ARTICLES

BOUNDS AND POLICIES FOR DYNAMIC ROUTING IN LOSS NETWORKS

DIMITRIS BERTSIMAS and THALIA CHRYSSIKOU

Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received April 1996; accepted October 1997)

We consider the problem of maximizing a weighted sum of expected rewards in steady-state in multiclass loss networks under dynamic routing and admission control, with Poisson arrivals and exponentially distributed holding times. By aggregating the underlying Markov decision process, we derive linear programming relaxations that contain the achievable performance region under all admissible policies and lead to a series of progressively tighter upper bounds. These relaxations allow stronger bounds at the expense of higher computational times. We further propose a series of routing and admission control policies from the relaxations that outperform, in computational experiments, other heuristic policies, such as the dynamic alternative routing with trunk reservation and the least busy alternative routing, variations of which are used in practice. The suboptimality guarantees defined as best bound/best policy range from 0 to 2.5% under symmetry conditions (symmetric network topology, arrival rates, link capacities, rewards), and from 4% to 10% under asymmetry conditions. We discuss the qualitative behavior of these policies and obtain insights about their performance.

A multiclass loss network is a stochastic network that routes multiple types of calls/messages that may differ in their arrival processes, holding times and generated rewards. Loss networks have been widely used as models for computer and telecommunications networks, local area networks, multiprocessor interconnection architectures, database structures, mobile radio and broadband packet networks. The importance of loss networks has led to extensive research in the last two decades. For a comprehensive review see Kelly (1991) and Ross (1995).

The two key problems that arise in the area of loss networks are performance analysis and optimization. The main performance analysis question is to characterize the performance of a *particular* policy of accepting and routing calls. The main optimization problem is to determine an optimal policy for accepting and routing calls in the network that maximizes a linear combination of expected rewards in steady-state.

With regard to analyzing the loss network performance given a particular admission control and dynamic routing policy, the central question is to estimate the probability that a call is blocked. Even for a policy that always accepts calls (maximum packing) and uses a static, i.e., a priori determined, routing policy the computation of the blocking probabilities is $\sharp P$-complete (Louth et al. 1994), a strong indication of the intractability of the problem. Given the intractability of the problem, simulation, approximations, and asymptotics as the size of the network increases are the primary tools for performance questions (see Lazarev and Starobinets 1977, Kelly 1990, Key 1990, Hunt and Laws 1992, Mitra and Gibbens 1992). Kelly (1986) and Mitra et al. (1993) develop approximate procedures for estimating blocking probabilities based on solving Erlang's formula under the assumption of independent blocking. Simulation of specific routing policies has also been widely used (see, for example, Weber 1964, Inoue et al. 1989, *IEEE Communications Magazine* 1990, Gibbens 1990, Mitra and Seery 1991, Ross and Wang 1992, Gibbens et al. 1993, Ash and Huang 1994).

With regard to optimizing the performance of loss networks, there are both *admission control* and *dynamic routing* decisions involved. An *admission control* policy determines whether to accept an incoming call, while a *dynamic routing* policy determines the route for each accepted call. It is well known that an optimal policy must have a mechanism for rejecting arriving calls, since a strategy that always accepts a call whenever it is possible to carry it (maximum packing) may carry significantly less traffic than a policy that occasionally does not admit arriving calls. Miller (1969) and Lippman (1975) show that the optimal policy for the significantly simpler single link problem is to use trunk reservation, which is an easily implemented control mechanism that gives priority to chosen traffic streams. Hunt and Laws (1993) prove that for large, completely connected and symmetric networks, the asymptotically optimal policy is the "Least Busy Alternative" (LBA) routing strategy with trunk reservation. Marbukh (1981) reports results on the analysis of the LBA strategy, while Ott and Krishnan (1992) propose a policy improvement algorithm starting with an initial fixed routing scheme. Stacey

Subject classifications: Programming, dynamic/optimal control: upper bounds for performance of loss network. Communication heuristics: policies for dynamic routing.
Area of review: TELECOMMUNICATIONS.

0030-364X/99/4703-0379 $05.00

and Songhurst (1987) and Gibbens et al. (1988) develop a heuristic algorithm, the dynamic alternative routing with trunk reservation (DAR), variants of which are currently being used in practice.

In order to compare the efficiency of various heuristic policies, Gibbens and Kelly (1990) and Kelly (1994) derive bounds on the performance of *any* dynamic routing policy. These performance bounds under Poisson arrival processes for calls and exponentially distributed holding times, involve the solution of a linear programming problem. For symmetric and fully connected networks with many alternative paths the bounds are known to be tight (see Kelly 1994).

In the present paper we propose a new mathematical programming approach for providing upper bounds on the performance of loss networks and deriving new heuristic policies using information from the bounds. In particular our contributions are as follows:

1. We propose a method to construct a series of progressively tighter bounds on the performance of loss networks. The technique can be seen as a generalization of the work of Bertsimas et al. (1994) for multiclass queueing networks to loss networks. It can also be seen as designing appropriate aggregations (relaxations) of the underlying Markov decision process. Its power stems from the fact that it takes into account interactions among the various classes of calls. In extensive computational experiments we demonstrate that the second order relaxation leads to almost identical bounds with the bounds derived in Kelly (1994), while the third order relaxation improves upon them especially for the case of asymmetric network topology and/or asymmetric parameters (arrival rates, link capacities, rewards). The major benefit of these relaxations is that they allow stronger bounds at the expense of higher computational times.

2. Using information from the bounds, we propose several heuristic policies. To the best of our knowledge, there is no systematic approach to derive policies from relaxations. Our work represents a proposal in this direction. The derived policies are quite close to the bounds, outperform policies that have been proposed in the literature (DAR, LBA) and variants of which are used in practice. We discuss the qualitative behavior of these policies and obtain insights about their performance.

The rest of the paper is structured as follows. In § 1, we formally define the loss network problem and the class of policies that we consider. In § 2, we introduce the technique to obtain bounds for the case of the single link problem in a way that illustrates the basic ideas without excessive notation. In § 3, we derive a series of bounds for loss networks. In § 4, we derive heuristic policies from the relaxations of the previous section. In § 5, we report computational results on the effectiveness of the new bounds and policies and discuss their behavior. Finally, in § 6, we include some concluding remarks.

1. PROBLEM FORMULATION

Consider a loss network that is represented by a complete directed graph $G = (V, A)$ with $N = |V|$ nodes and $|A| = N(N-1)$ ordered pairs of nodes (i, j). Calls of type (i, j) need to be routed from node i to node j and carry a reward w_{ij}. Arriving calls of type (i, j) may be routed either directly on link (i, j) or on a route $r \in R(i, j)$ (a path in G), where $R(i, j)$ is the set of alternative routes for calls of type (i, j).

We assume that $R(i, j)$ consists of only paths from i to j that visit only one intermediate node. Let C_{ij} be the capacity of the direct link (i, j). Note that if the network is not fully connected, then the missing links have $C_{ij} = 0$. Let $S(i, j) = \{(i, j)\} \cup R(i, j)$ be the set of routes for calls of type (i, j). When a call of type (i, j) arrives at node i, it can be routed through route r only if there is at least one free circuit on each link of the route. If it is accepted, it generates a revenue of w_{ij} and simultaneously holds one circuit on each link of the route r for the holding period of the call. Incoming calls of type (i, j) arrive at the network according to a Poisson process of rate λ_{ij}, while their holding period is assumed to be exponentially distributed with rate μ and independent of earlier arrivals, holding times, and the route used to carry the call. The problem is to find an admission control and dynamic routing policy to maximize the total expected reward in steady-state.

Let $n_{ij}^r(t)$ be the number of calls of type (i, j) routed through r that are present in the network at time t. The vector $\vec{n}(t)$ consisting of $n_{ij}^r(t)$, $(i, j) \in A$, $r \in S(i, j)$, is called the state of the system at time t. A policy is called *Markovian* if each decision is determined solely as a function of the current state of the system. Under a Markovian policy the loss network under study evolves as a continuous-time Markov chain. It is well known that it is sufficient to restrict our attention to Markovian policies (see, for example, Heyman and Sobel 1984).

Let $n_{ij}^r = E[n_{ij}^r(t)]$, where the expectation is taken with respect to the invariant distribution. We denote with $\vec{n}$ the vector of n_{ij}^r. We are interested in determining a scheduling policy that maximizes a linear reward function of the form

$$\sum_{(i,j)\in A} w_{ij} \left(\sum_{r \in S(i,j)} n_{ij}^r \right).$$

As the policies vary, the vectors $\vec{n}$ vary. The set of achievable vectors $\vec{n}$ is called the *achievable region*. If we can characterize the achievable region exactly, then we can transform the control problem to a mathematical programming problem. Our objective in the next two sections is to give approximate characterizations of the achievable region, i.e., to derive linear programming problems whose feasible regions contain the achievable region. In this way we obtain a series of progressively tighter upper bounds on the optimal expected reward.

2. SINGLE LINK: APPROXIMATE AND EXACT CHARACTERIZATIONS

Our goal in this section is to obtain a set of bounds on the performance of any Markovian dynamic routing policy for the problem of one link and two classes of arriving calls. The advantage of the single link case is that it illustrates simply and without excessive notation the ideas we will use for the general network case. Moreover, in this case we are able to explicitly characterize the achievable region. This characterization corresponds to modeling the problem as a Markov decision process.

Consider a single link of capacity C, with two different types of traffic. Let λ_1 and λ_2 be the arrival rates, μ be the service rate, and w_1 and w_2 be the rewards generated by accepting a type 1 and type 2 call, respectively. We are interested in maximizing the expected reward $\sum_{i=1}^{2} w_i n_i$, where n_i is the expected value of the number of calls of type i. Under any Markovian policy we consider the collection of times at which either an arrival or a completion of a call occurs. If t is an arrival epoch, we denote by $A_i(t)$ the decision to accept a call of type i at time t. Similarly, we denote by $\bar{A}_i(t)$ the decision to reject such a call.

2.1. The Max-Flow Bound

If we consider no interaction among the two classes of calls, then the following linear programming problem:

$$\text{maximize} \sum_{i=1}^{2} w_i n_i \tag{1}$$

subject to

$$\sum_{i=1}^{2} n_i \leqslant C,$$

$$n_i \leqslant \lambda_i/\mu, \quad i = 1, 2,$$

$$n_i \geqslant 0, \quad i = 1, 2,$$

provides an upper bound on the optimal expected reward (Gibbens and Kelly 1990, Franks and Rishel 1973). Note that the offered traffic λ_i/μ represents the expected number of calls in a $M/M/\infty$ system. Notice that while the linear program (1) provides a bound on the optimal reward, it does not provide a policy.

