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AN APPLICATION OF INVARIANT SETS TO GLOBAL DEFINABILITY[1]

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Abstract. Vaught's "*-transform method" is applied to derive a global definability theorem of M. Makkai from a classical theorem of Lusin.

In a recent article [10], R. L. Vaught introduced a construction which connects the invariant descriptive set theory of "logic actions" with the model theory of the infinitary language $L_{\omega_1\omega}$. In this paper we will use Vaught's construction to give a short derivation of a recent global definability theorem of M. Makkai [8] from a classical theorem of Lusin on countable-to-one continuous functions.

Makkai's theorem may be stated as follows. Assume that ρ is an arbitrary countable similarity type. Let $\boldsymbol{P}$ be a new n-ary relation symbol and let $\rho_1 = \rho + \boldsymbol{P}$ be the corresponding expansion of ρ. Given a sentence $\sigma \in L_{\omega_1\omega}(\rho_1)$ and a ρ-structure $\mathfrak{A} = (A, \underset{\sim}{S})$, let $M_\sigma(\mathfrak{A}) = \{P \subseteq A^n : (\mathfrak{A}, P) \models \sigma\}$.

THEOREM 1 (MAKKAI). *For each sentence $\sigma \in L_{\omega_1\omega}(\rho_1)$ the following are equivalent:*

(i) *For every countable ρ-structure $\mathfrak{A}$, $M_\sigma(\mathfrak{A})$ is countable.*

(ii) *There exists a set $\Phi = \{\varphi_i(v_1 \cdots v_{n+k_i}) : i \in \omega\} \subseteq L_{\omega_1\omega}(\rho)$ such that*

$$\sigma \models \bigvee_{i\in\omega} \exists v_{n+1} \cdots v_{n+k_i} \forall v_1 \cdots v_n(\boldsymbol{P}(v_1 \cdots v_n) \leftrightarrow \varphi_i).$$

Theorem 1 is the infinitary version of the well-known Chang-Makkai Theorem (cf. [3, 5.3.6]). This first-order result is easily derived from the infinitary version using H. J. Keisler's theory [5] of approximations to infinitary formulas—see Remark II below.

We will derive Makkai's theorem from the following theorem of Lusin. $f: B \to Y$ is *countable-to-one* if the preimage of every point in Y is countable. A topological space is *Polish* if it is separable and completely metrizable.

(1) *Let B be a Borel subset of a Polish space X and suppose f is a countable-to-one, continuous function on B to a metric space Y. Then there is a collection $\{B_i : i \in \omega\}$ of Borel sets such that $B = \bigcup_{i\in\omega} B_i$ and each $f{\restriction}_{B_i}$ is one-one.*

A variant of (1) is stated in Kuratowski [6, §39,VII, Corollary 5]. A proof may be found in Lusin [7]. Since this last reference is somewhat obscure, we have included a sketch of a proof of (1) in Remark IV below.

The central proof of this paper was included in the author's Ph. D. dissertation which was written under the supervision of R. L. Vaught. Thanks are also due to John Burgess for a stimulating conversation regarding Proposition 2.

Before proceeding with our proof of Theorem 1, we summarize the material from Vaught [10] which we require.

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ρ is viewed as a pair $(\rho', \mathcal{K}_\rho)$ where $\mathcal{I}$, $\mathcal{K}_\rho$ are disjoint countable sets and $\rho': \mathcal{I} \to \omega$.

$$X_\rho = \prod_{i \in \mathcal{I}} 2^{\omega^{\rho'(i)}} \times \omega^{\mathcal{K}_\rho}$$

is the topological space formed over the discrete spaces 2, ω. We identify each ρ-structure $(\mathfrak{A}, \underset{\sim}{S})$ with $\underset{\sim}{S} \in X_\rho$ to view X_ρ as the set of ρ-structures having universe ω. In the typical case of one binary relation, $X_\rho = 2^{\omega \times \omega}$. Note that X_ρ is Polish.

