Improved Randomized Approximation Algorithms for Lot-Sizing Problems

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1 Introduction

We consider in this paper multi-product, lot-sizing problems that arise in manufacturing and inventory systems. We describe the problem in a manufactruring setting. There is a set N of products. For each product $j \in N$ there is a set π_j (called predecessors of product j) of products consumed in producing product j. We define the product network G to be a directed network with node set N and arc set $A = \{(i,j) : i \in \pi_j\}$. In other words, the network G corresponds to the flow of materials in the system and contains no circuit.

External demand d_i for product i is assumed to be constant in time. Clearly in order to satisfy the demand orders should be placed for the products dynamically in time. If an order is placed for product i, an ordering cost K_i is incurred. Moreover, an incremental echelon holding cost h_i is incurred per unit time the item spends in inventory. The production rate is assumed to be infinite. The objective is to schedule orders for each of the products over an infinite horizon so as to minimize long-run average cost.

As the optimal dynamic policy can be very complicated, the research community (see for instance Roundy [18, 19], Jackson, Maxwell and Muckstadt [10], Muckstadt and Roundy [14]) has focused on stationary and nested policies defined as follows: Orders are placed periodically in time at equal intervals, for each of the products in the system (stationary policies). If product j precedes product i, then an order is placed for product j only when an order is placed for product i at the same time (nested policies). Therefore, under a stationary and nested policy the objective is to decide the period T_i that an order is placed. The reason stationary and nested policies are attractive is that they are easy to implement. Muckstadt and Roundy [14] discuss in detail the rationale of using order intervals T_i as variables.

The problem of designing an optimal stationary and nested policy can then be formulated (see [18]) as the following nonlinear integer programming problem.

$$(P_{NS}) \quad = \min \quad G(T) \equiv \sum_{i \in N} \left(\frac{K_i}{T_i} + H_i T_i \right)$$

$$\frac{T_i}{T_j} \in \{1, 2, 3 \ldots\} \text{ if } (i,j) \in A,$$

$$T_i = k_i T_L \text{ for each } i,\ k_i \in \mathcal{Z}_+$$

Period T_L is called the base period and it can be constant or allowed to vary, depending on the model. The coefficient H_i is given by $H_i = (h_i - \sum_{j\in\pi_i} h_j)D_i$ and D_i represents the aggregate demand, which is calculated recursively starting from products with $s_i = \emptyset$ by $D_i = d_i + \sum_{k\in s_i} D_k$ (s_i is the set of successors of product i).

We consider the following convex relaxation of the problem:

$$(P_R) \quad Z_R = \min \sum_{j\in N}\left(\frac{K_i}{T_i} + H_i T_i\right)$$
$$T_i \geq T_j \text{ if } (i,j) \in A,$$
$$T_i \geq T_L \text{ for each } i.$$

Notice that the constraints $T_i \geq T_j$ model the condition that policies are nested.

As the objective function is convex, the relaxation (P_R) can be solved in polynomial time using interior point algorithms or the algorithm by Hochbaum and Shanktikumar [9]. For systems with special structure the runnning time can be improved substantially. For instance, if G is a tree, Jackson and Roundy [11] show that the relaxed problem can be solved in $O(n \log n)$ time, where $n = |N|$. When G corresponds to a star graph, Queyranne [15], and also Lu Lu and Posner [12] showed that the relaxed problem can be solved in $O(n)$ time, using a linear time median finding algorithm.

Regarding approximation algorithms, Roundy ([18, 19]), and Maxwell and Muckstadt [13] showed how to round an optimal solution of the relaxed problem (P_R) to a feasible solution for (P_{NS}). The policies constructed are called **power-of-two policies**, where each T_i is of the form $2^{p_i}T_L$, where p_i is integer. Let Z_H be the value of the heuristic used. They obtained the following bounds:

1. If T_L is not fixed, but subject to optimization, then

$$\frac{Z_H}{Z_R} \leq \frac{1}{\sqrt{2}\log 2} \approx 1.02.$$

2. If T_L is fixed, then

$$\frac{Z_H}{Z_R} \leq \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) \approx 1.06.$$

The technique used is deterministic rounding and convex duality. The technique utilizes the properties of the optimal relaxed solution. In both cases the bounds are tight. These results are often referred in the literature as 98% and 94% effective lot-sizing policies respectively.

These results have been extensively studied and extended to other versions of lot-sizing problems: finite production rates (Atkins, Queyranne and Sun [1]), individual capacity bounds of the form $2^{l_i}T_L \leq T_i \leq 2^{u_i}T_L$ and more general cost structures (Zheng [21]), and backlog (Atkins et al. [1]). All these extensions use determinisitc rounding to generate power-of-two policies with the same 94% and 98% bounds.

In this paper, we propose a new approach to these lot-sizing problems that uses randomized rounding. This design technique has been used extensively by the discrete optimization community. It was first introduced by Raghavan and Thompson [16], and was used subsequently for a variety of other combinatorial problems. See for instance Goemans and Williamson [7, 8], Bertsimas and Vohra [3], Bertsimas et al. [2]. Our contributions in this paper are as follows:

1. We propose new 94% and 98% randomized rounding algorithms for Problem (P_{NS}) under both the fixed and the variable base period models. Our proof is simple and unlike the original deterministic rounding does not depend on the structure of the optimal solution. Roundy's 98% algorithm can be obtained by derandomizing our algorithm. However, derandomizing the 94% algorithm leads to a different deterministic algorithm. The randomized rounding method is interesting in its own right as it introduces dependencies in the rounding process and generates random variables with distributions with nonlinear density functions.