2.2. An Approximate Characterization

The problem can be modeled as a continuous time Markov decision process (MDP),

$$(n_1(t), n_2(t)).$$

Although this is a manageable description, we consider a partial description in order to illustrate the idea of our method.

We consider first a partial description of the state space by considering the individual MDPs $(n_i(t))$. The probability flow equations corresponding to these MDPs provide a bound on the optimal reward.

In the following theorem, we define as decision variables (in a linear programming sense) the quantities,

$$x(i, a) = P\{n_i(t) = a, A_i(t)\}, \; y(i, a) = P\{n_i(t) = a\},$$

where we assume that the loss network is operating under the stationary distribution.

Theorem 1. *The solution value of the following linear program*

$$\text{maximize} \sum_{i=1}^{2} w_i \sum_{a=0}^{C} a y(i, a) \tag{2}$$

subject to

$$\lambda_i x(i, a) + \mu a y(i, a)$$

$$= \lambda_i x(i, a-1) + \mu(a+1) y(i, a+1), \quad \forall i, a, \tag{3}$$

$$\sum_{a=0}^{C} y(i, a) = 1, \quad i = 1, 2,$$

$$\sum_{i=1}^{2} \sum_{a=0}^{C} a y(i, a) \leqslant C, \tag{4}$$

$$\sum_{a=0}^{C} a y(i, a) \leqslant \lambda_i/\mu, \quad i = 1, 2,$$

$$x(i, a) \leqslant y(i, a), \quad i = 1, 2, a = 0, \ldots, C-1,$$

$$x(i, a), y(i, a) \geqslant 0,$$

provides an upper bound on the optimal expected reward.

Proof. We will prove only Equation (3); the other equations are obvious from the definition of the variables. We consider the continuous Markov decision process (MDP) $(n_i(t))$. Equation (3) represents the probability flow equation of the MDP process (see Heyman and Sobel 1984, p. 244). The LHS of (3) is the rate of probability flow out of state $n_i(t) = a$, where the RHS is the rate of probability flow into state $n_i(t) = a$. □

Notice that the above linear program allows only limited "interaction" between the two classes of calls, in the sense that variables corresponding to different classes participate in the same constraints. Only the capacity constraint (4) allows the two classes to interact in the previous relaxation. Notice, however, that the interaction between the classes occurs only through the expected values n_i exactly in the same way as in the max-flow bound of the previous section. There is, however, an important difference: Unlike the solution of (1), the solution of relaxation (2) provides a policy as it specifies the probabilities $x(i, a) = P\{n_i(t) = a, A_i(t)\}$.

2.3. An Exact Characterization

As we have already mentioned, the problem can be modeled as a continuous time Markov decision process (MDP), $(n_1(t), n_2(t))$. We define as variables (in a linear programming sense) the quantities,

$$x(i, a, b) = P\{n_1(t) = a, n_2(t) = b, A_i(t)\},$$

$$y(a, b) = P\{n_1(t) = a, n_2(t) = b\}.$$

Theorem 2 (Miller 1969). *The solution value of the following linear program*

$$\text{maximize } w_1 \sum_{a=0}^{C} a \sum_{\{b \mid a+b \leqslant C\}} y(a, b)$$

$$+ w_2 \sum_{b=0}^{C} b \sum_{\{a \mid a+b \leqslant C\}} y(a, b) \tag{5}$$

subject to

$$\sum_{i=1}^{2} \lambda_i x(i, a, b) + \mu(a+b) y(a, b)$$
$$= \lambda_1 x(1, a-1, b) + \lambda_2 x(2, a, b-1) + \mu(a+1) y(a+1, b) + \mu(b+1) y(a, b+1), \quad a+b \leqslant C,$$
$$\sum_a \sum_b y(a, b) = 1,$$
$$x(i, a, b) \leqslant y(a, b), \quad i = 1, 2, a+b \leqslant C,$$
$$x(i, a, b), y(a, b) \geqslant 0,$$

is equal to the optimal expected reward.

Note that in the above formulation the capacity constraints are absent; they are automatically satisfied because the quantities $x(i, a, b)$ and $y(a, b)$ are defined only for $a + b \leqslant C$.

As Miller (1969) and Lippman (1975) demonstrated, the optimal policy is indeed a trunk reservation policy. Theorem 2 captures this characterization as follows: Assuming that $w_1 < w_2$, there exists a number a^*, such that $x(1, a, b) = 0$ for all $a + b \geqslant a^*$, while $x(1, a, b) = 1$ for all $a + b < a^*$. Moreover, $x(2, a, b) = 1$ for all $a, b (a + b \leqslant C)$. The number a^* is called the *trunk reservation parameter*.

The linear programming problems (1), (2), and (5) have $O(1)$, $O(C)$ and $O(C^2)$ variables, respectively. These relaxations suggest that we can formulate inexact relaxations that provide bounds on the optimal expected reward at significant computational savings. Moreover, some of these relaxations (Relaxation (2)) provide heuristic policies. Although the single link problem is a very tractable problem to solve exactly, these insights carry over to the network problem as we show in the next section.

3. APPROXIMATE POLYHEDRAL CHARACTERIZATIONS FOR LOSS NETWORKS

In this section, we derive a series of bounds on the performance of a loss network $G = (V, A)$. Let $n_{ij}^r(t)$ be the number of calls of type (i, j) that are routed through route r at time t. For an arrival epoch of type (i, j) at time t, we denote by $A_{ij}^r(t)$ the decision to accept a call of type (i, j) through route $r \in S(i, j)$ at time t. Similarly, we denote by $\bar{A}_{ij}(t)$ the decision to reject such a call. It is clear that the admission control and dynamic routing of calls can be formulated as an MDP using as variables

$$x_{ij}(\vec{a}, r) = P\{\vec{n}(t) = \vec{a}, A_{ij}^r(t)\}.$$

Although an MDP using these variables gives the exact optimal policy, because of the huge dimensions of the resulting linear program such an approach is not realistic. For this reason, we will consider different tractable relaxations that give rise to concrete policies. We first review one relaxation that has been proposed in the literature.

3.1. The Max-Flow Bound

Gibbens and Kelly (1990) prove that under all dynamic routing policies, the optimal expected reward in steady-state is bounded above by the following linear programming problem:

$$\text{maximize} \sum_{(i,j)\in A} \sum_{r\in S(i,j)} w_{ij} n_{ij}^r$$

subject to

$$n_{ij}^{(i,j)} + \sum_{r\in R(i,j)} n_{ij}^r \leqslant \lambda_{ij}/\mu, \quad \forall (i, j) \in A,$$
$$n_{ij}^{(i,j)} + \sum_{r\in R(k,l):(i,j)\in r} n_{kl}^r \leqslant C_{ij}, \quad \forall (i, j) \in A,$$
$$n_{ij}^{(i,j)} \geqslant 0, \; n_{ij}^r \geqslant 0, \quad \forall (i, j) \in A, r \in R(i, j).$$

The variable $n_{ij}^{(i,j)}$ denotes the direct flow from node i to node j and n_{ij}^r denotes the alternative flow through route r. It has been proven (Key 1990) that the bound can be achieved asymptotically for fully connected networks with symmetric parameters, as the network size and offered traffic increase, by the adaptive policy that routes traffic according to the solution of the above linear programming problem.

3.2. An Approximate Characterization: First-Order Relaxation

We consider again a partial description of the state space by considering the individual MDPs $(n_{ij}(t))$, where $n_{ij}(t)$ represents the total number of calls using link (i, j) at time t. The probability flow equations corresponding to these MDPs provide a bound on the optimal reward. In the Theorem 3, we define as decision variables the following quantities:

$$x_{ij}(a, (i, j)) = P\{n_{ij}(t) = a, A_{ij}^{(i,j)}(t)\},$$
$$x_{ij}(a, (k, i, j)) = P\{n_{ij}(t) = a, A_{kj}^{(k,i,j)}(t)\},$$
$$y_{ij}(a) = P\{n_{ij}(t) = a\}, \quad z_{ij}(r) = E[n_{ij}^r(t)].$$

Theorem 3. *The optimal expected reward in the loss network G is bounded above by the optimal value of the following linear problem*:

$$\text{maximize} \sum_{(i,j)\in A} w_{ij} \sum_{r\in S(i,j)} z_{ij}(r) \tag{6}$$

subject to

$$\mu a y_{ij}(a) = \lambda_{ij} x_{ij}(a-1, (i, j)) - \lambda_{ij} x_{ij}(a, (i, j)) + \sum_{k:(k,i,j)\in S(k,j)} \lambda_{kj} x_{ij}(a-1, (k, i, j)) - \sum_{k:(k,i,j)\in S(k,j)} \lambda_{kj} x_{ij}(a, (k, i, j)) + \sum_{l:(i,j,l)\in S(i,l)} \lambda_{il} x_{ij}(a-1, (i, j, l)) - \sum_{l:(i,j,l)\in S(i,l)} \lambda_{il} x_{ij}(a, (i, j, l)) + \mu(a+1) y_{ij}(a+1), \quad \forall (i, j) \in A, a = 0, \ldots, C_{ij}, \tag{7}$$

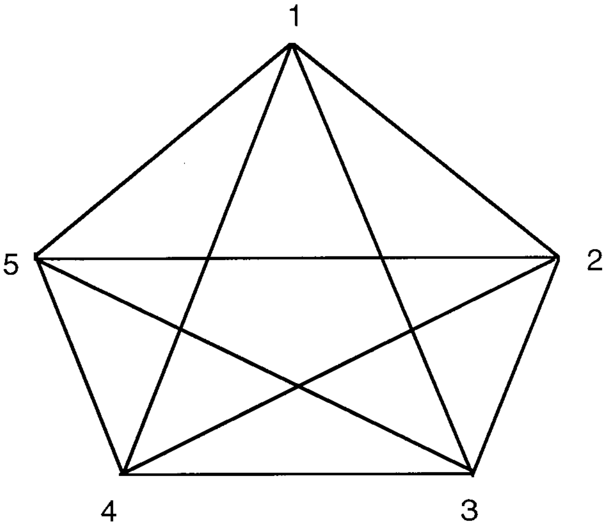

Figure 1. Five-node fully connected network.

