An Optimization Approach to Credit Risk

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Abstract

This paper sheds new light on the relation between the survival probability distribution and the price of a credit risky security. Using no-arbitrage arguments and a variety of optimization methods, we produce bounds on the survival probability distribution of a defaultable security from existing prices of comparable credit risky securities. We address the problem of inaccuracies in the estimation of discount factors, key parameters of the model, by using robust optimization methods. We also propose an approach for classification of risky bonds based on integer programming. Finally, we show the effectiveness of the suggested methods by performing a number of computational experiments with simulated and real market data.

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1 Introduction

A *credit* (or equivalently, *default*) *risky security* is a security that has a nonzero probability of defaulting on its payments due to the possibility of financial distress of the issuer. A fundamental example of a credit risky security is a corporate bond. A corporate bond promises its holder a fixed stream of payments, but may default on its promise.

Two sets of variables determine the prices of credit risky bonds. The first set of variables is associated with unpredictability in the movement of *market* variables. Interest rate risk is one such example: it affects every bond, defaultable or not. The second set of variables describes the sources of risk associated with the *credit* characteristics of the security, such as the risk of default, and the risk of insufficient recovery in the event of default.

The evaluation of credit risk has been receiving increasing attention in the finance literature in recent years. Most generally, approaches can be placed in two categories: firm value models and intensity-based models. In firm value models default occurs when the stochastic process describing the value of the firm crosses a certain boundary. Intensity-based models are concerned with the stochastic process for the market price of the risky security rather than with the process for the value of the issuer. Frequently, the stochastic process for credit spreads is modeled directly, and the default process for credit risky securities is assumed to be some variation of a Poisson process.

Intensity-based models have gained popularity over firm-value models, because the latter suffer from some problems, such as lack of a clear definition of "firm value," and difficulty of estimating the firm value for bond issuers that are not necessarily also stock issuers, e.g. municipalities. The interested reader is referred to Jarrow, Lando, and Turnbull [17], Duffie and Singleton [10], Das [5, 6], Davis [8], and Pliska [9] for a comprehensive coverage of the different models of default risk.

Recovery rates, most generally defined as the fraction of promised payments that the defaulting parties are able to make as they announce default, are usually modeled as a function of the value of the firm, or as exogenous variables that must be estimated from historical data or from observed prices. Duffie and Singleton [10] treat the recovery rate of a bond as a percentage of the *market* value of the bond just before default. Das [6] and Davis [8] define the recovery rate as a percentage of the *face* value of the bond that is recovered in the event of default.

The subject of recovery rate estimation has received extensive treatement in the literature, but not with a large success. Recovery rates are difficult to model mathematically, because they depend on a number of legal and political factors, and because their stochastic behavior does not fit easily into any of the "convenient" stochastic processes typically used to model financial data, such as Wiener or Poisson. Das [6] attempts to describe the recovery rate behavior with a stochastic process defined on the range [0,1]; however, to our knowledge, there have been no studies indicating that this process provides an accurate description.

The stochastic modeling of default rates and recovery rates involves making assumptions that are not always easy to validate. Our overall objective in this paper is to demonstrate that a lot of information can be extracted from existing market prices assuming only a relatively weak condition: non-existence of arbitrage in credit risky markets. Our methodology employs techniques from linear, nonlinear, robust, and integer programming, as well as duality theory.

The contributions and structure of this paper can be summarized as follows:

1. Using linear programming and duality theory, we compute bounds on the survival probability distribution and the price of a credit risky bond given data on market prices of similar credit risky bonds (Section 2);

2. We address the uncertainty and estimation inconsistencies in discount factors, key parameters of the model, by applying robust optimization methods (Section 3);

3. We propose an approach for classification of risky bonds based on integer programming (Section 4);

4. We show the effectiveness of the suggested methods by performing a number of computational experiments with simulated and real market data (Section 5).

2 Evaluation of Credit Risk

This section presents the pricing formulas for credit risky bonds. The unknown parameters in the pricing formulas are the survival probabilities and the recovery rate of the bonds. We discuss how market prices of defaultable bonds can be used to compute bounds on the fair prices and the survival probability distribution of risky bonds with similar credit characteristics. The bounds on the survival probability distribution of credit risky bonds can be used to compute bounds on the fair premium of credit default swaps (Pachamanova [19]). We demonstrate explicitly that an arbitrage opportunity exists if a set of survival probabilities cannot be found.

2.1 Pricing Credit Risk When the Recovery Rate Is Zero

Consider a bond with a face value (also called *principal* or *notional*) of 1. The bond makes its coupon payments $c_1, \ldots, c_n$ at times $t = 1, \ldots, n$. At the last coupon payment date, $t = n$, the payment includes the principal. If all payments are guaranteed to be made, the fair price of the bond can be written as

$$B_0 = \sum_{j=1}^{n} c_j df_j + 1 \cdot df_n, \tag{1}$$

where df_j is the discount factor at time j. One possible representation of df_j is $\frac{1}{(1+y)^j}$, where y is the default-free yield.[1]

Now suppose that there is a risk that the bond issuer may default on the coupon payments.

[1]It should be noted that for an illiquid security, as is the case with most corporate bonds, y may consist of the risk-free rate plus an illiquidity premium. The latter is difficult to estimate; however, a frequently used approximation for y is the LIBOR discount rate.

The time of default, τ, is uncertain. We assume that if default occurs between payment dates $j-1$ and j, the coupon payment c_j and all subsequent payments are not made. The probability that a payment is collected on date t, i.e., the probability that the bond has not defaulted by time t, is given by the survivor function $\pi_t = P(\tau > t)$. At this time, we will assume that no part of the notional is recovered once default occurs, i.e., that the *recovery rate* is zero.

Harrison and Kreps [15] show that in frictionless markets, the non-existence of arbitrage implies the existence of a risk-neutral martingale measure that uniquely determines a valid linear pricing rule for all assets. Jouini and Kallal [18] extend these results to include transaction costs: they show that non-existence of arbitrage in the presence of transaction costs is equivalent to the existence of a martingale measure that prices a process lying between the bid and the ask processes. In order to apply these results to markets in which credit risk is present, we assume that a set of credit risky securities that span the credit risky markets is traded. A cash flow can then be valued by computing its expected value under the risk-neutral survival probability measure $\boldsymbol{\pi}$. Hence, the fair value of a credit risky cash flow c is $0 \cdot (1 - \pi_t) + c \cdot df_t \cdot \pi_t = c \cdot df_t \cdot \pi_t$, and the price of a credit risky bond with a face value of 1 and payments of $c_1, c_2, \ldots, c_n + 1$ at times $t = 1, \ldots, n$, is given by

$$B_1 = \sum_{j=1}^{n} c_j df_j \pi_j + 1 \cdot df_n \pi_n. \tag{2}$$

Note that in Equation (2), the survival probabilities π_j are not directly observable, and are thus unknown. Clearly, they are a decreasing function of time, i.e., $1 \geq \pi_1 \geq \ldots \geq \pi_n \geq 0$.

Consider a collection of m credit risky bonds that have the same probability of default. They can be bonds from the same issuer, or from the same rating class and industry. Let $\{p_i^{bid}, p_i^{ask}\}$ be the observed bid and the ask price of the ith bond, $i = 1, \ldots, m$. Let the sorted list of coupon payment dates for bonds $1, \ldots, m$ be $\mathcal{N} = \{1, \ldots, n\}$, and let $\mathcal{N}_i = \{t_i^1, \ldots, T_i\}$, $\mathcal{N}_i \subseteq \mathcal{N}$, denote the index of the dates in $\mathcal{N}$ on which bond i makes coupon payments. The last date index in $\mathcal{N}_i$, T_i, represents the index of the date on which bond i matures, and the nominal is paid back. For simplicity (this is consistent with existing bonds in the market), we assume that on each date $j \in \mathcal{N}_i$, bond i pays out the same coupon c_i.