$\omega!$ is the group of permutations of ω. Given $n \in \omega$, let ${}^{\underline{n}}\omega$ be the set of one-one functions on n to ω. Given $s \in {}^{\underline{n}}\omega$, let $[s] = \{f \in \omega!: s \subseteq f\}$. The collection $\{[s]: n \in \omega, s \in {}^{\underline{n}}\omega\}$ forms a countable basis for the relative product topology on $\omega!$. With this topology, $\omega!$ is completely metrizable and a topological group. $\omega!$ acts continuously on X_ρ according to the map J_ρ: $(g, \underset{\sim}{S}) \mapsto g\underset{\sim}{S}$ where $(\omega, g\underset{\sim}{S})$ is the usual isomorph of $(\omega, \underset{\sim}{S})$ under g. $\mathcal{J}_\rho = (\omega!, X_\rho, J_\rho)$ is the *canonical logic action of type* ρ. $gB = \{g\underset{\sim}{S}: \underset{\sim}{S} \in B\}$. B is *invariant* if $gB = B$ for every $g \in \omega!$.

The central feature of Vaught's method is the *-transform defined as follows. Fix an invariant subspace $X \subseteq X_\rho$. Given $B \subseteq X$ and $\underset{\sim}{S} \in X$, let $B^{\underset{\sim}{S}} = \{g: g\underset{\sim}{S} \in B\}$. Define

$$B^* = \{\underset{\sim}{S} \in X: B^{\underset{\sim}{S}} \text{ is comeager}\},$$

$$B^\Delta = \sim(\sim B)^* = \{\underset{\sim}{S}: B^{\underset{\sim}{S}} \text{ is not meager}\}.$$

More generally, for $n \in \omega$, $s \in {}^{\underline{n}}\omega$, define

$$B^{*[s]} = \{\underset{\sim}{S}: B^{\underset{\sim}{S}} \cap [s] \text{ is comeager in } [s]\},$$

$$B^{*n} = \{(\underset{\sim}{S}, t): t \in {}^{\underline{n}}\omega \ \&\ \underset{\sim}{S} \in B^{*[t]}\}.$$

It is easily seen that for arbitrary B, $\langle B_i, i \in \omega\rangle$

(i) B^* and B^Δ are invariant,

(ii) $B = B^* = B^\Delta$ if and only if B is invariant,

(iii) $(\bigcup_i B_i)^\Delta = \bigcup_i B_i^\Delta$.

Also, for B Borel,

(iv) $B^\Delta = \bigcup_{n \in \omega} \bigcup_{s \in {}^{\underline{n}}\omega} B^{*[s]}$.

An n-formula of $L_{\omega_1\omega}(\rho)$ is a formula with free variables included in $\{v_0, \ldots, v_{n-1}\}$. Given an n-formula φ, we set $[\![\varphi^{(n)}]\!] = \mathrm{Mod}(\varphi) \cap X \times \omega^n = \{(\underset{\sim}{S}, \bar{a}): \underset{\sim}{S} \in X \ \&\ (\omega, \underset{\sim}{S}) \models \varphi(\bar{a})\}$. A sentence is a 0-formula, $[\![\varphi]\!] = [\![\varphi^{(0)}]\!]$. An inductive argument based on (iii), (iv) shows:

(2) For every $n \geq 0$ and every (relative) Borel set $B \subseteq X$, there is an n-formula φ of $L_{\omega_1\omega}(\rho)$ such that $B^{*n} = [\![\varphi^{(n)}]\!]$.

Now we are prepared to prove Theorem 1 from (1).

PROOF OF THEOREM 1. (ii) $\Rightarrow$ (i) is obvious. Assume (i).

Since the set of isomorphism types of finite ρ_1-structures is countable and every such isomorphism type is definable, we may assume that all models of σ are infinite.

Let $B = [\![\sigma]\!] \subseteq X_{\rho_1}$ and let $\pi: [\![\sigma]\!] \to X_\rho$ be the projection $(\underset{\sim}{S}, P) \mapsto \underset{\sim}{S}$. $[\![\sigma]\!]$ is Borel, π is continuous, and for each $\underset{\sim}{S}$, $\pi^{-1}(\underset{\sim}{S}) = M_\sigma((\omega, \underset{\sim}{S}))$ is countable. Thus, we may apply (1) to obtain Borel sets B_i, $i \in \omega$ such that $[\![\sigma]\!] = \bigcup_{i \in \omega} B_i$ and each $\pi_{\restriction B_i}$ is one-one.