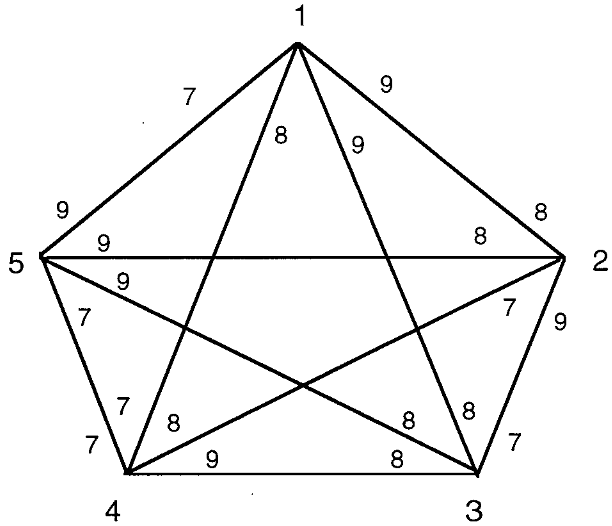

Figure 2. Asymmetric rates λ_{ij}.

$$\sum_{r \in S(k,l):(i,j) \in r} z_{kl}(r) = \sum_{a=0}^{C_{ij}} a y_{ij}(a), \quad \forall (i, j) \in A,$$

$$\sum_{a=0}^{C_{ij}} y_{ij}(a) = 1, \quad \forall (i, j) \in A,$$

$$\sum_{a=0}^{C_{ij}} a y_{ij}(a) \leqslant C_{ij}, \quad \forall (i, j) \in A,$$

$$\sum_{r \in S(i,j)} z_{ij}(r) \leqslant \lambda_{ij}/\mu, \quad \forall (i, j) \in A,$$

$$x_{ij}(a, (i, j)), x_{ij}(a, (k, i, j)) \leqslant y_{ij}(a),$$
$$\forall k: (k, i, j) \in S(k, j), \forall a,$$

$$x_{ij}(a, (i, j, l)) \leqslant y_{ij}(a), \quad \forall l: (i, j, l) \in S(i, l), \forall a,$$

$$x_{ij}(a, (i, j)), x_{ij}(a, (k, i, j)), \geqslant 0,$$
$$\forall (i, j) \in A, a = 0, \ldots, C_{ij},$$

$$x_{ij}(a, (i, j, l)), y_{ij}(a), z_{ij}(r) \geqslant 0,$$
$$\forall (i, j) \in A, a = 0, \ldots, C_{ij}.$$

Proof. We consider the continuous Markov decision process MDP ($n_{ij}(t)$). By equating the rate of probability flow out of the state ($n_{ij}(t) = a$) to the rate of probability flow into the state ($n_{ij}(t) = a$) we obtain:

$$\lambda_{ij} P\{n_{ij}(t) = a, A_{ij}^{(i,j)}(t)\}$$
$$+ \sum_{k:(k,i,j) \in S(k,j)} \lambda_{kj} P\{n_{ij}(t) = a, A_{kj}^{(k,i,j)}(t)\}$$
$$+ \sum_{l:(i,j,l) \in S(i,l)} \lambda_{il} P\{n_{ij}(t) = a, A_{il}^{(i,j,l)}(t)\} + \mu a P\{n_{ij}(t) = a\}$$
$$= \lambda_{ij} P\{n_{ij}(t) = a - 1, A_{ij}^{(i,j)}(t)\}$$
$$+ \sum_{k:(k,i,j) \in S(k,j)} \lambda_{kj} P\{n_{ij}(t) = a - 1, A_{kj}^{(k,i,j)}(t)\}$$
$$+ \sum_{l:(i,j,l) \in S(i,l)} \lambda_{il} P\{n_{ij}(t) = a - 1, A_{il}^{(i,j,l)}(t)\}$$
$$+ \mu(a + 1) P\{n_{ij}(t) = a + 1\}.$$

Rearranging leads to (7). Apart from the capacity and offered traffic constraints, the other relations are obvious from the definition of the variables. □

The number of variables used in the previous relaxation of the achievable performance region is $O(NRC)$, where N is the number of demands (i, j), R is the maximum number of alternative paths for every link and C is the largest link capacity. Notice that the above relaxation allows limited "interaction" among the various classes of calls (i, j) in the network. Interaction of various classes occurs in the link capacity constraints.

3.3. Aggregate Characterization: Second-Order Relaxation

In this section we consider a more detailed aggregation of the underlying MDP. In particular, we consider the aggregated MDP, $(n_{ij}^1(t), n_{ij}^2(t))$, where the superscript (1) represents the direct calls and the superscript (2) represents all the alternative flow that uses link (i, j). We denote by $A_{ij}^1(t)$ the decision to accept a call of type (i, j) directly at time t and $A_{ij}^2(t)$ the decision to accept any call that uses link (i, j) (i.e., a call from k to j through i for some k, or a call from i to l through j from some l). We define the following variables:

$$x_{ij}(a, b, \theta) = P\{n_{ij}^1(t) = a, n_{ij}^2(t) = b, A_{ij}^{\theta}(t)\},$$
$$\theta = 1, 2,$$
$$y_{ij}(a, b) = P\{n_{ij}^1(t) = a, n_{ij}^2(t) = b\}, \quad z_{ij}(r) = E[n_{ij}^r(t)].$$

Let

$$\lambda_{ij}^1 = \lambda_{ij} \text{ and } \lambda_{ij}^2 = \sum_{k:(k,i,j) \in S(k,j)} \lambda_{kj} + \sum_{l:(i,j,l) \in S(i,l)} \lambda_{il}.$$

Using similar arguments as before, we obtain the following result.

Table I
Numerical Results for the Symmetric Network Topology with $w_{ij} = 1$

Instances	$C_{ij} = 10$ $\lambda_{ij} = 9$	$C_{ij} = 10$ Figure 2	$C_{ij} = 10$ Figure 3	$C_{ij} = 10$ Figure 4	Figure 5 $\lambda_{ij} = 7$	Figure 5 Figure 2	Figure 5 Figure 3	Figure 5 Figure 4
Max-Flow	180	161	177	200	140	161	172	186
Relax. (6)	180	161	176.3006	196.9953	140	160.7972	170.1667	183.3176
Relax. (8)	153.5349	147.7839	152.1222	162.4981	135.3189	141.5745	145.7076	155.1323
Kelly (1994)	153.5349	147.7839	152.1222	162.4981	135.3189	141.5745	145.7076	155.1323
Relax. (9)	153.5349	147.7170	152.0445	162.4843	135.1272	141.3890	145.5892	155.0787
DAR	150.3910	141.7375	148.3200	161.5540	126.5590	136.1390	141.9740	154.0055*
	± 0.0799	± 0.0804	± 0.0958	± 0.0810	± 0.1013	± 0.1165	± 0.0779	± 0.0611
DYN	152.6375*	144.6585	150.3555*	162.3525*	126.9810	136.3105	142.0090	153.8275
	± 0.0971	± 0.0829	± 0.0696	± 0.0626	± 0.1473	± 0.0992	± 0.0485	± 0.0745
STATIC	151.8570	145.2255*	148.9275	160.3025	131.9275*	138.3345*	142.6140*	151.6790
	± 0.1639	± 0.1419	± 0.1659	± 0.1811	± 0.1519	± 0.1542	± 0.1102	± 0.1527

Theorem 4. *The optimal expected reward in the loss network G is bounded above by the optimal value of the following linear problem*:

$$\text{maximize} \sum_{(i,j)\in A} w_{ij}\left(\sum_{a=0}^{C_{ij}} a \sum_{b:a+b\leqslant C_{ij}} y_{ij}(a, b) + \sum_{r\in R(i,j)} z_{ij}(r)\right) \quad (8)$$

subject to

$$\lambda_{ij}^1 x_{ij}(a, b, 1) + \lambda_{ij}^2 x_{ij}(a, b, 2) + (a+b)\mu_{ij} y_{ij}(a, b) = \lambda_{ij}^1 x_{ij}(a-1, b, 1) + \lambda_{ij}^2 x_{ij}(a, b-1, 2) + (a+1)\mu y_{ij}(a+1, b) + (b+1)\mu y_{ij}(a, b+1), \quad \forall (i, j),\ (a, b): a+b \leqslant C_{ij},$$

$$\sum_{r\in R(k,l):(i,j)\in r} z_{kl}(r) = \sum_{b=0}^{C_{ij}} b \sum_{a:a+b\leqslant C_{ij}} y_{ij}(a, b), \quad \forall (i, j) \in A,$$

$$\sum_{a=0}^{C_{ij}} \sum_{b:a+b\leqslant C_{ij}} y_{ij}(a, b) = 1, \quad \forall (i, j) \in A,$$

$$\sum_{a=0}^{C_{ij}} a \sum_{b:a+b\leqslant C_{ij}} y_{ij}(a, b) + \sum_{b=0}^{C_{ij}} b \sum_{a:a+b\leqslant C_{ij}} y_{ij}(a, b) \leqslant C_{ij}, \quad \forall (i, j) \in A,$$

$$\sum_{a=0}^{C_{ij}} a \sum_{b:a+b\leqslant C_{ij}} y_{ij}(a, b) + \sum_{r\in R(i,j)} z_{ij}(r) \leqslant \lambda_{ij}/\mu, \quad \forall (i, j) \in A,$$

$$x_{ij}(a, b, \theta) \leqslant y_{ij}(a, b), \quad \forall (i, j),\ \theta,\ (a, b): a+b \leqslant C_{ij},$$

$$x_{ij}(a, b, \theta),\ y_{ij}(a, b),\ z_{ij}(r) \geqslant 0, \quad \forall (i, j),\ \theta,\ (a, b): a+b \leqslant C_{ij}.$$

The number of variables used in the above characterization is $3NC^2 + NR$, where N is the number of demands (i, j) and C is the largest link capacity.

In Relaxation (8) we have expanded the analysis used on the single-link problem to the network case and we have treated each link as if it were in isolation. This formulation takes into account at every link *both* the direct and the aggregate alternative flow. This particular way of interaction among classes, although implemented quite differently, was introduced in Kelly (1994) using dynamic programming arguments. The degree of interaction between the classes of calls in Relaxation (8) is higher than in Relaxation (6). In the computational section we will see that Relaxation (8) leads to tighter bounds than the ones derived from Relaxation (6).