Theorem 1 *If there are no arbitrage opportunities, the prices of the m bonds satisfy the following system of inequalities:*

$$\mathbf{p}^{bid} \leq \mathbf{A} \cdot \boldsymbol{\pi} \leq \mathbf{p}^{ask} \tag{3}$$
$$1 \geq \pi_1 \geq \ldots \geq \pi_n \geq 0,$$

where $\mathbf{p}^{bid} \in \Re^{m \times 1}$ and $\mathbf{p}^{ask} \in \Re^{m \times 1}$ are the vectors of bid and ask bond prices, and a_{ij} is defined as:

$$a_{ij} = \begin{cases} c_i df_j & \text{if } j \in \mathcal{N}_i \setminus \{T_i\}, \\ (c_i + 1) df_j & \text{if } j = T_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof:

For the purpose of deriving a contradiction, suppose we cannot find a set of survival probabilities $\boldsymbol{\pi} \in \Re^{n \times 1}$ such that system (3) is satisfied. We will show that we can construct an arbitrage strategy. Let $\mathbf{A}_j$ denote the jth column of $\mathbf{A}$. If the system (3) is infeasible, then by the Farkas lemma the following system is feasible:

$$(\mathbf{p}^{ask})'\mathbf{x}^+ + (\mathbf{p}^{bid})'\mathbf{x}^- + s_0 \quad < \quad 0 \tag{4}$$

$$(\mathbf{x}^+)'\mathbf{A}_j + (\mathbf{x}^-)'\mathbf{A}_j + s_{j-1} - s_j \quad = \quad 0, \; j = 1, 2, \ldots, n \tag{5}$$

$$\mathbf{x}^+ \geq \mathbf{0}, \; \mathbf{x}^- \quad \leq \quad \mathbf{0} \tag{6}$$

$$s_j \quad \geq \quad 0, \; j = 0, 1, \ldots, n. \tag{7}$$

Eqs. (5) and (7) can be combined to obtain

$$\sum_{i=1}^{m} (x_i^+ a_{ij} + x_i^- a_{ij}) + s_{j-1} = s_j \geq 0, \; j = 1, 2, \ldots, n.$$

The vectors $\mathbf{x}^+$ and $\mathbf{x}^-$ represent long and short positions in bonds that an investor could take at time 0, and s_0 represents the long position in a riskless zero-coupon bond. The value $(\mathbf{p}^{ask})'\mathbf{x}^{+} + (\mathbf{p}^{bid})'\mathbf{x}^{-} + s_0$ is the value of the portfolio today. Since it is negative, this means that the amount invested in short positions is higher than the amount invested in long positions, i.e., the investor has a positive amount of cash on hand today. The equations in the constraints guarantee that the difference of the cash income from coupons and the cash paid out for coupons on bonds that have been sold short plus the amount invested in the riskless asset at the previous coupon payment date can be made nonnegative at each of the n coupon payment dates.

Therefore, if the bonds in the portfolio survive, the strategy $(\mathbf{x}^{+}, \mathbf{x}^{-}, s_0)$ yields a net riskless nonnegative cash inflow from coupon payments at each coupon payment date. If they default, there is no positive cash inflow from coupons from bonds in positive positions $((\mathbf{x}^{+})'\mathbf{A}_j = 0)$, but also there is no negative outflow $((\mathbf{x}^{-})'\mathbf{A}_j = 0)$ for coupon payments to owners of bonds held in short positions. Therefore, there exists an arbitrage opportunity: the investor is guaranteed that independent of whether the bonds he holds default or survive, at each coupon date j he holds cash $s_j \geq 0$, while at the same time he starts off with a strictly positive amount of cash on hand at time 0. □

2.2 Pricing Credit Risk with Nonzero Recovery Rate

Prices of credit risky bonds incorporate not only information about the probability of default, but also about the expected amount that can be recovered in case of default. We denote the post-default recovery rate expressed as a fraction of the notional by L, $0 \leq L \leq 1$. The probability of obtaining the recovery rate L at time j equals the probability that the default occurs between $j-1$ and j, which is $(\pi_{j-1} - \pi_j)$.

The value of a risky bond B_2 with coupon payments $c_1, \ldots, c_n$ at dates $t = 1, \ldots, n$, with a face value of 1, and a recovery rate of L, can be computed from the following formula:

$$B_2 = \sum_{j=1}^{n} c_j df_j \pi_j + df_n \pi_n + L \sum_{j=1}^{n} df_j (\pi_{j-1} - \pi_j), \tag{8}$$

where $\pi_0 = 1$. Since the term involving L is always nonnegative, Eq. (8) implies that the possibility of recovering some amount in the event of default increases the price of the credit risky bond.

We consider a collection of m risky bonds with the same default probability, and we use the same notation as in the previous section. However, now we assume that each bond has a recovery rate of L_i, $i = 1, \ldots, m$. Similarly to Theorem 1, under the assumption of no-arbitrage, the m bond prices satisfy:

$$\begin{array}{rcccl} p_1^{bid} & \leq & \sum_{j \in \mathcal{N}_1} c_1 df_j \pi_j + df_{T_1} \pi_{T_1} + L_1 \sum_{j \in \{0\} \cup \mathcal{N}_1} df_j (\pi_{j-1} - \pi_j) & \leq & p_1^{ask} \\ & & \vdots & & \\ p_m^{bid} & \leq & \sum_{j \in \mathcal{N}_m} c_m df_j \pi_j + df_{T_m} \pi_{T_m} + L_m \sum_{j \in \{0\} \cup \mathcal{N}_m} df_j (\pi_{j-1} - \pi_j) & \leq & p_m^{ask} \\ & & 1 \geq \pi_1 \geq \ldots \geq \pi_n \geq 0. & & \end{array}$$

The above system can be written as:

$$\begin{aligned} \mathbf{p}^{bid} \leq \mathbf{q} + \bar{\mathbf{A}} \cdot \boldsymbol{\pi} \leq \mathbf{p}^{ask} \\ 1 \geq \pi_1 \geq \ldots \geq \pi_n \geq 0, \end{aligned} \tag{9}$$

with the vector $\mathbf{q} \in \Re^{m \times 1}$ defined as $q_i = L_i df_{t_i^1}$ and

$$\bar{a}_{ij} = \begin{cases} c_i df_j - L_i df_j + L_i df_{j+1} & \text{for } j \in \mathcal{N}_i \setminus \{T_i\}, \\ (c_i + 1) df_j - L_i df_j & \text{for } j = T_i, \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Optimal bounds

Given the information on bid and ask prices of the collection of the m bonds, in this section we calculate (a) the underlying survival probability distribution of the collection of bonds, and (b) the fair price of a *test* risky bond with similar credit characteristics.

We give the test bond an index of 0, and expand the set of coupon payment dates $\mathcal{N}$ to include the coupon payment dates of the test bond. The new collection of coupon payment dates is $\mathcal{N}^+ = \mathcal{N}_0 \cup \mathcal{N}_1^+ \cup \ldots \cup \mathcal{N}_m^+$, where the sets $\mathcal{N}_0$ and $\mathcal{N}_i^+$, $i = 1, \ldots, m$, consist of the indices of the coupon payment dates of the 0th and ith bond in the total collection of coupon dates, respectively. $\mathcal{N}^+$ is now $\{1, \ldots, n^+\}$, and the system of inequalities (9) can be written as

$$\mathbf{p}^{bid} \leq \mathbf{q}^+ + \bar{\mathbf{A}}^+ \cdot \boldsymbol{\pi}^+ \leq \mathbf{p}^{ask}$$

$$1 \geq \pi_1^+ \geq \ldots \geq \pi_n^+ \geq 0.$$

The $\{i, j\}$th entry of the matrix $\bar{\mathbf{A}}^+ \in \Re^{m \times n^+}$ is defined as

$$\bar{a}_{ij}^+ = \begin{cases} c_i df_j - L_i df_j + L_i df_{j+1} & \text{if } j \in \mathcal{N}_i^+ \setminus \{T_i^+\}, \\ (c_i + 1) df_j - L_i df_j & \text{if } j = T_i^+, \\ 0 & \text{otherwise.} \end{cases}$$

where T_i^+, $i = 1, \ldots, m$ is the index of the maturity date of the ith bond in $\mathcal{N}^+$. $\boldsymbol{\pi}^+ \in \Re^{n^+ \times 1}$ is the vector of survival probabilities at each coupon date in the collection $\mathcal{N}^+$. We also define $q_0 = L_0 df_{t_0^1}$ and

$$a_{0j} = \begin{cases} c_0 df_j - L_0 df_j + L_0 df_{j+1} & \text{for } j \in \mathcal{N}_0 \setminus \{T_0\}, \\ (c_0 + 1) df_j - L_0 df_j & \text{for } j = T_0, \\ 0 & \text{otherwise.} \end{cases}$$

