\circ}, \tag{A18}$$

$$P_{45^\circ} < S_{45^\circ}, \tag{A19}$$

$$P_{90^\circ} < S_{90^\circ}. \tag{A20}$$

The three possible anomalous conditions all correspond to reversals of the roles of the qP and qSV velocities. Condition (A19) is physically more obvious than (A16).

With the condition $A_{11} > A_{33}$ (the normal situation), four anomalous cases exist:
Case A. $S_{90^\circ} > P_{90^\circ}$;
Case B. $S_{45^\circ} > P_{45^\circ}$ (note that the slowness surface is independent of the sign of a ((11) in (16)), so normal media and this case can only be distinguished by the polarization);
Case C. $S_{0^\circ} > P_{0^\circ}$ and $S_{90^\circ} > P_{90^\circ}$;
Case D. $S_{45^\circ} > P_{45^\circ}$ and $S_{90^\circ} > P_{90^\circ}$.
As $P_{0^\circ} > P_{90^\circ}$, the case of $S_{0^\circ} > P_{0^\circ}$ alone is impossible as it implies Case C, but in general can be obtained from Case A. The situation $S_{0^\circ} > P_{0^\circ}$ and $S_{45^\circ} > P_{45^\circ}$ is impossible as it also implies $S_{90^\circ} > P_{90^\circ}$, but in general can be obtained from Case D.

In Fig. 19 we have illustrated the four anomalous situations. In each case where the roles of the velocities are reversed, the polarizations are anomalous in the same directions, i.e. if $S_{x^\circ} > P_{x^\circ}$, then the faster wave has transverse polarization at x°. The critical features in understanding the behaviour of ξ are the signs of the endpoints of $t_1 \cos 2\psi$ and the midpoint of t_2 in (A3). In Cases B and C, intermediate pure mode directions always exist (in Case B, it is anomalous). In the normal case and Cases A and D, an intermediate pure direction may or may not exist (see (A14)). With ξ varying significantly, the sensitivity (46) is more complicated and in general all four terms are significant.

For brevity, we do not consider the case $A_{11} < A_{33}$. The analysis is very similar to the above, and the results equivalent.

References

Červený V. 1972. Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophysical Journal of the Royal Astronomical Society* **29**, 1–13.

Červený V. 1982. Direct and inverse kinematic problems for inhomogeneous anisotropic media—linearization approach. *Contributions of the Geophysical Institute of the Slovak Academy of Science* **13**, 127–133.

Červený V. and Jech J. 1982. Linearized solutions of kinematic problems of seismic body waves in inhomogeneous slightly anisotropic media. *Journal of Geophysics* **51**, 96–104.

Chapman C.H. and Pratt R.G. 1992. Traveltime tomography in anisotropic media—I. Theory. *Geophysical Journal International* **109**, 1–19.

Costa J.C. 1993. *Modelagem sísmica e inversão na presença de anisotropia* (*Seismic modelling and inversion in the presence of anisotropy*). Ph.D. thesis, Universidade Federal do Pará, Belém, Paráz, Brasil.

Dellinger J.A. 1991. *Anisotropic seismic wave propagation.* Ph.D. thesis, Stanford University.

Every A.G. and Sachse W. 1992. Sensitivity of inversion algorithms for recovering elastic constants of anisotropic solids from longitudinal wavespeed data. *Ultrasonics* **30**, 43–48.

Helbig K. and Schoenberg M. 1987. Anomalous polarization of elastic waves in transversely isotropic media. *Journal of the Acoustical Society of America* **81**, 1235–1245.

Hornby B.E. 1995. *The elastic properties of shales.* Ph.D. thesis, Cambridge University.

Jech J. and Pšenčík I. 1989. First-order perturbation method for anisotropic media. *Geophysical Journal International* **99**, 369–376.

Jones L.E.A. and Wang H.F. 1981. Ultrasonic velocities in Cretaceous shales from the Williston basin. *Geophysics* **46**, 288–297.

Miller D.E. and Chapman C.H. 1991. Incontrovertible evidence of anisotropy in crosswell data. 61st SEG meeting Houston, Expanded Abstracts, 825–828.

Miller D.E., Leaney S. and Borland W.H. 1994. An *in situ* estimation of anisotropic elastic moduli for a submarine shale. *Journal of Geophysical Research* **99**, 21659–21665.

Miller D.E. and Spencer C. 1994. An exact inversion for anisotropic moduli from phase slowness data. *Journal of Geophysical Research* **99**, 21651–21657.

Musgrave M.J.P. 1970. *Crystal Acoustics.* Holden-Day, Inc.

Neighbours J.R. and Schacher G.E. 1967. Determination of elastic constants from sound-velocity measurements in crystals of general symmetry. *Journal of Applied Physics* **38**, 5366–5375.