3.4. Approximate Characterization: Third-Order Relaxation

In this section we consider a similar relaxation as in the previous one, but we now augment the state space by introducing the decision to accept on a *specific* alternative route that uses link (i, j). This new approximation of the performance region allows interaction between the direct calls using link (i, j) and the calls that are routed alternatively using a specific route consisting of the link (i, j). We define the following decision variables:

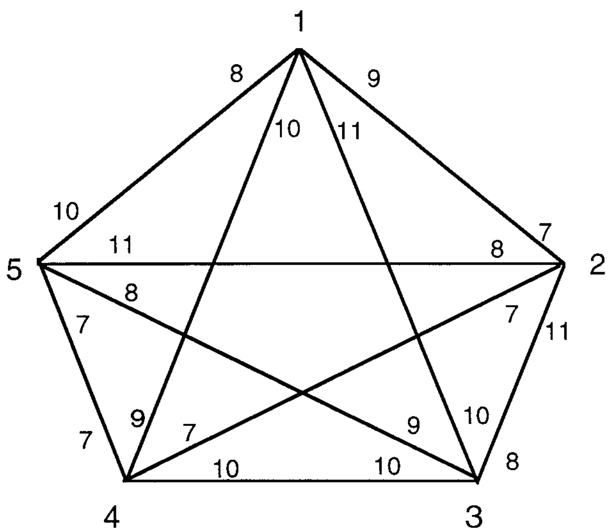

Figure 3. Asymmetric rates λ_{ij}.

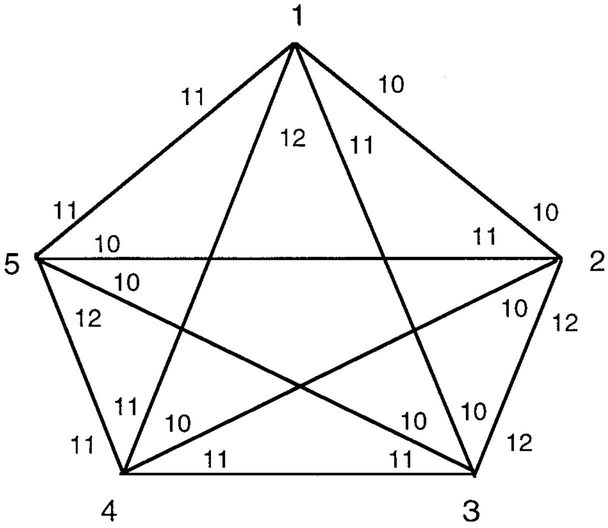

Figure 4. Asymmetric rates λ_{ij}.

$$x_{ij}(a,b,(i,j)) = P\{n_{ij}^1(t)=a,\ n_{ij}^2(t)=b,\ A_{ij}^{(i,j)}(t)\},$$
$$x_{ij}(a,b,(k,i,j)) = P\{n_{ij}^1(t)=a,\ n_{ij}^2(t)=b,\ A_{kj}^{(k,i,j)}(t)\},$$
$$x_{ij}(a,b,(i,j,l)) = P\{n_{ij}^1(t)=a,\ n_{ij}^2(t)=b,\ A_{il}^{(i,j,l)}(t)\},$$
$$y_{ij}(a,b) = P\{n_{ij}^1(t)=a,\ n_{ij}^2(t)=b\},$$
$$z_{ij}(r) = E[n_{ij}^r(t)].$$

Theorem 5. *The optimal expected reward in the loss network G is bounded above by the optimal value of the following linear problem*:

$$\text{maximize} \sum_{(i,j)\in A} w_{ij}\left(\sum_{a=0}^{C_{ij}} a \sum_{b:a+b\le C_{ij}} y_{ij}(a,b) + \sum_{r\in R(i,j)} z_{ij}(r)\right) \quad (9)$$

subject to

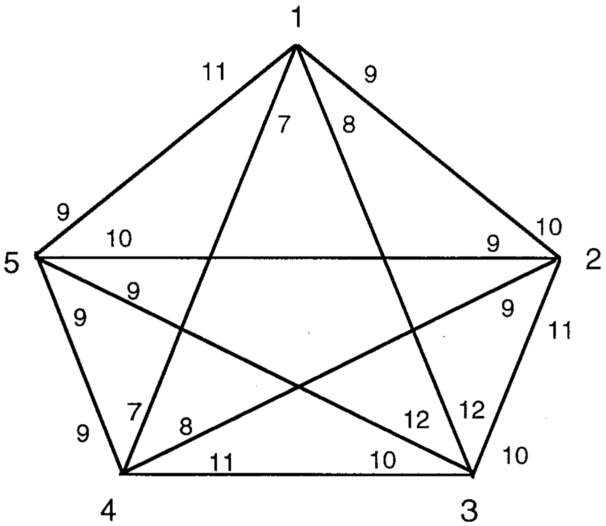

Figure 5. Asymmetric capacities C_{ij}.

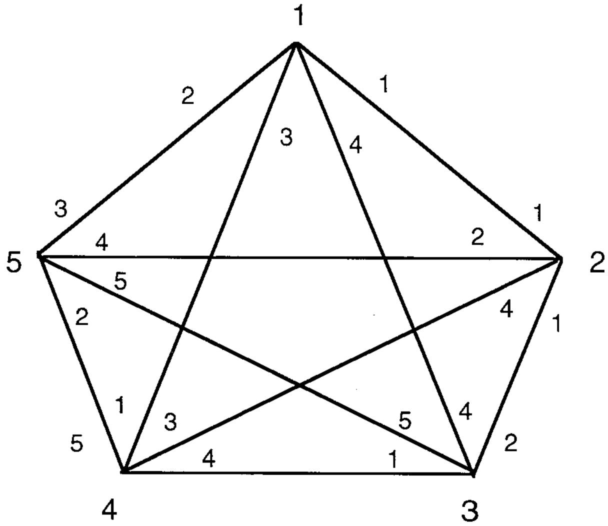

Figure 6. Asymmetric rewards w_{ij}.

$$\begin{aligned}
\mu(a+b)y_{ij}(a,b) &= \lambda_{ij}x_{ij}(a-1,b,(i,j)) \\
&- \lambda_{ij}x_{ij}(a,b,(i,j)) \\
&+ \sum_{k:(k,i,j)\in S(k,j)} \lambda_{kj}x_{ij}(a,b-1,(k,i,j)) \\
&+ \sum_{k:(k,i,j)\in S(k,j)} \lambda_{kj}x_{ij}(a,b,(k,i,j)) \\
&+ \sum_{l:(i,j,l)\in S(i,l)} \lambda_{il}x_{ij}(a,b-1,(i,j,l)) \\
&- \sum_{l:(i,j,l)\in S(i,l)} \lambda_{il}x_{ij}(a,b,(i,j,l)) \\
&+ \mu(a+1)y_{ij}(a+1,b) + \mu(b+1)y_{ij}(a,b+1), \quad (10)
\end{aligned}$$

$\forall(i,j),\ (a,b):a+b\le C_{ij}$,

$$\sum_{r\in R(k,l):(i,j)\in r} z_{kl}(r) = \sum_{b=0}^{C_{ij}} b \sum_{a:a+b\le C_{ij}} y_{ij}(a,b),\ \forall(i,j),$$

$$\sum_{a=0}^{C_{ij}} \sum_{b:a+b\le C_{ij}} y_{ij}(a,b) = 1, \quad \forall(i,j)\in A,$$

$$\sum_{a=0}^{C_{ij}} a \sum_{b:a+b\le C_{ij}} y_{ij}(a,b) + \sum_{b=0}^{C_{ij}} b \sum_{a:a+b\le C_{ij}} y_{ij}(a,b) \le C_{ij},$$

$\forall(i,j)\in A$,

$$\sum_{a=0}^{C_{ij}} a \sum_{b:a+b\le C_{ij}} y_{ij}(a,b) + \sum_{r\in R(i,j)} z_{ij}(r) \le \lambda_{ij}/\mu,$$

$\forall(i,j)\in A$,

$$\sum_{a,b} x_{ik}(a,b,(i,k,j)) = \sum_{a,b} x_{kj}(a,b,(i,k,j)), \quad (11)$$

$\forall(i,j)\in A,\ k:(i,k,j)\in S(i,j)$,

$x_{ij}(a,b,(i,j)),\ x_{ij}(a,b,(k,i,j) \le y_{ij}(a,b)$,

$k:(k,i,j)\in S(k,j),\ a+b\le C_{ij}$,

$x_{ij}(a,b,(i,j,l) \le y_{ij}(a,b)$,

$l:(i,j,l)\in S(i,l),\ a+b\le C_{ij}$,

$x_{ij}(a,b,(i,j)),\ x_{ij}(a,b,(k,i,j) \ge 0$,

$x_{ij}(a,b,(i,j,l)),\ y_{ij}(a,b),\ z_{ij}(r)\ge 0.$

Table II
Numerical Results for the Symmetric Network Topology with Asymmetric w_{ij}

Instances	$C_{ij} = 10$ $\lambda_{ij} = 9$	$C_{ij} = 10$ Figure 2	$C_{ij} = 10$ Figure 3	$C_{ij} = 10$ Figure 4	Figure 5 $\lambda_{ij} = 7$	Figure 5 Figure 2	Figure 5 Figure 3	Figure 5 Figure 4
Max-Flow	513	462	505	575	399	462	496	547
Relax. (6)	513	462	504.3006	568.1215	399	461.7972	492.2673	540.6047
Relax. (8)	475.7227	446.0763	470.7498	512.4231	393.3802	433.4687	453.1082	489.1450
Kelly (1994)	475.7227	446.0763	470.7498	512.4231	393.3802	433.4687	453.1082	489.1450
Relax. (9)	475.2197	445.9519	470.1752	511.0087	393.2768	432.7652	451.7711	487.6016
DAR	428.7290	407.4830	421.2355	458.3565	361.2795	390.6085	406.7500	439.2390
	± 0.3282	± 0.3521	± 0.3048	± 0.2486	± 0.3300	± 0.3617	± 0.2883	± 0.2386
DYN	430.2970	404.5310	422.0240	448.7120	362.9500	390.0040	408.5395	433.5815
	± 0.3621	± 0.3383	± 0.3114	± 0.3176	± 0.3992	± 0.3892	± 0.2805	± 0.3670
STATIC	436.7510*	420.7865*	439.3345*	466.8990*	378.5880*	406.8750*	418.7945*	445.8550*
	± 0.5270	± 0.3933	± 0.4152	± 0.5214	± 0.4567	± 0.5769	± 0.7922	± 0.7179

Proof. We consider the continuous Markov decision process (MDP) $(n^1_{ij}(t), n^2_{ij}(t))$. Equation (10) represents conservation of probability flow out of and into state $(n^1_{ij}(t) = a, n^2_{ij}(t) = b)$. The other equations are obvious from the definition of the variables. □

The number of variables in the linear program (9) is $2NRC^2$. Note that we can overcome the difficulty imposed by the large capacity by scaling the actual link capacity and changing its measure so as to represent a block of circuits.