Under the assumption of no-arbitrage, the following theorem follows:

Theorem 2 (Optimal bounds)

(a) The optimal solutions π_j^{min} (π_j^{max}), $j = 1, \ldots, n$, of the optimization problems

$$\begin{aligned} \min\ (\max) \quad & \pi_j \\ \text{s.t.} \quad & \mathbf{p}^{bid} \leq \mathbf{q} + \bar{\mathbf{A}} \cdot \boldsymbol{\pi} \leq \mathbf{p}^{ask} \\ & 1 \geq \pi_1 \geq \ldots \geq \pi_n \geq 0. \end{aligned} \tag{10}$$

provide bounds on the underlying survival probability distribution of the set of m credit risky bonds with recovery rates of $L_1, \ldots, L_m$, whose market bid and ask prices are given by $\mathbf{p}^{bid}$ and $\mathbf{p}^{ask}$, respectively.

(b) The optimal objective function values of the optimization problems

$$
\begin{aligned}
\min \ (\max) \quad & q_0 + (\mathbf{a}^0)'\boldsymbol{\pi}^+ \\
\text{s.t.} \quad & \mathbf{p}^{bid} \leq \mathbf{q}^+ + \bar{\mathbf{A}}^+ \cdot \boldsymbol{\pi}^+ \leq \mathbf{p}^{ask} \\
& 1 \geq \pi_1^+ \geq \ldots \geq \pi_n^+ \geq 0
\end{aligned}
\tag{11}
$$

provide bounds on the price of a credit risky bond given observed bid and ask prices $\mathbf{p}^{bid}$ and $\mathbf{p}^{ask}$ of a collection of m risky bonds with the same credit characteristics.

If the recovery rate is not known, the problem of estimating bounds on the price and the survival probability distribution of a test risky bond becomes nonlinear. There are different ways to avoid working with a nonlinear formulation. For example, the initial guess for L_i, $i = 1, \ldots, m$, could be a number estimated from historical data. Moody's Investors Service Global Credit Research produces statistical tables periodically. One can check if this initial guess is consistent with market beliefs by using the market price of a bond with the same credit risk and seniority. The procedure is as follows:

One sets the recovery rate to a constant and increases it by a given step size, e.g. 0.1. One then computes the bounds on the fair price of the given risky bond for each recovery rate using the method from Theorem 2(b). Plotting the market price of the given risky bond versus the minimum and maximum bounds allows one to see if the market price falls outside these bounds for some values of L. The latter values of L are to be excluded from consideration. The estimate of L then can be taken to be the point that is the closest to the historial value for that industry and seniority, and at which the market price falls between the computed arbitrage bounds.

3 Addressing Uncertainty For The Discount Factors

The discount factors used in pricing formulas (1), (2), and (8) can be estimated either directly from default-free bond prices, or by inverting a yield curve such as the LIBOR. Both procedures, however, are prone to errors. In Section 3.1, we use robust optimization techniques to address the problem of multiple discount term structures that are consistent with the market. In Section 3.2, we address the issue of *infeasibility* in problems (10) and (11) that results from inaccuracies in discount factor estimation.

3.1 Robustness with Respect to Discount Factor Estimates

To estimate discount factors directly from bond prices, one solves an optimization problem in which the constraints require that the prices of riskless bonds computed using the (unknown) discount factors come close to bond prices observed in the market. The objective is to minimize some norm of the error between the estimated and the market bond prices (see, for example, Ioffe et al. [16]). In reality, a smooth curve for the discount factors is selected, and its parameters are estimated from the optimization problem. Usually more than one set of discount factors solve the optimization problem, depending on the choice of norm to be minimized, and on the expression for the approximating discount function.

Uncertainty about the exact discount term structure can arise also when the discount factors are estimated by interpolating and extrapolating a yield curve at each coupon payment date. An error in the estimation of this yield curve can translate into an error in the discount factor estimates, and can propagate across all discount factor estimates.

One possible way to include considerations of the inaccuracy in discount factor estimates is to treat the discount factors in problems (10) and (11) as uncertain coefficients. We can then take a *robust polyhedral optimization* approach. Robust optimization is a methodology that requires the optimal solution to an optimization problem to remain feasible for *any* realization of the uncertain coefficient data within pre-specified uncertainty sets. Below, we provide a brief review of the methodology, and suggest choices for such uncertainty sets.

Let $\mathbf{x} \in \Re^{n\times 1}$ be a vector of variables, $\mathbf{c} \in \Re^{n\times 1}$ be a vector of certain objective function coefficients, $\tilde{\mathbf{A}} \in \Re^{m\times n}$ be a matrix of uncertain coefficients that varies in a polytope $P^A = \{\text{vec}(\tilde{\mathbf{A}}) \mid \mathbf{G}\cdot\text{vec}(\tilde{\mathbf{A}}) \leq \mathbf{d}\}$, and P^x be defined by constraints that include only certain coefficients. Note that $\text{vec}(\mathbf{A}) \in \Re^{n\cdot m\times 1}$ denotes a vector that consists of the rows of matrix $\mathbf{A}$. Although the entries of the cost vector $\mathbf{c}$ are not allowed to be uncertain, this requirement is not as restrictive as it initially appears, since one can introduce an additional variable to replace the objective function, and write an equivalent problem where data uncertainty is present only in the constraints. The robust counterpart of the linear programming problem

$$\max\left\{\mathbf{c}'\mathbf{x} \mid \tilde{\mathbf{A}}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in P^x\right\}$$

can be stated as

$$\max\left\{\mathbf{c}'\mathbf{x} \mid \tilde{\mathbf{A}}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in P^x,\ \forall\tilde{\mathbf{A}} \in P^A\right\}.$$

When the optimization problem is linear and the pre-specified uncertainty sets are polyhedral, the robust counterpart of the optimization problem is a compact, efficiently solvable linear program (Ben-Tal and Nemirovski [1, 2], El Ghaoui et al. [12, 13], Bertsimas et al. [3]). We state this as a theorem below. The proof can be found in Bertsimas et al. [3], and is omitted.