$[\![\sigma]\!]$ is invariant, so

$$[\![\sigma]\!] = [\![\sigma]\!]^{\Delta} = (\bigcup_i B_i)^{\Delta} = \bigcup_i B_i^{\Delta} = \bigcup_i \bigcup_n \bigcup_{s \in \underline{n}_\omega} B_i^{*[s]}.$$

By (2), there is a set $\Psi = \{\psi_{im}(v_{n+1} \cdots v_{n+m}): i, m \in \omega\} \subseteq L_{\omega_1\omega}(\rho_1)$ such that for each $m, i \in \omega$, $(\underset{\sim}{S}, P) \in X_{\rho_1}$, and $s \in \omega^m$,

(3) $$(\omega, \underset{\sim}{S}, P, s) \models \psi_{im} \text{ if and only if } s \in {}^{\underline{m}}\omega \;\&\; (\underset{\sim}{S}, P) \in B_i^{*[s]}.$$

It follows that $\sigma \models \bigvee_{i\in\omega} \bigvee_{m\in\omega} \exists v_{n+1} \cdots v_{n+m} \psi_{im}$. We claim that for every $i, m \in \omega$, $s \in {}^{\underline{m}}\omega$, $\underset{\sim}{S} \in X_\rho$, $P_1, P_2 \in 2^{\omega^n}$

(4) $$[(\omega, \underset{\sim}{S}, P_1, s) \models \psi_{im} \;\&\; (\omega, \underset{\sim}{S}, P_2, s) \models \psi_{im}] \Rightarrow P_1 = P_2.$$

This suffices since (ii) then follows by the Beth definability theorem for $L_{\omega_1\omega}$.

The following computation together with (3) establishes (4):

$$\begin{aligned}(\underset{\sim}{S}, P_1), (\underset{\sim}{S}, P_2) \in B_i^{*[s]} &\Rightarrow B_i^{(\underset{\sim}{S}, P_1)} \cap B_i^{(\underset{\sim}{S}, P_2)} \cap [s] \text{ is comeager in } [s] \\ &\Rightarrow (\exists g \in [s])[(g\underset{\sim}{S}, gP_1), (g\underset{\sim}{S}, gP_2) \in B_i] \\ &\Rightarrow (\exists g \in \omega!)[gP_1 = gP_2] \\ &\Rightarrow P_1 = P_2. \qquad \square\end{aligned}$$

REMARKS. I. Given σ as in the theorem, it is apparent that each $M_\sigma((\omega, \underset{\sim}{S}))$ is an analytic (Σ_1^1) subset of 2^{ω^n}. It follows immediately from a famous theorem of Suslin (cf. [6, p. 479]), that condition (i) is equivalent to

(i′) For every $\underset{\sim}{S} \in X_\rho$, $M_\sigma((\omega, \underset{\sim}{S}))$ does not include a perfect subset.

II. In [5], H. J. Keisler developed a theory of finitary approximations to formulas of $L_{\omega_1\omega}$ which applies to the present situation as follows: Suppose a sentence $\sigma \in L_{\omega\omega}(\rho_1)$ satisfies (i). Applying Theorem 1, let Ψ be the infinitary sentence exhibited in (ii). We may assume that Ψ is in negation-normal form. Then by Keisler's results there is a sentence $\psi \in L_{\omega\omega}$ which "approximates" Ψ, such that $\sigma \models \psi$. Any such ψ has the form

$$\bigvee_{i\in I} \exists v_{n+1} \cdots v_{n+k_i} \forall v_1 \cdots v_n(\boldsymbol{P}(v_1, \ldots, v_n) \leftrightarrow \varphi_i)$$

where I is finite and each $\varphi_i \in L_{\omega\omega}(\rho)$. This constitutes a proof of the finitary Chang-Makkai Theorem (C-M) [3, 5.3.6]. The pair of derivations "(1) $\Rightarrow$ Theorem 1", "Theorem 1 $\Rightarrow$ C-M" adds to the list of cases where a result in first-order logic can be derived from an analogous topological theorem using Vaught's transform and Keisler's approximations. The canonical example of this phenomenon is the pair of derivations "Suslin separation $\Rightarrow$ Lopez-Escobar Interpolation" (implicit in [10]), and "Lopez-Escobar Interpolation $\Rightarrow$ Craig Interpolation" (implicit in [5]). For another example, see [9, §2]. J. W. Addison has been the primary advocate of the study of such analogies, see e.g. [1].