The methodology that we have developed so far leads to a polyhedral set that contains the performance region of loss networks and takes into account interactions between calls routed directly or alternatively. We can easily generalize the proposed method and by augmenting the aggregated state space of the underlying MDP, we can obtain relaxations that lead to tighter bounds at the expense of higher computational requirements. By continuing in this manner, we can recapture the exact formulation of the problem as an MDP.

A different approach to bounding the performance of loss networks is developed by Kelly (1994). This bounding method is based on aggregating all alternative flows in the loss network as follows. In every link (i, j) there are two types of flows: the direct flow and all the alternative flow present in the link. Using the exact solution via dynamic programming for the single link case, Kelly (1994) derives bounds on the optimal reward. As we have already mentioned, our second relaxation is close in spirit with this approach. In fact, our computational results of Section 5 indicate that the bounds are extremely close, although not identical.

3.5. Symmetric Loss Networks

We now restrict our attention on applications of the methodology presented in the previous section for the simpler case of a symmetric, fully connected loss network. This kind of network has received much attention and there is now a considerable literature and understanding of the

Table III
Trunk Reservation Parameters for the Symmetric Network Topology

Link	$C_{ij} = 10$ $\lambda_{ij} = 9$	$C_{ij} = 10$ Figure 2	$C_{ij} = 10$ Figure 3	$C_{ij} = 10$ Figure 4	Figure 5 $\lambda_{ij} = 7$	Figure 5 Figure 2	Figure 5 Figure 3	Figure 5 Figure 4
1-2	3	3	3	4	2	3	3	4
1-3	3	3	4	4	3	4	5	5
1-4	3	3	4	5	3	3	5	6
1-5	3	2	3	4	2	2	3	4
2-1	3	3	2	4	2	3	2	4
2-3	3	3	4	5	2	3	4	4
2-4	3	2	2	4	2	2	2	4
2-5	3	3	3	4	2	3	3	4
3-1	3	3	4	4	2	2	3	3
3-2	3	2	3	5	2	2	3	5
3-4	3	3	4	4	2	3	4	4
3-5	3	3	3	4	2	2	3	3
4-1	3	2	3	4	3	3	4	5
4-2	3	3	2	4	3	3	3	4
4-3	3	3	4	4	2	3	3	4
4-5	3	3	2	4	2	2	2	4
5-1	3	3	4	4	2	3	4	4
5-2	3	3	4	4	2	3	4	4
5-3	3	3	3	4	2	3	3	4
5-4	3	2	2	5	2	2	2	5

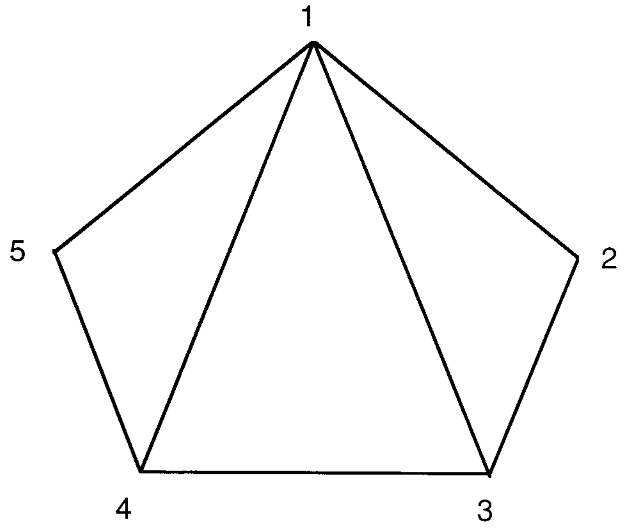

Figure 7. Five-node asymmetric network.

behavior of dynamic routing policies applied to it (Mitra et al. 1993, Mitra and Gibbens 1992, Gibbens and Kelly 1990). As the number of nodes and alternative routes increases, the bounds obtained by trunk reservation policies tend to be tight. Asymptotically and when there is symmetry in all input parameters, the optimal policy is of the threshold type: An incoming call (i, j) is routed directly if there is at least one free circuit or through a two-link alternative path, provided that the number of calls currently in progress in both links of the path is below an appropriately chosen trunk reservation parameter.

Consider a symmetric, fully connected network with N nodes, where $\lambda_{ij} = \lambda$, $\mu = 1$, $w_{ij} = 1$ and $C_{ij} = C$. By symmetry, there are just two types of flows, one carried on a direct route and another carried on a two-link alternative route. Let $n^1(t)$ be the number of calls per link that are routed directly at time t and $n^2(t)$ be the calls per link that are routed alternatively at time t. The dimension of the problem is significantly smaller than before. As a result, we are able to solve large-scale networks very efficiently.

By defining the following decision variables:

$$z(\theta) = E[n^{\theta}(t)], \quad \theta = 1, 2,$$

the max-flow bound on the expected reward in steady-state is given by the following problem:

$$\text{maximize}(|A|z(1) + |A|(N-2)z(2)) \tag{12}$$

subject to

$$z(1) + (N-2)z(2) \leq \lambda,$$
$$z(1) + 2(N-2)z(2) \leq C,$$
$$z(1), z(2) \geq 0,$$

where $|A| = N(N-1)$ is the number of the ordered pair of nodes.

The decision variables introduced in Relaxation (6) are reduced by symmetry to the following quantities:

$$x(a, \theta) = P\{n(t) = a, A^{\theta}(t)\}, \quad \theta = 1, 2,$$
$$y(a) = P\{n(t) = a\},$$

where $n(t)$ is the total number of calls present in a link at time t. Then, the optimal expected reward is bounded above by the optimal value of the following linear program:

$$\text{maximize}(|A|z(1) + |A|(N-2)z(2)) \tag{13}$$

subject to

$$\begin{aligned} ay(a) = {} & \lambda x(a-1, 1) - \lambda x(a, 1) \\ & + 2(N-2)\lambda x(a-1, 2) - 2(N-2)\lambda x(a, 2) \\ & + (a+1)y(a+1), \quad a = 0, \ldots, C, \end{aligned}$$
$$z(1) + 2(N-2)z(2) = \sum_{a=0}^{C} ay(a),$$
$$\sum_{a=0}^{C} y(a) = 1,$$
$$z(1) + 2(N-2)z(2) \leq C,$$
$$z(1) + (N-2)z(2) \leq \lambda,$$
$$x(a, \theta) \leq y(a), \quad \theta = 1, 2, a = 0, \ldots, C,$$
$$x(a, \theta), y(a), z(1), z(2) \geq 0, \quad \theta = 1, 2, a = 0, \ldots, C.$$

Because by symmetry there is no distinction between alternative calls using a link, Relaxations (8) and (9) are equivalent. Defining the following variables:

$$x(a, b, \theta) = P\{n^1(t) = a, n^2(t) = b, A^{\theta}(t)\}, \quad \theta = 1, 2,$$
$$y(a, b) = P\{n^1(t) = a, n^2(t) = b\},$$

the average reward per unit time is bounded above by the problem:

$$\text{maximize}(|A|z(1) + |A|(N-2)z(2)) \tag{14}$$

subject to

$$\begin{aligned} & \lambda x(a, b, 1) + 2(N-2)\lambda x(a, b, 2) + (a+b)y(a, b) \\ & = \lambda x(a-1, b, 1) + 2(N-2)\lambda x(a, b-1, 2) \\ & \quad + (a+1)y(a+1, b) + (b+1)y(a, b+1), \end{aligned}$$
$$z(1) = \sum_{a=0}^{C} a \sum_{b:a+b\leq C} y(a, b),$$
$$2(N-2)z(2) = \sum_{b=0}^{C} a \sum_{a:a+b\leq C} y(a, b),$$
$$\sum_{a,b:a+b\leq C} y(a, b) = 1,$$
$$z(1) + 2(N-2)z(2) \leq C,$$
$$z(1) + (N-2)z(2) \leq \lambda,$$
$$x(a, b, \theta) \leq y(a, b), \quad \theta = 1, 2, \forall (a, b): a + b \leq C,$$
$$x(a, b, \theta), y(a, b), z(1), z(2) \geq 0, \quad \theta = 1, 2, \forall (a, b): a + b \leq C.$$

The number of variables needed for Relaxations (12), (13), and (14) is $O(1)$, $O(C)$, and $O(C^2)$, respectively.

Table IV
Numerical Results for the Asymmetric Network Topology with $w_{ij} = 1$

Instances	$C_{ij} = 10$ $\lambda_{ij} = 9$	$C_{ij} = 10$ Figure 8	$C_{ij} = 10$ Figure 9	$C_{ij} = 10$ Figure 10	Figure 11 $\lambda_{ij} = 7$	Figure 11 Figure 8	Figure 11 Figure 9	Figure 11 Figure 10
Max-Flow	126	112	125	140	98	112	121	131
Relax. (6)	126	112	123.8698	134.5948	98	109.9550	117.0262	125.8031
Relax. (8)	107.3013	102.7700	107.1094	114.5898	92.0074	97.8346	102.2940	109.2020
Kelly (1994)	107.3013	102.7900	107.1094	114.5898	93.2482	98.2800	102.2940	109.2020
Relax. (9)	107.2207	102.6398	106.9730	114.5704	93.1246	97.9900	102.1123	109.1606
DAR	105.2025	98.6599	104.4455	114.1055*	89.5632	94.6840	99.8781	108.5875*
	± 0.0783	± 0.0931	± 0.0847	± 0.0491	± 0.0792	± 0.0948	± 0.0362	± 0.0764
DYN	105.1120	98.8338	104.3900	113.8245	88.4288	94.7754	99.9837	108.5030
	± 0.0629	± 0.0909	± 0.0671	± 0.0670	± 0.0765	± 0.0742	± 0.0988	± 0.0616
STATIC	105.9030*	101.1520*	105.1465*	112.8550	91.7282*	96.4015*	100.5060*	107.1290
	± 0.1707	± 0.1331	± 0.1270	± 0.1626	± 0.1188	± 0.1262	± 0.0960	± 0.1452

4. HEURISTIC POLICIES

Clearly, an exact formulation of the problem as an MDP leads to an exact admission control and dynamic routing policy, that is deterministic. Given complete knowledge of the state space, we would accept an incoming call (i, j) through r according to the probabilities

$$x_{ij}(\vec{a}, r) = P\{\vec{n}(t) = \vec{a}, A^r_{ij}(t)\}.$$

Note that under complete information $x_{ij}(\vec{a}, r)$ would be either 0 or 1. Given that the quantities $x_{ij}(\vec{a}, r)$ are not known, due to the huge dimensions of the resulting linear program, we propose instead policies that use only partial information of the state space. In particular, we propose policies that naturally arise from the third-order relaxation we proposed.