Theorem 3 *Let* $\mathbf{x} \in \Re^{n\times 1}$ *be a vector of variables;* $\mathbf{c} \in \Re^{n\times 1}$, $\mathbf{b} \in \Re^{m\times 1}$, $\mathbf{G} \in \Re^{l\times(m\cdot n)}$, *and* $\mathbf{d} \in \Re^{l\times 1}$ *be given data, and* $\tilde{\mathbf{A}} \in \Re^{m\times n}$ *be a matrix of uncertain coefficients. Then, the problem*

$$\begin{aligned} \max\quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.}\quad & \tilde{\mathbf{A}}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in P^x \\ & \forall\tilde{\mathbf{A}}\ \textit{such that}\ \mathbf{G}\cdot \textit{vec}(\tilde{\mathbf{A}}) \leq \mathbf{d} \end{aligned} \tag{12}$$

is equivalent to the linear programming problem

$$
\begin{aligned}
\max \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & (\mathbf{p}^i)'\mathbf{G} = (\mathbf{x}_i)', \quad && i = 1, \ldots, m \\
& (\mathbf{p}^i)'\mathbf{d} \le b_i, && i = 1, \ldots, m \\
& \mathbf{p}^i \ge 0, && i = 1, \ldots, m \\
& \mathbf{x} \in P^x,
\end{aligned}
\tag{13}
$$

where $\mathbf{p}^i \in \Re^{l \times 1}$ *and* $\mathbf{x}_i \in \Re^{(m \cdot n) \times 1}$, $i = 1, \ldots, m$, *are vectors of variables.* $\mathbf{x}_i$ *contains* $\mathbf{x}$ *in entries* $(i-1) \cdot n + 1$ *through* $i \cdot n$, *and zero everywhere else.*

Bertsimas et al. [3] discuss a number of convenient formulations of polyhedral uncertainty sets. For example, if we have information about the nominal value $\check{a}_{ij}$ and the standard deviation σ_{ij} of each uncertain coefficient $\tilde{a}_{ij}$, we can define a polyhedral uncertainty set of the kind

$$
P^A = \left\{ \tilde{a}_{ij} \;\Big|\; \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|\tilde{a}_{ij} - \check{a}_{ij}|}{\sigma_{ij}} \le \Gamma \right\}, \tag{14}
$$

where Γ is a *robustness budget* parameter that controls the tradeoff between robustness and optimality, and can be linked to probabilistic guarantees. In effect, we require protection against all movements from the uncertain coefficients that are within a total distance Γ from their nominal values. We normalize these movements by the standard deviations of the corresponding coefficients, because intuitively, if a coefficient has a small standard deviation, it should not vary by a lot. If it does, then it should consume more of the robustness budget, since this is an event with low probability. The robust counterpart of problem (12) with this definition of a polyhedral set P^A is given by (see Bertsimas et al. [3])

$$
\begin{aligned}
\max \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & (\mathbf{x}_i)'\text{vec}(\check{\mathbf{A}}) + \Gamma \cdot \left\| \mathbf{\Sigma}^{\frac{1}{2}} \mathbf{x}_i \right\|_\infty \le b_i, \quad i = 1, \ldots, m \\
& \mathbf{x} \in P^x,
\end{aligned}
\tag{15}
$$

where $\mathbf{\Sigma} = [\sigma_{ij}]$ and $\| \, . \, \|_\infty$ denotes the infinity norm.[2]

We next present examples of polyhedral uncertainty sets for the discount factors $\tilde{df}$. Theorem 3 can then be applied to produce a linear robust counterpart formulation of optimization problems (10), (11).

1. The simplest model is to have ranges of possible values for the discount factors at each time period:

$$\underline{df}_t \leq \tilde{df}_t \leq \overline{df}_t, \tag{16}$$

   where $\underline{df}_t$ and $\overline{df}_t$ are known bounds on the discount factor values.

2. If an estimate of the range of the instantaneous rate at time t, $[\underline{r}_t, \overline{r}_t]$, is available, we can impose the following restriction on the discount factors:

$$\underline{\delta}_t \tilde{df}_{t-1} \leq \tilde{df}_t \leq \overline{\delta}_t \tilde{df}_{t-1}, \tag{17}$$

   where $\underline{\delta}_t$ and $\overline{\delta}_t$ are of the form $e^{-\overline{r}_t \Delta t}$ and $e^{-\underline{r}_t \Delta t}$, respectively.

3. Alternatively, we can consider an estimate $\hat{r}_t$ of the instantaneous rate at time t, and incorporate uncertainty through additional nonnegative variables $\tilde{\epsilon}_t$ and $\tilde{\eta}_t$ that control the deviation from the estimated values of the discount factors:

$$\delta_t \tilde{df}_{t-1} - \tilde{\eta}_t \leq \tilde{df}_t \leq \delta_t \tilde{df}_{t-1} + \tilde{\epsilon}_t. \tag{18}$$

   Here δ_t equals $e^{-\hat{r}_t \Delta t}$, and we require that

$$\begin{aligned} \sum_{i=1}^{n} \frac{\tilde{\epsilon}_i}{\sigma_{\epsilon_i}} &\leq \Gamma_\epsilon \text{ and} \\ \sum_{i=1}^{n} \frac{\tilde{\eta}_i}{\sigma_{\eta_i}} &\leq \Gamma_\eta, \end{aligned}$$

   where σ_{ϵ_i} and σ_{η_i} are the standard deviations of $\tilde{\epsilon}_i$ and $\tilde{\eta}_i$, respectively. The estimation errors may be larger for discount factors at later dates, as errors propagate from earlier

[2]The infinity norm $\|\mathbf{x}\|_\infty$ of a vector $\mathbf{x} \in \Re^{n \times 1}$ is defined as $\max_{j=1,\ldots,n} |x_j|$.

dates. This can be modeled by increasing the standard deviations of $\tilde{\epsilon}_t$ and $\tilde{\eta}_t$ as t increases.

We study the effect of making the optimal bounds problem robust with respect to the estimates of the discount factors in computational experiments in Section 5.

3.2 Adjusting the Discount Factors to Prevent Infeasibility

In computational experiments with real data, we have encountered the problem that problems (10), (11) were sometimes infeasible. Theorem 2 would imply in this case the existence of arbitrage opportunities. However, infeasibility may be caused also by errors in the nominal estimates of the discount factors.

It is possible to avoid the infeasibility by removing the bonds that are the source of the problem. However, the group of bonds with similar credit characteristics available in the market is usually small, and we would like to keep as many of the original bonds in the data set as possible. Our idea for dealing with the infeasibility problem is to perturb the data just enough so that we end up with a feasible set without throwing out a number of bonds.

For this purpose, we consider the following problem:

Given data $(\mathbf{A}_0, \mathbf{b}_0)$, for which the system $\mathbf{A}_0\mathbf{x} \leq \mathbf{b}_0,\ \mathbf{x} \geq 0$ is infeasible, find the minimum (in the sense outlined below) perturbation matrix $\mathbf{D}_\epsilon$ such that the matrix $\hat{\mathbf{A}}_\epsilon = \mathbf{A}_0 + \mathbf{D}_\epsilon$ has the property that the system $\hat{\mathbf{A}}_\epsilon\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0$ has a solution. Here $\mathbf{A}_0$, $\hat{\mathbf{A}}_\epsilon$ and $\mathbf{D}_\epsilon$ are in $\Re^{m_0 \times n_0}$, $\mathbf{b}_0 \in \Re^{m_0 \times 1}$, and $\mathbf{x} \in \Re^{n_0 \times 1}$. We want

$$\|\mathbf{D}_\epsilon\| = \inf_{\{\mathbf{D}:\ (\mathbf{A}_0+\mathbf{D})\mathbf{x} \leq \mathbf{b}_0,\ \mathbf{x} \geq 0 \text{ has a solution}\}} \|\mathbf{D}\|, \tag{19}$$

i.e., we require that the perturbation be "minimum" in the operator norm sense. The operator norm of a matrix $\mathbf{D}$ is defined as $\max_{\{\mathbf{x}:\ \|\mathbf{x}\|_1=1\}} \|\mathbf{D}\mathbf{x}\|_\infty$.