III. If the set B in (1) is only assumed to be analytic, the result still holds with each B_i required only to be Borel relative to B. This follows immediately from the following classical fact [7, p. 247]:

(5) *Assume that X, Y are Polish, B is an analytic subset of $X \times Y$, and that each X-cross-section of B is countable. Then B is included in a Borel set C twith the same property.*

Let E be an analytic equivalence relation on $X \times Y$ (such as the isomorphism relation on $X_\rho \times 2^{\omega^n} = X_{\rho_1}$), and suppose B is an E-invariant analytic set with countable cross-sections. By (5) and the Invariant First Separation Theorem (cf. [10, (10)], B is included in an invariant Borel set with the same property. It follows that the conclusion of Theorem 1 holds provided only that σ is an existential second-order ($\Sigma_1'^1$) sentence.

IV. As promised, we now sketch a proof of (1). We have introduced one modification to Lusin's argument (the use of the boundedness theorem) which will make it easier to extract the effective version of (1) in Remark V.

First of all, it is known (cf. [6, pp. 447, 443]):

(6) *If B is a Borel subset of a Polish space, then there exists a G_δ subset G of 2^ω and a continuous, one-one map of G onto B.*

Thus, since such a G is Polish, we may assume in (1) that B is Polish.

Now fix f, B, and Y as in (1). Given $C \subseteq B \times Y$ and $y \in Y$, let $C_y = \{x: (x, y) \in C\}$. Let G be the graph of f and for each $\alpha \in \omega_1$ and $y \in Y$, let G_y^α be the αth Cantor-Bendixson derivative of G_y. Let $B^\alpha = \{x: x \in G^\alpha_{f(x)}\}$. Let $\mathscr{H}$ be a countable basis for the topology of B and for each $U \in \mathscr{H}$ and each α, let $B^{\alpha,U} = \{x \in B^\alpha: \{x\} = G^\alpha_{f(x)} \cap U\}$. Then for any α,

$$B^{\alpha+1} = B^\alpha \sim \bigcup_{U \in \mathscr{H}} B^{\alpha,U} \quad \text{and} \quad B = \bigcup_{\beta \le \alpha} \bigcup_{U \in \mathscr{H}} B^{\alpha,U} \cup B^{\alpha+1}.$$

Clearly, f is one-one on each $B^{\alpha,U}$. An easy induction on α shows that each $B^{\alpha,U}$ is Borel. Thus, it suffices to establish

(7) For some $\alpha \in \omega_1$, $B^\alpha = \varnothing$.

Since f is continuous, each G_y is closed. It follows that for every $y \in Y$, there exists $\alpha \in \omega_1$ such that $G_y^\alpha = \varnothing$; let α_y be the least such α. (7) is equivalent to the assertion that $\sup\{\alpha_y: y \in Y\} < \omega_1$.

Let $D = \{R \in 2^{\omega\times\omega}$: for some $y \in Y$, (ω, R) is a linear order which can be embedded in $(\alpha_y, \in)\}$. Fix a recursive bijection F: $2^\omega \to (2^\omega)^{\omega\times\omega}$ and write $F(z) = \langle z_{ij}: i, j \in \omega\rangle$. Given $z \in 2^\omega$ let $z_i = \{z_{ij}: j \in \omega\}$. Given $y \in Y$, any subset of G_y has Cantor-Bendixson rank less than or equal to α_y. If follows that the assertion "$R \in D$" is equivalent to the assertion that R is a linear order and for some $y \in Y$ and some $z \in 2^\omega$

$$\begin{aligned} &\bigcup_{i\in\omega} z_i \subseteq G_y \;\&\; (\forall k, i)\,(k <_R i \to z_i \subsetneq z_k) \\ (8) \qquad &\;\&\; (\forall i,j)\,(z_{ij} \text{ isolated in } z_i \to (\forall k)\,(k >_R i \to z_{ij} \notin z_k)) \\ &\;\&\; (\forall i,j,k)\,(z_{ij} \notin z_k \to (\exists p)\,(p <_R k \;\&\; z_{ij} \text{ is isolated in } z_p)). \end{aligned}$$

Each of the clauses in (8) defines a Borel condition on R, y, and z. For example, the third clause may be written:

$$\bigwedge_{i,j,k} ((\bigvee_{U\in\mathscr{H}} (z_{ij} \in U \;\&\; \bigwedge_{p\neq j} (z_{ip} \notin U))) \to \bigwedge_{k,l} R(i,k) = 1 \to z_{ij} \neq z_{kl})).$$

Thus, D is an analytic set of well-orderings. It follows from the classical boundedness theorem [6, p. 501] that the order-types in D, and hence the ordinals $\{\alpha_y: y \in Y\}$ are bounded by some countable ordinal. (7) follows immediately.