4.1. Dynamic Heuristic Policies Arising from the Third-Order Relaxation

Our goal in this section is to derive efficient heuristic policies from the solution of the third-order relaxation described in Section 3.4. After solving this relaxation, we obtain the following joint probabilities:

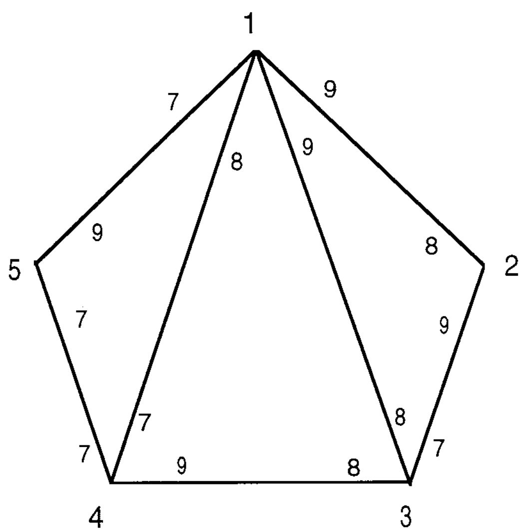

Figure 8. Asymmetric rates λ_{ij}.

$$x_{ij}(a, b, (i, j)) = P\{n^1_{ij}(t) = a, n^2_{ij}(t) = b, A^{(i,j)}_{ij}(t)\},$$

$$x_{ik}(a_1, b_1, (i, k, j)) = P\{n^1_{ik}(t) = a_1, n^2_{ik}(t) = b_1, A^{(i,k,j)}_{ij}(t)\},$$

$$x_{kj}(a_2, b_2, (i, k, j)) = P\{n^1_{kj}(t) = a_2, n^2_{kj}(t) = b_2, A^{(i,k,j)}_{ij}(t)\},$$

$$y_{ij}(a, b) = P\{n^1_{ij}(t) = a, n^2_{ij}(t) = b\}.$$

Given the state of the network, we propose to accept an incoming call (i, j) on the direct link with probability:

$$P\{A^{(i,j)}_{ij}(t) | n^1_{ij}(t) = a, n^2_{ij}(t) = b\} = \frac{x_{ij}(a, b, (i, j))}{y_{ij}(a, b)}.$$

Moreover, we propose to accept an incoming call (i, j) on the randomly selected alternative path (i, k, j) for some k with probability that depends on the traffic on links (i, k) and (k, j). We could use as an estimate of this probability the quantity:

$$A = P\{A^{(i,k,j)}_{ij}(t) | n^1_{ik}(t) = a, n^2_{ik}(t) = b\} = \frac{x_{ik}(a, b, (i, k, j))}{y_{ik}(a, b)},$$

or

$$B = P\{A^{(i,k,j)}_{ij}(t) | n^1_{jk}(t) = a, n^2_{jk}(t) = b\} = \frac{x_{jk}(a, b, (i, k, j))}{y_{jk}(a, b)}.$$

Other possibilities is to use $\min(A, B)$ or $(A + B)/2$. We have found empirically that using $(A + B)/2$ as an estimate of the probability of acceptance in an alternate route leads to better results. One could conceivably extend the above proposal and find the best weighted combination of A and B.

As we will see in Section 5, the obtained set of policies have some interesting properties. In the heuristic solution, alternative routing is only used for an intermediate level of network traffic; this is consistent with the comments in Gibbens and Kelly (1990). In addition, we obtain a two-dimensional *trunk reservation parameter* for every link in

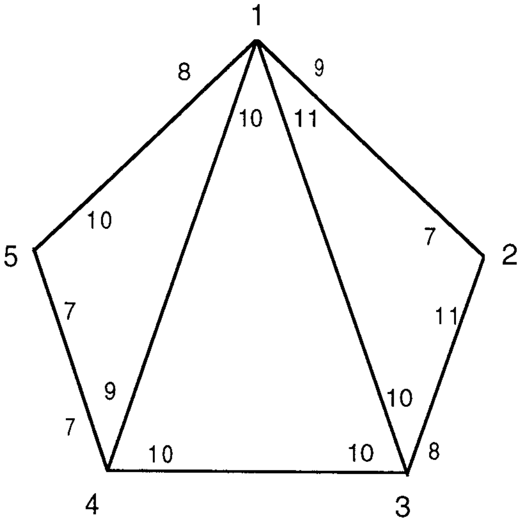

Figure 9. Asymmetric rates λ_{ij}.

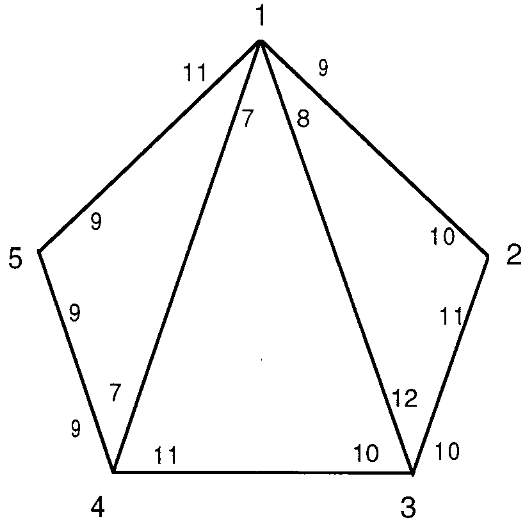

Figure 11. Asymmetric capacities C_{ij}.

the network that suggests that we actually perform a multilevel threshold policy in every link.

The probability of acceptance on a alternative route is nonzero only if the number of busy circuits in both links used is below a certain integer that depends on the level of direct *and* alternative flow on those links. In this sense the proposed policies provide an effective mechanism for controlling the unrestricted use of alternative routing that may severely degrade the performance of the network.

4.2. A Static Heuristic Policy Arising from the Third-Order Relaxation

In this section we propose a static heuristic policy that arises from the third-order relaxation. The proposed policy is static in the sense that it does not depend on the state of the system. The policy is described as follows.

Accept an incoming call (i, j) on the direct link with probability:

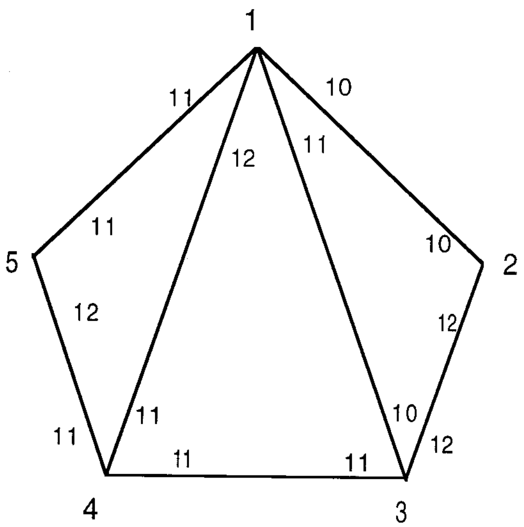

Figure 10. Asymmetric rates λ_{ij}.

$$P\{A_{ij}^{(i,j)}(t)\} = \sum_{(a,b)} P\{n_{ij}^1(t) = a,\, n_{ij}^2(t) = b,\, A_{ij}^{(i,j)}(t)\}$$
$$= \sum_{(a,b)} x_{ij}(a, b, (i, j)).$$

Accept an incoming call (i, j) on the randomly chosen alternative path (i, k, j) for some k with probability:

$$P\{A_{ij}^{(i,k,j)}(t)\} = \sum_{(a,b)} P\{n_{ik}^1(t) = a,\, n_{ik}^2(t) = b,\, A_{ij}^{(i,k,j)}(t)\}$$
$$= \sum_{(a,b)} P\{n_{kj}^1(t) = a,\, n_{kj}^2(t) = b,\, A_{ij}^{(i,k,j)}(t)\}$$
$$= \sum_{(a,b)} x_{ik}(a, b, (i, k, j))$$
$$= \sum_{(a,b)} x_{kj}(a, b, (i, k, j)).$$

The last equality was imposed as constraint (11) in the third-order relaxation. An advantage of the above static policy is that it does not require any knowledge of the state of the network links. In the next section we illustrate that in many cases the static policy yields better performance results than dynamic policies arising in the literature.

5. COMPUTATIONAL RESULTS

In this section we provide computational results in order to evaluate the performance of our bounding techniques and proposed heuristic policies for multiclass loss networks. We restrict our attention to complete as well as sparsely connected networks under conditions of symmetric and asymmetric problem parameters. We address the following questions:

1. The quality of the proposed bounds and their comparison with those proposed in Kelly (1994), which represent, to the best of our knowledge, the state of the art.
2. The quality of the proposed policies and their comparison with those of dynamic alternative routing with trunk reservation (DAR) (Gibbens and Kelly 1990),

Table V
Numerical Results for the Asymmetric Network Topology with Asymmetric w_{ij}

Instances	$C_{ij} = 10$ $\lambda_{ij} = 9$	$C_{ij} = 10$ Figure 8	$C_{ij} = 10$ Figure 9	$C_{ij} = 10$ Figure 10	Figure 11 $\lambda_{ij} = 7$	Figure 11 Figure 8	Figure 11 Figure 9	Figure 11 Figure 10
Max-Flow	306	273	309	345	238	273	299	328
Relax. (6)	306	273	306.4716	334.6542	238	270.7000	291.8802	317.7180
Relax. (8)	281.0646	262.3801	283.7627	308.7852	230.3609	252.4361	272.5380	294.1770
Kelly (1994)	281.0646	262.3801	283.7627	308.7852	232.2351	253.1035	272.5380	294.1770
Relax. (9)	280.5751	262.1257	283.2337	307.1803	231.7851	252.7518	271.4671	292.2671
DAR	255.3430	241.8830	255.0190	277.1125	214.4990	229.9370	243.7195	263.2080
	± 0.1711	± 0.2275	± 0.2826	± 0.1309	± 0.2478	± 0.2802	± 0.1339	± 0.2043
DYN	259.0200	242.4255	258.8705	282.6800*	213.8480	231.9250	248.0025	266.8305
	± 0.3188	± 0.2896	± 0.2374	± 0.1747	± 0.2656	± 0.2498	± 0.2771	± 0.2211
STATIC	261.6175*	254.3705*	264.8295*	279.4915	223.7170*	237.1780*	252.8445*	272.7070*
	± 0.9320	± 0.4307	± 0.3847	± 0.5479	± 0.4637	± 0.9864	± 0.7866	± 0.4628

which represent, to the best of our knowledge, one of the most effective policies proposed in the literature. Under the DAR policy a call is first offered to the direct route and is accepted if there is a free circuit. If the direct link has reached its full capacity, the call is next offered to a randomly chosen two-link alternative route and is accepted if the number of free circuits on each link is greater than the link's trunk reservation parameter t (i.e., the number of free circuits is at least $C - t - 1$). The calculation of the trunk reservation parameter is done as described in Gibbens and Kelly (1990). Otherwise, the call is rejected and lost from the system.