Freund and Vera [14] have solved the problem of finding the minimum such perturbation. Their solution is summarized by the following theorem:

Theorem 4 *For any $\epsilon > 0$, the matrix $\mathbf{D}_\epsilon$ that solves problem (19) is given by*

$$\mathbf{D}_\epsilon = -(\tilde{\gamma} + \epsilon)\mathbf{e}\mathbf{e}',$$

where the scalar $\tilde{\gamma}$ is a part of the optimal solution $(\tilde{\mathbf{x}}, \tilde{\gamma}, \tilde{r})$ to the following linear programming problem:

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & \mathbf{A}_0\mathbf{x} - \gamma\mathbf{e} \leq r\mathbf{b}_0 \\ & \mathbf{e}'\mathbf{x} = 1 \\ & r \geq 0, \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{20}$$

where $r \in \Re$, $\gamma \in \Re$ and $\mathbf{x} \in \Re^{n_0 \times 1}$ are variables, and $\mathbf{e} \in \Re^{n_0 \times 1}$ is a vector of ones.[3]

4 An Integer Programming Approach to Classification of Risky Bonds

In this section, we propose an integer programming approach to classify credit risky bonds. We are given m bonds with initial assignments to K_0 classes represented by binary numbers $z_{i,k_0(i)}$ that equal 1 if bond i belongs to class $k_0(i)$, and 0 otherwise. The problem we address is to classify risky bonds in K classes in a way that minimizes the change from an existing classification represented by the vector $\mathbf{z_0}$, given only price and coupon data for these bonds. The key idea of the method is that bonds of the same class should have the same survival probabilities. In other words, survival probabilities are class dependent.

We define binary variables $z_{i,k}$ with value 1 if bond i is assigned to class k, and 0 otherwise. We use $\boldsymbol{\pi}^i \in \Re^{n \times 1}$, $i = 1, \ldots, m$ and $\mathbf{v}^k \in \Re^{n \times 1}$, $k = 1, \ldots, K$ to denote the vector of survival probabilities for bond i and class k respectively, where n is the total number of coupon

[3] Note that problem (20) is always feasible.

payment dates for all m bonds. The idea of the method is to require that if $z_{i,k} = 1$, then $\boldsymbol{\pi}^i = \mathbf{v}^k$.

The data given are as in Section 2, i.e., we know the prices p_i^{bid}, p_i^{ask}, the coupon payments c_i, the recovery rate L_i, and the discount factors df_t. From the data we calculate q_i, the ith entry of the vector $\mathbf{q} \in \Re^{m\times 1}$, and $\bar{\mathbf{a}}_i$, the ith row of the matrix $\bar{\mathbf{A}}$ of discounted coupon payments as defined in Section 2. The proposed model is as follows.

$$\begin{array}{lll}
\min & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} |z_{i,k} - z_{i,k_0(i)}| & \\
\text{s.t.} & p_i^{bid} \le q_i + \bar{\mathbf{a}}_i'\boldsymbol{\pi}^i \le p_i^{ask}, & i = 1,\ldots,m \\
& |\boldsymbol{\pi}^i - \mathbf{v}^k| \le (1 - z_{i,k})\cdot\mathbf{e}, & i = 1,\ldots,m,\ k = 1,\ldots,K \\
& \displaystyle\sum_{k=1}^{K} z_{i,k} = 1, & i = 1,\ldots,m \\
& 1 \ge \pi_1^i \ge \ldots \ge \pi_n^i \ge 0, & i = 1,2,\ldots,m \\
& 1 \ge v_1^k \ge \ldots \ge v_n^k \ge 0, & k = 1,2,\ldots,K \\
& z_{i,k} \in \{0,1\}. &
\end{array} \tag{21}$$

Absolute value is used to denote entry-wise absolute differences in the vectors $\boldsymbol{\pi}^i$ and $\mathbf{v}^k$.

The first set of constraints requires that the price of bond i falls within the bid-ask spread observed in the market if the default probabilities assigned to bond i are $\boldsymbol{\pi}^i$. The constraints $|\boldsymbol{\pi}^i - \mathbf{v}^k| \le (1 - z_{i,k})\cdot\mathbf{e}$, $i = 1,\ldots,m$, $k = 1,\ldots,K$, ensure that all bonds assigned to class k are assigned the same default probability distribution. If $z_{i,k}$ equals 0, the latter constraint becomes irrelevant. Finally, $\sum_k z_{i,k} = 1\ \forall i,k$ limits the number of assignments to only one class per bond.

An attractive aspect of the model is that it allows the user to decide which bonds are similar. Thus, for example, if our initial assignment of bonds to classes leads to an infeasible system (9), the model can be applied to re-assign classes, and thus re-assign survival probabilities, so that the resulting system is feasible.

5 Computational Results

5.1 Controlled Experiments

In our first set of simulated experiments, we are interested in (a) the effect of the number, maturity, bid-ask spread and recovery rate of the bonds in the collection to the quality of the bounds, and (b) the effectiveness of the robust optimization approach outlined in Section 3 and the integer programming approach in Section 4.

We assume that defaults follow a Poisson distribution with parameter $\lambda = 0.02$. This implies that the survival probability distribution for the bond issuer is of the form $\Pr(\tau > t) = e^{-\lambda t}$. A value of 0.02 for λ makes the probability of survival at the end of the first year 98%. The survival probability decreases to 67% for a time period of 20 years.

Quality of Bounds.

Having computed prices of bonds using the survival distribution described above, we produce a random bid-ask spread by drawing from a uniform distribution with end points 0 and 0.01 (this ensures that on average, we are half a percent higher or lower than the correct price). We then calculate the minimum and maximum bounds on the price of a test bond as well as bounds on the survival probabilities using the methods described in Section 2.

Number of bonds. In Table 1 we report bounds on the price of a test bond with maturity of 7.5 years and semi-annual coupon payment of 8%: computations for 5, 10, and 20 bonds in the data set, with maturities ranging from 1 to 20 years, and semi-annual coupon payments ranging from 5% to 9%. The results indicate that the bounds improve as the number of bonds in the data set increases. Even a small number of bonds, e.g. five bonds, produces bounds with a spread within 5% of the real test bond price, which can be sufficiently accurate for many purposes. One can also observe that that the marginal increase in the accuracy of bounds is greater for a small number of bonds than it is for a large one. Twenty bonds in the data set produce almost as tight bounds on the test price as fifty or a hundred do. Note that because of the randomness in generating the bid-ask spread of the bonds in the data set, the tightness of the bounds may vary from trial to trial.

In Figure 1 we present the bounds on the survival probabilities. Figures 1(a)-(c) illustrate the computed bounds for data set sizes of five, ten, and twenty bonds, and recovery rate of zero, and Figures 1(d)-(f) show the bounds when the recovery rate is 80%. Again, the number of bonds in the data set influences the quality of the bounds, while the tightness of the bounds increases with the number of bonds in the data set. The greatest marginal improvement is observed for small data sizes, while for data set sizes greater than twenty bonds, the bounds do not improve significantly. We notice that the bounds become wider as the number of years increases. The reason for this is that there are fewer bonds whose market price values constrain the possible values for the survival probabilities in later years. By the end of the twentieth year, most bonds have matured.

Maturity. In Table 2 we report bounds on a test bond that has shorter maturity (7.5 years) than the bonds in the data set (10-20 years). By contrast, the maturity of the bonds in the data set used for generating the results in Table 1, is 1-20 years. The bounds on the test price in Table 2 relative to the bounds in Table 1 are wider. Even twenty bonds with maturities greater than the maturity of the test bond are not sufficient to produce bounds as tight as the bounds produced by only the five bonds maturing both before and after the test bond in Table 1.

Bid-Ask Spread. We run the experiment with the same data sets of bonds as in Table 1, but deviations from the real price are now drawn from a uniform distribution on $[0, 0.03]$ instead of $[0, 0.01]$. The results are presented in Table 3. The bounds on the price of the test bond are noticeably wider than the bounds in Table 1.