V. In [2, IV, 4.6], J. Barwise shows that Theorem 1 (in the strong form of III) relativizes to any admissible fragment $L_A \subseteq L_{\omega_1\omega}$. The argument establishing (2) is highly effective (see [10, §5]), so our method can be used to derive Barwise's 4.6 from a suitable strengthening of (1).

By adding some extra remarks, we can modify the classical arguments to establish the required admissible versions of (1) and (5). Let $\boldsymbol{A}$ be a countable admissible set and let $X = X_{\rho_1}$, $Y = X_\rho$, where ρ, ρ_1 are $\boldsymbol{A}$-finite and $\rho \subseteq \rho_1$. Let $f = \pi$ be the projection map $X_{\rho_1} \to X_\rho$. Elements of X_{ρ_1} will be denoted $\underset{\sim}{T} = (\underset{\sim}{R}, \underset{\sim}{S})$ with $\underset{\sim}{S} = \pi(\underset{\sim}{T})$. A *Borel ρ_1-name* is a propositional (variable-free) sentence of $L_{\omega_1\omega}(\rho_1^+)$, where ρ_1^+ is the result of adding constant symbols $\mathbf{0}$, $\mathbf{1}$, ... to ρ_1. $\boldsymbol{\Sigma}_1^1$, $\boldsymbol{\Pi}_1^1$ ρ_1-names are defined similarly. Given a ρ_1-name σ, let $[\sigma] = \{T\colon (\omega, T, 0, 1, \ldots) \models \sigma\}$.

The admissible versions of (1) and (5) are:

($5_{\boldsymbol{A}}$) *Suppose σ is a $\boldsymbol{\Sigma}_1^1$-ρ_1-name such that $\sigma \in \boldsymbol{A}$ qnd each $\underset{\sim}{S}$-cross-section of $[\sigma]$ is countable. Then there is a Borel ρ_1-name $\psi \in \boldsymbol{A}$ such that $[\sigma] \subseteq [\psi]$ and $[\psi]$ has countable cross-sections.*

($1_{\boldsymbol{A}}$) *Suppose ψ is a Borel ρ_1-name such that $\psi \in \boldsymbol{A}$ and each $\underset{\sim}{S}$-cross-section of $[\psi]$ is countable. Then there is a sequence $\Phi = \langle \varphi_i\colon i \in I\rangle \in \boldsymbol{A}$ of Borel ρ_1-names, such that $[\psi] = \bigcup_{i\in I} [\varphi_i]$ and each $\underset{\sim}{S}$-cross-section of each $[\varphi_i]$ has at most one element.*

Consider ($5_{\boldsymbol{A}}$). We may assume that $\boldsymbol{A}$ is the admissible closure of $\{\sigma\}$. Given $\underset{\sim}{S}$, let $\boldsymbol{A}[\underset{\sim}{S}]$ be the smallest admissible set including $\boldsymbol{A}$ and containing $\underset{\sim}{S}$. $\mathrm{HYP}_{\boldsymbol{A}}(\underset{\sim}{S}) = \boldsymbol{A}[\underset{\sim}{S}] \cap X_{\rho_1}$, $[\sigma]_S = [\sigma] \cap \pi^{-1}\{\underset{\sim}{S}\}$, $H = \{\underset{\sim}{T}: \underset{\sim}{T} \in \mathrm{HYP}_{\boldsymbol{A}}(\pi(\underset{\sim}{T}))\}$. Then H is $\boldsymbol{\Pi}_1^1[\boldsymbol{A}]$ (i.e. $H = [\theta]$ for some $\boldsymbol{\Pi}_1^1$-name $\theta \in \boldsymbol{A}$, and since each $[\sigma]_{\underset{\sim}{S}}$ is countable, $[\sigma] \subseteq H$ (cf. [2, IV, 4.4]). ($5_{\boldsymbol{A}}$) follows by the Barwise interpolation theorem.