3. The role of the following factors to the quality of the bounds and policies: (a) asymmetry in the topology of the network, (b) asymmetry of the data, and (c) size of the network.
4. Understanding of the qualitative behavior of the proposed policies and the reasons for their success.

5.1. Symmetric Network Topology Examples

Consider the five-node fully connected network of Figure 1. We calculate the following bounds:

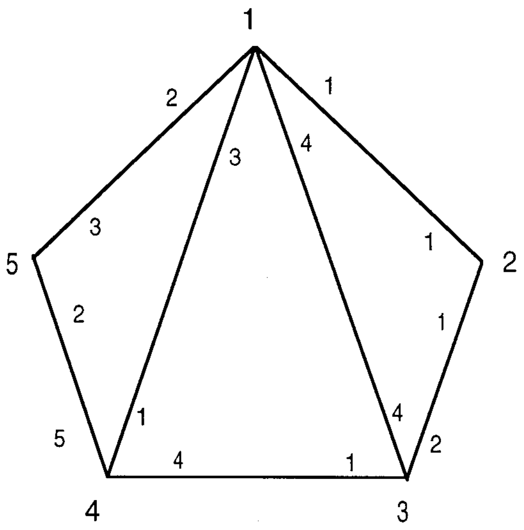

Figure 12. Asymmetric rewards w_{ij}.

1. The Max-Flow Bound presented in § 3.1.
2. The upper bound derived in § 3.2 using Relaxation (6).
3. The aggregate upper bound obtained in § 3.3 using Relaxation (8).
4. The upper bound derived in § 3.4 with Relaxation (9).
5. The upper bound developed in Kelly (1994).

We also present the performance of the following policies:

1. The dynamic alternative routing scheme with trunk reservation (DAR) (Gibbens et al. 1988, Stacey and Songhurst 1987).
2. The dynamic policy of § 4.1 using average values for the calculation of probabilities in an alternative route.
3. The static policy proposed in § 4.2.

We consider examples with both symmetric and asymmetric problem parameters (arriving rates λ_{ij}, link capacities C_{ij}, and generated rewards w_{ij}). The service rate μ is assumed to be one in all instances considered. We have calculated all the bounds using the linear programming package CPLEX, and we have simulated all the proposed policies. In all the tables we report both the estimated simulated value and its standard deviation.

Table I compares the upper bounds and heuristic policies obtained for the case of symmetric rewards throughout the network ($w_{ij} = 1$). We consider both symmetric and asymmetric arrival rates and link capacities. The arrival rates used in Table I are described in Figures 2, 3, and 4, while the capacities are described in Figure 5. The arrival rates increase from Figure 2 to Figure 4, and hence the traffic becomes heavier.

Table II compares the bounds and policies obtained for the case of asymmetric rewards, which are described in Figure 6. The trunk reservation parameters used for the simulation of the DAR policy are given in Table III; they are assumed to be the same for symmetric and asymmetric rewards, even though the notion of trunk reservation has to be rethought in the context of differing weights.

Both tables suggest that the bounds from Relaxations (8) and the one considered by Kelly (1994) are the same, while Relaxation (9) is slightly better. Somewhat surprisingly, the static policy improves upon the performance of

Table VI
Trunk Reservation Parameters for the Asymmetric Network Topology

Link	$C_{ij} = 10$ $\lambda_{ij} = 9$	$C_{ij} = 10$ Figure 8	$C_{ij} = 10$ Figure 9	$C_{ij} = 10$ Figure 10	Figure 11 $\lambda_{ij} = 7$	Figure 11 Figure 8	Figure 11 Figure 9	Figure 11 Figure 10
1-2	3	3	3	4	2	3	3	4
1-3	3	3	4	4	3	4	5	5
1-4	3	3	4	5	3	3	5	6
1-5	3	2	3	4	2	2	3	4
2-1	3	3	2	4	2	3	2	4
2-3	3	3	4	5	2	3	4	4
3-1	3	3	4	4	2	2	3	3
3-2	3	2	3	5	2	2	3	5
3-4	3	3	4	4	2	3	4	4
4-1	3	2	3	4	3	3	4	5
4-3	3	3	4	4	2	3	3	4
4-5	3	2	2	4	2	2	2	4
5-1	3	3	4	4	2	3	4	4
5-4	3	2	2	5	2	2	2	5

the dynamic alternative routing strategy, while the performance of the proposed dynamic policy is comparable to DAR. Under symmetric rewards (Table I), the bounds and the policies are very close to each other. The suboptimality guarantees defined as best bound/best policy range from 0 to 2.5%. Generally, as the traffic becomes heavier the suboptimality guarantee improves. However, for asymmetric rewards (Table II) the suboptimality guarantee is less tight, ranging from 4% to 10%. Especially for asymmetric rewards (Table II), the improvement of the static policy over DAR is quite noticeable.

5.2. Asymmetric Network Topology Examples

Consider the five-node network of Figure 7. We present examples with symmetric as well as asymmetric system parameters λ_{ij}, C_{ij}, and w_{ij}; the service rate μ is assumed to be one in all instances considered.

Table IV compares the bounds and policies obtained for the case of $w_{ij} = 1$. We consider both symmetric network parameters as well as asymmetric arrival rates and link capacities in Figures 8 through 11. The arrival rates increase from Figure 8 to Figure 10, and hence the traffic becomes heavier. Table V compares the bounds and policies obtained for the case of asymmetric rewards shown in Figure 12. The trunk reservation parameters are given in Table VI.

The conclusions from this study are qualitatively the same as before. Tables IV and V suggest that the bounds from Relaxations (8) and the one considered by Kelly (1994) are very close, although not identical, while Relaxation (9) is slightly better. Again the static policy is stronger and sometimes significantly, especially under conditions of asymmetry. The degree of the improvement in both the bound (9) over the bound in Kelly (1994) and the static policy over DAR increases as the asymmetry increases.

We have also considered larger instances of the problem (Figures 13 through 14), where similar results are obtained for medium traffic load conditions (see Table VII). In addition to the policies, we also present the least busy alternative routing strategy with trunk reservation (LBA) (Gibbens and Kelly 1990). The trunk reservation parameters are given in Table VIII.

5.3. Insights from the Computations

The computational results suggest that the bounds from Relaxations (8), (9) and Kelly (1994) are very close, with Relaxation (9) being the tightest but only slightly. In the case of asymmetric rewards, the improvement in the performance bound is larger as the traffic becomes heavier (see last column of Table V). The closeness of the bounds is not surprising because the proposed approach based on aggregations of the underlying MDP and the one proposed in Kelly (1994) treat each link as if it were in isolation and consider that the interaction among calls in the network is captured through the *total* alternative flow present in a link.

The main advantage of the proposed method is that it provides a natural way to obtain heuristic policies using partial information of the network state space. Particularly

Table VII
Numerical Results for $C_{ij} = 10$ and Asymmetric Rates (Figure 13)

	Max-Flow	Relax. (6)	Relax. (8)	Kelly (1994)	Relax. (9)
$w_{ij} = 1$	342	342	311.0631	311.0631	310.6223
Figure 14	608	608	575.6303	575.6303	575.4059
	DAR	DYN	STATIC	LBA	
$w_{ij} = 1$	299.0500 ± 0.1418	299.4615 ± 0.1777	305.0270* ± 0.1841	300.2005 ± 0.1305	
Figure 14	539.7685 ± 0.2963	540.8950 ± 0.3488	553.6180* ± 0.4683	541.1670 ± 0.2157	

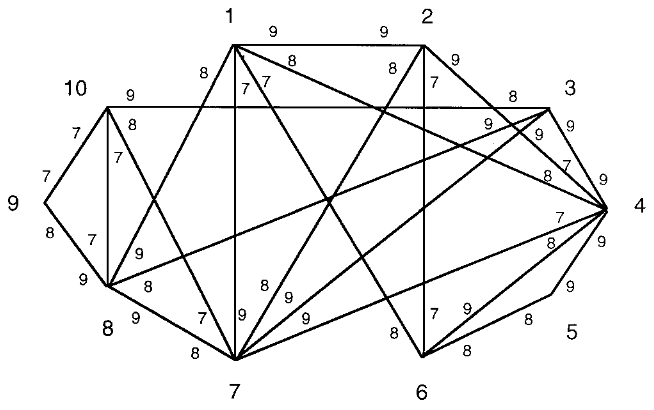

Figure 13. Asymmetric rates λ_{ij}.