The quality of the survival probability bounds is influenced by the magnitude of the spread in the bid and ask prices of the bonds in the data set. Instead of drawing from a uniform distribution on $[0, 0.01]$, we perform the experiment with the same twenty bonds from Figure

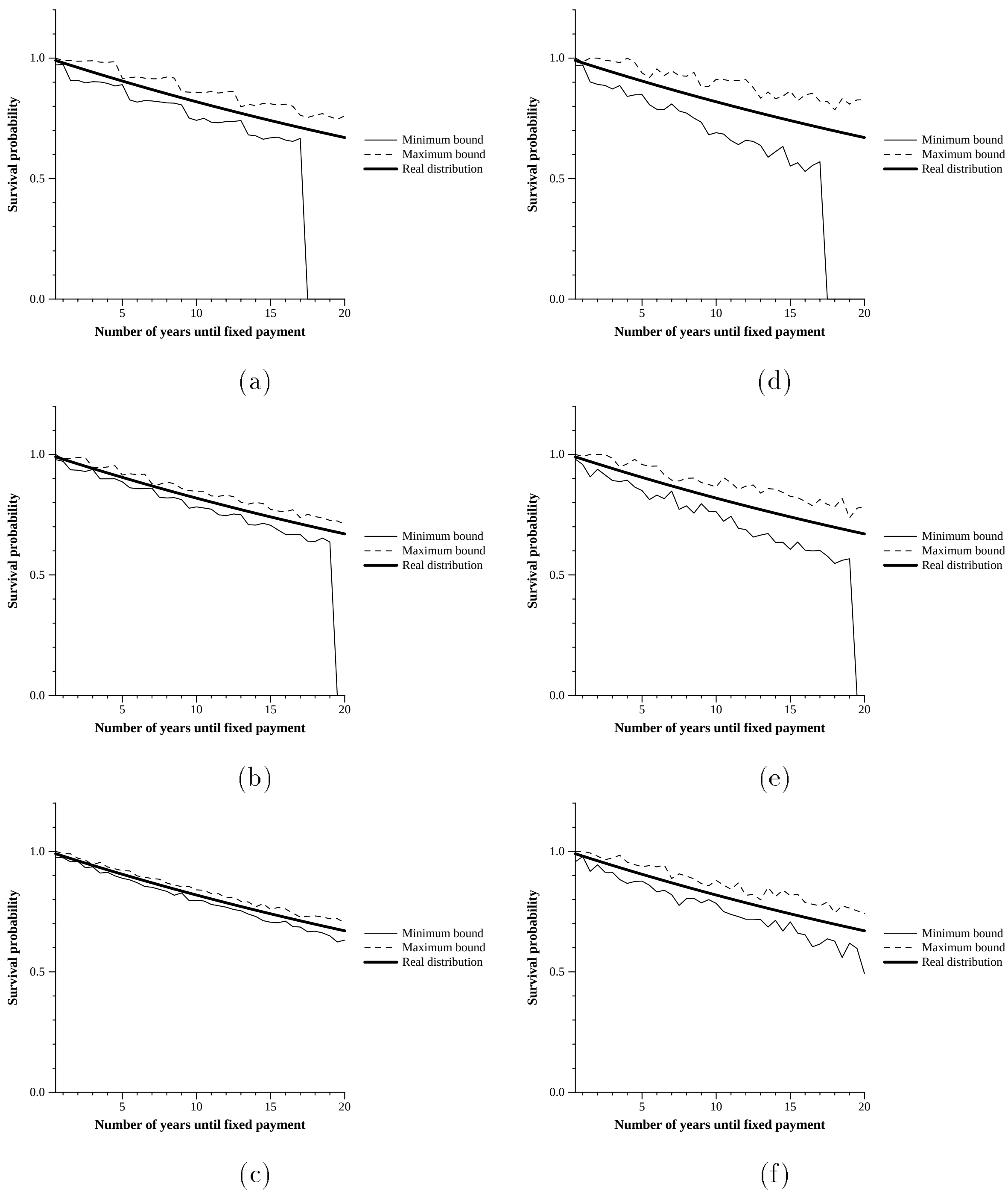


Figure 1: Bounds on the minimum and maximum value of the survival probability distribution for data sets with different number of bonds with maturities ranging from 1 to 20 years and semi-annual coupon payments ranging from 5% to 9%: (a) 5 bonds; (b) 10 bonds; (c) 20 bonds; (d), (e), (f) correspond to cases (a), (b), and (c), respectively, but the bond prices are generated with a recovery rate of 0.8.

Table 1: Bounds on the Price of a Test Bond as a function of the number of bonds in the data set and as a function of the recovery rate. The real price of the test bond is computed from pricing formula (8). "%" stands for (Max-Min)/Price.

L	Real Price	5 bonds			10 bonds			20 bonds		
		Min	Max	%	Min	Max	%	Min	Max	%
0.0	0.9910	0.9690	1.0257	5.73	0.9716	1.0065	3.52	0.9741	0.9998	2.60
0.1	1.0021	0.9776	1.0358	5.81	0.9817	1.0171	3.53	0.9813	1.0081	2.67
0.2	1.0133	0.9970	1.0503	5.27	0.9975	1.0244	2.66	0.9995	1.0218	2.20
0.3	1.0244	1.0042	1.0496	4.44	1.0120	1.0380	2.54	1.0104	1.0358	2.48
0.4	1.0355	1.0149	1.0618	4.53	1.0239	1.0398	1.54	1.0248	1.0479	2.23
0.5	1.0466	1.0328	1.0725	3.79	1.0362	1.0586	2.14	1.0380	1.0526	1.40
0.6	1.0577	1.0486	1.0759	2.58	1.0466	1.0678	2.01	1.0492	1.0617	1.18
0.7	1.0688	1.0520	1.0810	2.71	1.0611	1.0788	1.66	1.0563	1.0720	1.47
0.8	1.0799	1.0694	1.0990	2.74	1.0681	1.0880	1.84	1.0738	1.0891	1.41
0.9	1.0910	1.0765	1.1045	2.57	1.0827	1.0946	1.09	1.0848	1.0969	1.11
1.0	1.1021	1.0939	1.1136	1.79	1.0960	1.1050	0.81	1.0977	1.1036	0.53

1, but with a spread drawn from a uniform distribution on $[0, 0.03]$, which means that on average, 0.015 is added or subtracted from the true bond price. The bounds on the survival distribution, shown in Figure 2, are wider.

Recovery rate. The effect of the recovery rate depends on the relation between the bid-ask spread of the bonds and their price. The results in Tables 1, 2, and 3 appear to indicate that the difference between the minimum and the maximum bound on the price of the test bond decreases as the recovery rate L increases. However, the main reason for this is the way the bid-ask spreads were generated in these experiments. We added and subtracted uniform random numbers drawn from a constant range for the random bid-ask spread, not taking into

Table 2: Bounds on the Price of a Test Bond with maturity of 7.5 years and semi-annual coupon payment of 8%: computations for 5, 20, and 50 bonds in the data set, with maturities ranging from 10 to 20 years, and semi-annual coupon payments ranging from 5% to 9%.

L	Real	5 bonds			20 bonds			50 bonds		
	Price	Min	Max	%	Min	Max	%	Min	Max	%
0.0	0.9910	0.9576	1.1136	15.73	0.9621	1.1136	15.28	0.9642	1.0559	9.25
0.1	1.0021	0.9720	1.1136	14.13	0.9700	1.1136	14.33	0.9778	1.0678	8.98
0.2	1.0133	0.9839	1.1136	12.79	0.9854	1.1136	12.65	0.9912	1.0844	9.20
0.3	1.0244	0.9995	1.1136	11.13	0.9988	1.1136	11.20	1.0035	1.0653	6.03
0.4	1.0355	1.0157	1.1136	9.45	1.0114	1.1136	9.86	1.0181	1.0938	7.30
0.5	1.0466	1.0251	1.1136	8.45	1.0325	1.1136	7.75	1.0300	1.0846	5.21
0.6	1.0577	1.0378	1.1136	7.17	1.0430	1.1136	6.67	1.0441	1.0916	4.49
0.7	1.0688	1.0547	1.1136	5.51	1.0574	1.1136	5.26	1.0589	1.1065	4.45
0.8	1.0799	1.0627	1.1136	4.71	1.0683	1.1136	4.19	1.0723	1.1004	2.60
0.9	1.0910	1.0773	1.1136	3.33	1.0851	1.1136	2.61	1.0858	1.1055	1.80
1.0	1.1021	1.0950	1.1136	1.69	1.0990	1.1136	1.32	1.0977	1.1131	1.40

consideration the fact that a higher recovery rate produces a higher fair bond price. Since the test bond and the bonds in the collection were assumed to have the same recovery rate, when the bid-ask spread is relatively smaller as a percentage of price, the minimum and the maximum bounds on the test bond price are tighter. This tightening of the bounds as L increases does not happen if the range for the bid-ask spread is made proportional to the bond price.