The argument outlined in IV can be modified in a straightforward fashion to establish ($1_{\boldsymbol{A}}$):

One proof of (6) is just the topological version of the familiar technique of "adding Skolem predicates". An effective version is given in [9, Lemma 4.1]. Applying this lemma, we can reduce to the case where ρ_1 contains only 0-ary relation symbols (one for each name in the fragment generated by our original ψ) and ψ names a G_δ subset of X_{ρ_1}. X_{ρ_1} may be identified with 2^{ρ_1} and names for basic open subsets of X_{ρ_1} may be identified with finite partial functions from ρ_1 to 2. Thus, $\boldsymbol{A}$ contains the set $\hat{\mathscr{H}}$ of names for elements of the canonical basis $\mathscr{H}$ for X_{ρ_1}.

A $\boldsymbol{\Delta}_1^1$-name φ is a pair (θ_1, θ_2) where θ_1 is a $\boldsymbol{\Sigma}_1^1$-name, θ_2 is a $\boldsymbol{\Pi}_1^1$-name, and $[\theta_1] = [\theta_2] = [\varphi]$. Using the fact that $[\psi] \subseteq H$, one easily obtains a $\boldsymbol{\Delta}_1^1$-name for each $[\psi]^{0,U}$, ($U \in \mathscr{H}$) and for $[\psi]^1 = [\psi] \sim \bigcup_{U\in\mathscr{H}} [\psi]^{0,U}$. The definition involves ψ only as a parameter and can be used to generate, for any α_0, a sequence $\langle \varphi_{\alpha,\hat{U}}\colon \alpha \in \alpha_0, \hat{U} \in \hat{\mathscr{H}}\rangle$ of $\boldsymbol{\Delta}_1^1$-names, primitive recursive in ψ and α_0, such that $[\varphi_{\alpha,\hat{U}}] = [\psi]^{\alpha,U}$ for each α and $\hat{U}$. One then obtains a corresponding list $\langle \theta_{\alpha,\hat{U}}\colon \alpha < \alpha_0, \hat{U} \in \hat{\mathscr{H}}\rangle$ of Borel names by applying the Barwise interpolation theorem. Thus, it suffices to show that the ordinals $\{\alpha_{\underset{\sim}{S}}\colon \underset{\sim}{S} \in X_\rho\}$ have an upper bound in $\boldsymbol{A}$. This assertion, however, follows immediately from the admissible version of the boundedness theorem and the fact that D, as defined in Remark IV, has a $\boldsymbol{\Sigma}_1^1$-name in $\boldsymbol{A}$. (In the admissible version of (8), one works with a bijection $X_{\rho_1} \to X_{\rho_1}^{\omega\times\omega}$.) This establishes ($1_{\boldsymbol{A}}$).

VI. We close the paper with a remark on a related topic. It is a consequence of (1) that

(9) *If B, Y are as in* (1), *and f: $B \to Y$ is Borel measurable and counable-to-one, then the image of f is a Borel subset of Y.*

The following application of this result illustrates how the topological approach can simplify an argument from model theory, and extend it to a more general context. The special case of Proposition 2 for logic actions was proved by V. Harnik and M. Makkai [4] by a model-theoretic argument. Their paper also discusses several applications of that result.

Suppose G is a Borel subset of a Polish space G', and that G is equipped with a group structure such that the map $(g, h) \mapsto gh^{-1}$ is Borel measurable (i.e. G is a standard Borel group). Let B be a Borel subset of a Polish space Y, and suppose G acts on B according to a Borel map J: $(g, y) \mapsto gy$ (i.e. B is a standard Borel G-space). For $y \in B$, define $H_y = \{g : gy = y\}$. The action induces the equivalence $E_J = \{(y, y') : (\exists g \in G)(gy = y')\}$ on B.

PROPOSITION 2. *Assume G, B, Y are as described in the preceding paragraph. Suppose that for every $y \in B$, H_y is countable. Then E_J is a Borel subset of $Y \times Y$.*

PROOF. Consider the map J^*: $G \times B \to Y \times Y$ defined by $J^*(g, y) = (gy, y)$. Since J is Borel, so is J^*. By our hypothesis, J^* is countable-to-one. Since $E_J = J^*(G \times B)$ the conclusion follows immediately from (9). □

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