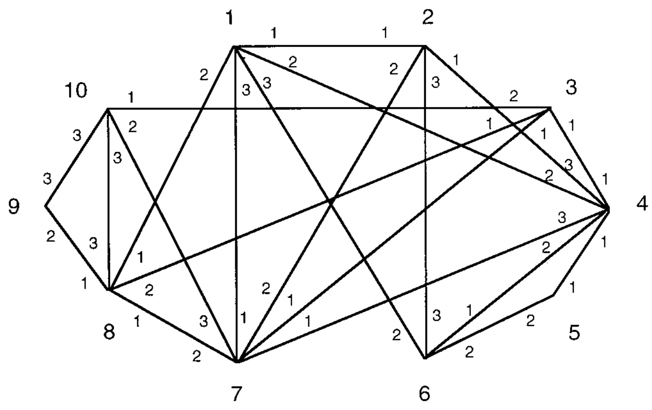

Figure 14. Asymmetric rewards w_{ij}.

for asymmetric rewards, the proposed static policy outperforms the DAR scheme by as much as 5%. We next attempt to understand qualitatively the behavior of our heuristic policies:

1. Dynamic routing of our heuristic policies is performed only for an intermediate range of overload (see Figure 15). As the traffic increases, alternative routing disappears and the proposed policy is to always accept on the direct link. This behavior is also present in the max-flow bound in Gibbens and Kelly (1990) and discussed as well in Gibbens and Kelly (1995).
2. For fixed traffic rates λ_{ij}, alternative routing is allowed only when the chosen route is moderately loaded (see Figure 16). When the alternative route is heavily loaded, the probability of acceptance on that route becomes zero, and the alternatively routed calls are rejected in anticipation of calls that are to be routed directly on the links that constitute the route.
3. The probability of acceptance of a call under our proposed heuristic policies depends on both the direct *and* alternative flow through a link; while in DAR, the probability of acceptance depends solely on the *total* flow. In addition, under the dynamic heuristic policy we obtain a two-dimensional *trunk reservation parameter* for every link in the network that seems to suggest that we actually perform a multilevel threshold policy in every link. The probability of acceptance on an alternative route is nonzero only if the number of busy circuits in both links used is below a certain integer that depends on the level of direct *and* alternative flow on those links. In this sense the proposed policies provide an effective mechanism for controlling the unrestricted use of alternative routing that may severely degrade the performance of the network. We believe that this property is the central reason that our heuristic policies outperform the DAR policy.
4. The proposed dynamic policy implicitly derives the trunk reservation parameter for every link; the probability of acceptance on an alternative route would be zero if the number of occupied circuits in the used links is above a certain integer (in Figure 16 this number is 7). Note though an important difference: The probability of acceptance when the number of calls is below the trunk reservation parameter is not necessarily one (in Figure 16 it is 0.73), as it is the case in DAR and other policies used in practice.
5. In the proposed static heuristic policy the acceptance probability of an arriving call is *not* always one, even though the link's capacity may not have been reached. Thus, even in the static policy there is a mechanism for controlling the arriving stream of calls in a way that improves the network's performance.

6. CONCLUDING REMARKS

In the present work, we propose a technique for approximating the region of achievable performance for multiclass loss networks, with Poisson arrivals and exponentially distributed holding times by considering aggregations of the MDP characterization of the problem. The method takes into account pairwise interaction between the directly and alternatively routed calls and by augmenting

Table VIII
Trunk Reservation Parameters for $C_{ij} = 10$ and Asymmetric Rates (Figure 13)

Link	1-2	1-4	1-6	1-7	1-8	2-1	2-4	2-6	2-7	3-4	3-7	3-8	3-10	4-1
t	3	3	2	2	3	3	3	2	3	3	3	3	3	3
Link	4-2	4-3	4-5	4-6	4-7	5-4	5-6	6-1	6-2	6-4	6-5	7-1	7-2	7-3
t	2	3	3	3	2	3	3	3	2	3	3	3	3	3
Link	7-4	7-8	7-10	8-1	8-3	8-7	8-9	8-10	9-8	9-10	10-3	10-7	10-8	10-9
t	3	3	2	3	3	3	3	2	3	2	3	3	2	2

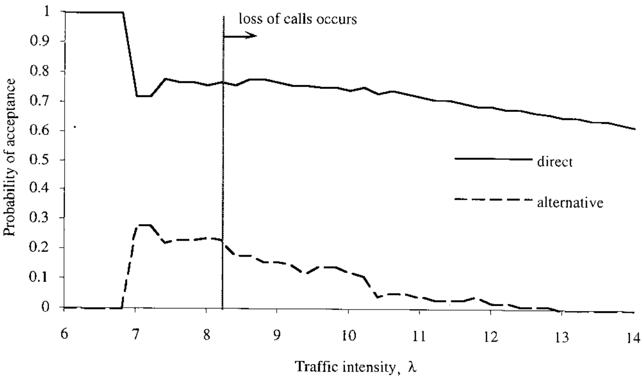

Figure 15. Asymmetric network topology with $C_{ij} = 10$, $w_{ij} = 1$, and $\lambda_{ij} = \lambda$. We plot the probability of acceptance on the direct route (1, 3)$P\{A_{13}^{(1,3)}(t)\}$ versus the probability of acceptance on the alternative routes (1, 2, 3) and (1, 4, 3), $P\{A_{13}^{(1,2,3)}(t)\} + P\{A_{13}^{(1,4,3)}(t)\}$.

the aggregated state space of the underlying MDP we can obtain relaxations that are expected to lead to tighter bounds at the expense of higher computational requirements.

For highly connected and symmetric networks the bounds are very close to those derived in Kelly (1994). As the degree of asymmetry in the problem parameters increases, the proposed bounds become stronger but only slightly. The use of the proposed method in a variety of more realistic settings remains to be explored and is a natural, as well as necessary, continuation of the present work.

The main advantage of the proposed method is that it provides a natural way to obtain heuristic policies using partial information of the network state space. We describe heuristic policies arising from the relaxations that improve upon the performance of heuristics used in practice. In addition, we obtain insight about the qualitative behavior of these policies that explain, to a certain degree, their success.

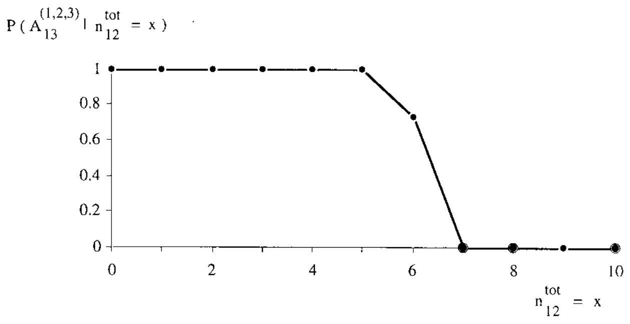

Figure 16. Probability of acceptance on (1, 2, 3) under $C_{ij} = 10$ and asymmetric rates (Figure 2).

ACKNOWLEDGMENT

This research was supported in part by Presidential Young Investigator award DDM-9158118, with matching funds from Draper Laboratory.

REFERENCES

Ash, G. R., and B. D. Huang. 1994. Comparative evaluation of dynamic routing strategies for a worldwide intelligent network. *The Fundamental Role of Teletraffic in the Evolution of Telecommunication Networks*, Proc. International Telecommunications Conference, 14.

Bertsimas, D., I. Ch. Paschalidis and J. Tsitsiklis. 1994. Optimization of multiclass queueing networks: polyhedral and nonlinear characterizations of achievable performance. *Ann. Applied Probability* **4**(1) 43–75.

Franks, R. L., and R. W. Rishel. 1973. Optimum network call-carrying capacity. *Bell System Technical J.* **52**(7) 1195–1214.

Gibbens, R. J., and F. P. Kelly. 1990. Dynamic routing in fully connected networks. *IMA J. Math. Control & Information* **7** 77–111.

——, and ——. 1995. Network programming methods for loss networks. *IEEE J. on Selected Areas in Communication* **13**(7) 1189–1198.

——, and ——, P. B. Key. 1988. Dynamic alternative routing—modelling and behaviour. *Proc. 12th International Teletraffic Congress*. North-Holland (Turin).

——, —— and S. R. E. Turner. 1993. Dynamic routing in multiparented networks. *IEEE/ACM Trans. on Networking* **1**(2) 261–270.

Heyman, D. P., and M. J. Sobel. 1984. *Stochastic Models in Operations Research*. Vol. ii: Stochastic Optimization. McGraw-Hill, New York.

Hunt, P. J., and C. N. Laws. 1992. Least busy alternative in queueing and loss networks. *Probability in Engineering and Informational Sci.* **6** 439–456.

——, and C. N. Laws. 1993. Asympotically optimal loss network control. *Math. Oper. Res.* **18** 880–900.

IEEE Communications Magazine. 1990. Advanced traffic control methods for circuit switched telecommunication networks. **28**(10).

Inoue, A., H. Yamamoto, and Y. Harada. 1989. An advanced large-scale simulation system for telecommunication networks with dynamic routing. *Networks* **89** 77–82.

Kelly, F. P. 1986. Blocking probabilities in large circuit-switched networks. *Advances in Applied Probability* **18**(2) 473–505.

——. 1990. Routing and capacity allocation in networks with trunk reservation. *Math. Oper. Res.* **14**(4) 771–793.

——. 1991. Loss networks. *Ann. Applied Probability* **1** 319–378.

——. 1994. Bounds on the performance of dynamic routing schemes for highly connected networks. *Math. Oper. Res.* **19** 1–20.

Key, P. B. 1990. Optimal control and trunk reservation in loss networks. *Probability in the Engineering & Informational Sci.* **4** 203–242.

Lazarev, V. G., and S. M. Starobinets. 1977. The use of dynamic programming for optimization of control in networks of communication of channels. *Eng. Cybern.* **3** 107–116.

Lippman, S. A. 1975. Applying a new device in the optimization of exponential queueing systems. *Oper. Res.* **23** 687–710.

Louth, G. M., M. Mitzenmacher, and F. P. Kelly. 1994. Computational complexity of loss networks. *Theoretical Computer Sci.* **125** 45–59.

Marbukh, V. V. 1981. Asymptotic investigation of a complete communications network with a large number of points and bypass routes. *Problemy Peradaăci Informacii* **16** 89–95.

Miller, B. 1969. A queueing reward system with several customer classes. *Management Sci.* **16** 234–245.

Mitra, D. 1987. Asymptotic analysis and computational methods for a class of simple, circuit-switched networks with blocking. *Advances in Appl. Probability* **19** 219–239.

——, and R. J. Gibbens. 1992. State-dependent routing on symmetric loss networks with trunk reservations: asymptotics, optimal design. *Ann. Oper. Res.* **35** 3–30.

——, —— and B. D. Huang. 1993. State-dependent routing on symmetric loss networks with trunk reservations. *IEEE Trans. on Communications* **41**(2) 400–411.

—— and J. B. Seery. 1991. Comparative evaluations of randomized and dynamic routing strategies for circuit-switched networks. *IEEE Trans. Communications* **39** 102–116.

Ott, T. J., and K. R. Krishnan. 1992. Separable routing: a scheme for state-dependent routing of circuit-switched telephone traffic. *Ann. Oper. Res.* **35** 43–68.

Pinsky, E., and A. E. Conway. 1992. Computational algorithms for blocking probabilities in circuit-switched networks. *Ann. Oper. Res.* **35** 31–42.

Ross, K. W. 1995. Multiservice Loss Models for Broadband Telecommunication Networks. Springer-Verlag.

—— and J. Wang. 1992. Monte-carlo summation applied to product-form loss networks. *Prob. Eng. Information Sci.* **6** 323–348.

Stacey, R. R., and D. J. Songhurst. 1987. Dynamic alternative routing in the British Telecom trunk network. *International Switching Symposium*, Phoenix, AZ.

Weber, J. H. 1964. A simulation study of routing and control in communication networks. *Bell Syst. Tech.* **43** 2639–2676.