Regarding survival probabilities, when the recovery rate is high, the prices of the bonds in the collection are high, and a wider feasible range for each particular survival probability π_t is available. This is the reason for the bounds on the survival probability distribution to become

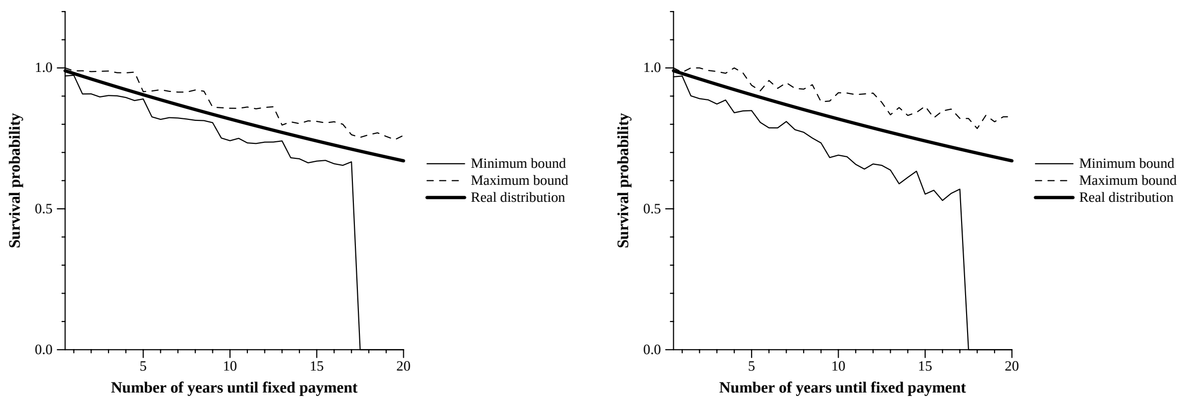
Table 3: Bounds on the Price of a Test Bond with maturity of 7.5 years and semi-annual coupon payment of 8%: computations for the 5, 10, and 20 bonds from Table 1, with maturities ranging from 1 to 20 years, and semi-annual coupon payments ranging from 5% to 9%. The random bid-ask spread is drawn from a uniform distribution on $[0, 0.03]$ rather than $[0, 0.01]$.

L	Real	5 bonds			10 bonds			20 bonds		
	Price	Min	Max	%	Min	Max	%	Min	Max	%
0.0	0.9910	0.9666	1.0396	7.36	0.9548	1.0238	6.96	0.9595	1.0091	5.00
0.1	1.0021	0.9560	1.0557	9.94	0.9796	1.0145	3.49	0.9694	1.0096	4.01
0.2	1.0133	0.9773	1.0507	7.24	0.9838	1.0301	4.57	0.9828	1.0219	3.86
0.3	1.0244	0.9949	1.0687	7.21	0.9970	1.0428	4.46	1.0062	1.0471	3.99
0.4	1.0355	1.0227	1.0753	5.08	1.0093	1.0510	4.02	1.0235	1.0423	1.81
0.5	1.0466	1.0168	1.0850	6.52	1.0169	1.0599	4.11	1.0351	1.0668	3.03
0.6	1.0577	1.0391	1.0890	4.73	1.0359	1.0787	4.04	1.0380	1.0670	2.73
0.7	1.0688	1.0444	1.1119	6.32	1.0467	1.0901	4.06	1.0459	1.0876	3.90
0.8	1.0799	1.0573	1.1136	5.21	1.0614	1.0907	2.71	1.0745	1.0958	1.98
0.9	1.0910	1.0710	1.1016	2.80	1.0753	1.1101	3.19	1.0804	1.0989	1.70
1.0	1.1021	1.0839	1.1136	2.70	1.0882	1.1071	1.71	1.0988	1.1098	1.01

less tight as the recovery rate increases. Note that this is not the case in the computation of bounds on the price of a test bond, because there a combination of survival probabilities participates in the objective function.

Robust Discount Factor Estimation.

We consider the robust optimization model from Eq. (16) using the same value of λ as before. We assume that the underlying interest rate $\check{r}$ that generates the risky bond prices is 6%. We compute the nominal prices for 25 risky bonds with assumed recovery rate of 0, coupons ranging from 5% to 9%, and maturities ranging from 3 years to 7 years. A random bid-ask



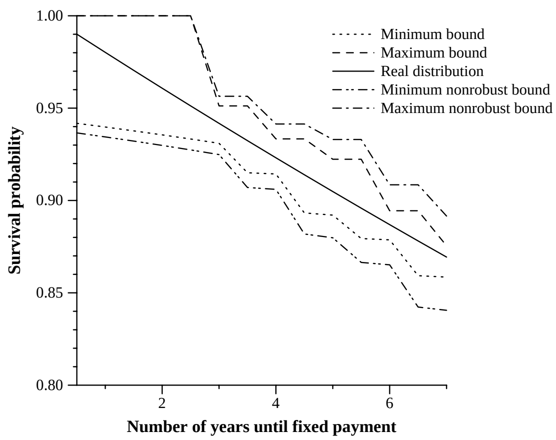
(a) (b)

Figure 2: Bounds on the minimum and maximum value of the survival probability distribution for the data set of 20 bonds (Figure 1, (c) and (f)) but for a random bid-ask spread on $[0, 0.03]$ rather than $[0, 0.01]$: (a) recovery rate of 0; (b) recovery rate of 0.8.

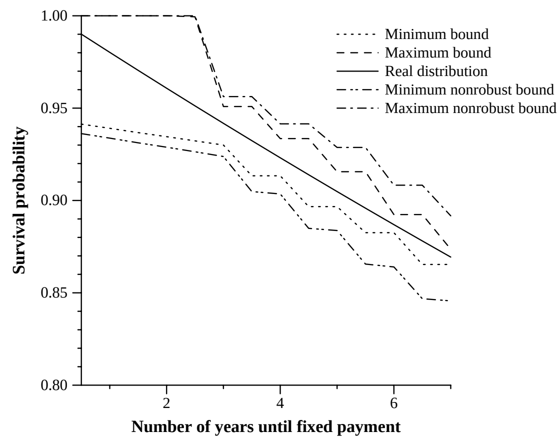
spread is created by adding and subtracting 0.01 plus a uniform random variable that takes values on $[0, 0.01]$.

We then study the bounds on the survival probabilities when upper and lower bounds on the discount factors are available. These are determined by three ranges of the interest rate: (a) $\underline{r}_t = 5.8\%$, $\overline{r}_t = 6.2\%$; (b) $\underline{r}_t = 5.6\%$, $\overline{r}_t = 6.4\%$; and (c) $\underline{r}_t = 5.9\%$, $\overline{r}_t = 6.3\%$. The discount factor ranges used in (16) are computed as $\underline{df}_t = e^{-\overline{r}_t t}$ and $\overline{df}_t = e^{-\underline{r}_t t}$.

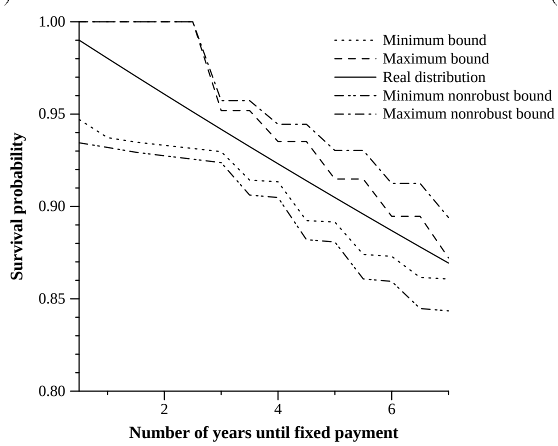
The results are presented in Figure 3. For comparison, we have plotted also the nonrobust bounds obtained with point estimates of the interest rate equal to its expected value. As expected, the robust bounds are tighter, because the robust linear programming problem has a smaller feasible region than the nonrobust one. The bounds become even tighter as the possible range for the interest rate increases. Finally, if the interest rate range is not symmetric around $\check{r}$, as is the case in (c), the computed probability bounds are not symmetric around the real value of survival probabilities.



(a)



(b)



(c)

Figure 3: Robust and nonrobust survival probability bounds obtained from formulation (16) for bond prices generated with a nominal interest rate of 6% and discount factors generated with: (a) $\underline{r} = 5.8\%$, $\overline{r} = 6.2\%$; (b) $\underline{r} = 5.6\%$, $\overline{r} = 6.4\%$; and (c) $\underline{r} = 5.9\%$, $\overline{r} = 6.3\%$.

Classification.

We performed a number of tests on the performance of the classification method from Section 4 with different values for recovery rates, bid-ask spread, and size of the data set. We simulated bond price data for different values of the parameter λ, and let the classification program assign bonds to groups. The program was able to differentiate between bonds for differences in the values of the paramater λ as small as 0.001.

Our experiments indicate that the size of the recovery rate does not influence the accuracy of the results. However, the impact of the bid-ask spread can be significant. An increase in the number of bonds in the data set does not change the results much, but it is recommendable to work with at least four or five bonds.

5.2 Examples with Real Market Data

Bounds on Prices and Survival Probabilities.

In this section, we consider a collection of bonds of an issuer of S&P rating AA. One such example is Southwestern Bell Tel Co. Southwestern Bell bonds traded on the market on July 2, 1999, are listed in Table 5.[4] The U.S. Treasury yield curve data for July 2, 1999 is given in Table 4.[5]

Table 4: U.S. Treasury Yield Curve Data, July 2, 1999

Bills/Notes/Bonds	3 mons	6 mons	1 year	2 yrs	5 yrs	10 yrs	30 yrs
Yield	4.67	4.97	5.06	5.57	5.70	5.80	6.01

We treat the first Southwestern Bell bond as a test bond, with the remaining four bonds defining the feasible set for problems (10) and (11). The results produced by solving problems (10) and (11) for different values of the recovery rate are given in Table 6. The market price

[4] Source: http://www.bondsonline.com.

[5] Source: Bloomberg L.P.

Table 5: Data on bonds from AA and BBB- Rating, July 2, 1999

Bond Number	S&P Rating	Issuer	Coupon(%)	Maturity	Yield	Price
0	AA	Southwestern	6.125	03-01-2000	5.229	1.0055
1	AA	Southwestern	6.375	04-01-2001	5.827	1.0087
2	AA	Southwestern	6.580	08-26-2003	6.247	1.0119
3	AA	Southwestern	7.000	07-01-2015	6.591	1.0400
4	AA	Southwestern	7.000	12-28-2020	6.932	1.0075
5	BBB-	Delta Air	10.375	02-01-2011	7.785	1.1950
6	BBB-	Delta Air	10.375	02-01-2011	7.785	1.1950
7	BBB-	Delta Air	9.000	05-15-2016	7.744	1.1169
8	BBB-	Delta Air	9.250	03-15-2022	7.951	1.1352
9	AA	Bell Atlantic	5.875	02-01-2004	5.951	0.9969

of the Southwestern Bell test bond (Table 5) falls within the arbitrage bounds computed with our linear programs for any value of the recovery rate except $L \in [0.75, 1]$. The assumption that the market price is a fair price leads us to believe that the market agents consider the possibility of recovering more than 75% of the notional in the event of default unlikely.

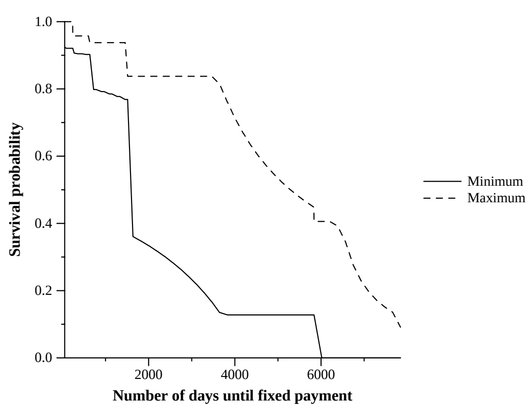


Figure 4: Bounds on the survival probability distribution of the Southwestern Bell bonds for a recovery rate of 63%, the average historical recovery rate for senior secured public debt.

Table 6: Bounds on the price of the Southwestern Bell Test Bond

Recovery Rate	Market Price	Min Bound	Max Bound	Difference	Difference as % of market price
0.0	1.0055	1.0001	1.0291	0.0289	2.88
0.1	1.0055	1.0002	1.0291	0.0290	2.87
0.2	1.0055	1.0002	1.0291	0.0290	2.87
0.3	1.0055	1.0003	1.0291	0.0288	2.86
0.4	1.0055	1.0003	1.0291	0.0288	2.86
0.5	1.0055	1.0004	1.0291	0.0287	2.85
0.6	1.0055	1.0006	1.0291	0.0285	2.83
0.7	1.0055	1.0008	1.0291	0.0283	2.81
0.8	1.0055	infeasible	infeasible		
0.9	1.0055	infeasible	infeasible		
1.0	1.0055	infeasible	infeasible		

We proceed to compute bounds on the survival probabilities for Southwestern Bell bonds. The total number of coupon dates for all six Southwestern Bell bonds traded in the market on July 2, 1999, is 90. The results for the 90 survival probabilities for assumed recovery rate of 63%[6] are plotted in Figure 4. By interpolating between the different values of the survival probabilities we obtain the approximate shape of the survival distribution in time for any value of the recovery rate. The gap between the bounds noticeably widens as the time period increases. The reason for tighter short term bounds is that estimation of survival probabilities in the immediate future is conducted with bonds with both long and short maturities, whereas only a small number of bonds with long maturities is available to estimate the survival probabilities in the long run.

[6] We take the average historic recovery rate value for senior secured public debt.

Classification.

We next apply the integer programming approach for assigning bonds to their correct rating class. Consider the mix of bonds with ratings AA and BBB- in Table 5. We assign the same rating to all of them at the beginning and let the program decide how to split them into two classes. To make sure that the assignment is consistent, we impose the additional constraints that bonds from the same issuer belong to the same class. The program correctly assigns bonds from Bell Telephone- New Jersey Inc. and Southwestern Bell Telephone Co. to the same rating class, and separates them from Delta Air bonds.

We note that the survival probabilities obtained after solving (21) are only indicative of the magnitude of the survival probabilities of each class. In order to obtain bounds for the survival probabilities for the particular class, we solve problem (10) once the bonds are assigned to their group by (21).

Regarding the accuracy of the classification, we feel that the data is often insufficient to distinguish between rating categories very close to one another, e.g. AA and AA-. However, our experiments indicate that the method works well when the difference in rating classes is greater.

6 Concluding Remarks

We have shown how to compute bounds on the price of a risky bond, and how to estimate bounds on the survival probability distribution of a collection of similar bonds, given their recovery rates, and assuming no-arbitrage. Robust optimization techniques allow for handling uncertainty in key parameters of the pricing model. We also proposed an approach to classify bonds based on integer programming.

All the optimization problems we have proposed can be solved easily by state of the art optimization solvers in a few seconds. While the bounds we derive do not model explicitly specific factors, they work directly with the data, and thus are appealing from a practical point of view.

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