2. We study a generalization of the fixed based model by allowing the base period T_L to vary over a finite set of choices $\{2^{k/p}T_L : k \text{ integer}\}$, with p, T_L fixed. We propose a randomized rounding algorithm that produces a power-of two policy with bound $\frac{2^{\frac{1}{p}}+1}{2\sqrt{2}p\left(2^{\frac{1}{p}}-1\right)}$, where p denote the number of points T_L is allowed to vary. For $p = 1$ and $p = \infty$, the bound reduces to 1.06 and 1.02 respectively. For the one warehouse, multi-retailer problem (OWMR), Lu Lu and Posner [12] have also obtained a similar bound for a class of integer-ratio policies.

3. For a general production distribution network under nested policies, we propose new convex relaxations and randomized rounding algorithms that use $T_i = 2^{p_i}T_L$ or $3 \cdot 2^{p_i}T_L$. This improves the bound for the fixed base period case from 1.06 to 1.043 and for the special case of Problem (OWMR) to 1.031.

4. Our techniques generalize to several other extensions considered in the literature (capacitated versions, submodular cost functions and multiple resource constrained problems)

2 Randomized rounding and lot-sizing problems

In this section, we introduce the key randomized rounding ideas used in this paper.

2.1 A new 94% approximation algorithm

In this section we consider the case of fixed base period T_L. We consider the following rounding scheme:

> **Algorithm A:**
> Let $T = (T_1, \ldots, T_n)$ be a feasible solution to relaxation (P_R), and $T_i = 2^{p_i} z_i T_L$, where $1 \leq z_i \leq 2$. Generate a point Y in the interval $[1, 2]$, with probability distribution $F(y) = \frac{y^2-1}{1+y^2/2}$. If $z_i < Y$, then $T_i^o = 2^{p_i} T_L$, else $T_i^o = 2^{p_i+1} T_L$.

The above rounding scheme always generates a feasible solution $(T_1^o, T_2^o, \ldots, T_n^o)$ to problem (P_{NS}). We only need to check that the precedence constraints $T_i \leq T_j$ are preserved. If $p_j > p_i$, then $T_j^o \geq T_i^o$. If $p_j = p_i$, then since $T_j \geq T_i$, we must have $z_j \geq z_i$. Hence $z_i \geq y$ only if $z_j \geq y$.

Theorem 1. *Given any feasible solution $(T_1, \ldots, T_n)$ to Problem (P_R) with cost $G(T)$, Algorithm A returns a power-of-two policy (with fixed base T_L) with an expected cost of not more than 1.06 $G(T)$.*

Proof: It is easy to see that

$$E(T_i^o) = 2^{p_i} T_L (1 - F(z_i)) + 2^{p_i+1} T_L F(z_i) = T_i \frac{1 + F(z_i)}{z_i}$$

$$= T_i \frac{3z_i}{z_i^2 + 2} \leq \frac{\sqrt{2} + 1/\sqrt{2}}{2} T_i \approx 1.06\, T_i.$$

Similarly,

$$E(\frac{1}{T_i^o}) = \frac{1}{2^{p_i} T_L}(1 - F(z_i)) + \frac{1}{2^{p_i+1} T_L} F(z_i) = \frac{1}{T_i}(1 - \frac{F(z_i)}{2}) z_i$$

$$= \frac{1}{T_i} \frac{3z_i}{z_i^2 + 2} \leq \frac{\sqrt{2} + 1/\sqrt{2}}{2} \frac{1}{T_i}.$$

The bound follows since the maximum value of the function $\frac{3z_i}{z_i^2+2}$ is at most $3\sqrt{2}/4$. □

Note that the distribution function $F(y)$ is chosen so that $(1+F(y))/y = y(1-F(y)/2) = 3y/(y^2+2)$. The maximum is attained at the point $y=\sqrt{2}$ with a value of $\frac{3\sqrt{2}}{4} \approx 1.06$. Furthermore, using the optimal solution to (P_R) as input to the rounding process, we obtain a 94% approximation algorithm to the original lot-sizing problem.
De-randomization. The above randomized algorithm can be made deterministic: Sort the z_i's in non-decreasing order, say $z_1 \leq z_2 \leq \ldots \leq z_n$. For all y in $[z_i, z_{i+1})$, the randomized algorithm returns the same solution. Hence, there are at most $n+1$ distinct solutions obtained. Thus the best solution can be obtained in $O(n \log n)$ time, which is the time needed for the sorting operation.

2.2 The 98% approximation algorithm revisited

The same insensitivity result can also be improved to a 98% guarantee, if one allows the base period T_L to vary, i.e., T_L is a variable in (P_R). In fact, Roundy's 98% algorithm [18, 19] already has this feature. We recast Roundy's algorithm into the following randomized rounding algorithm:

> **Algorithm B**:
> Let $T = (T_1, \ldots, T_n, T_L)$ be a feasible solution to (P_R), with $T_L > 0$. Let $T_i = 2^{p_i} T_L z_i$, where $1 \leq z_i \leq 2$. Generate a point Y in the interval $[1,2]$, with probability distribution $F(y) = \frac{\log y}{\log 2}$. If $Y > z_i$, then $T_i^o = 2^{p_i} \frac{Y}{\sqrt{2}}$ else $T_i^o = 2^{p_i+1} \frac{Y}{\sqrt{2}}$. Let $T_L^o = \frac{Y}{\sqrt{2} T_L}$.

The rounded solution T_i^o is chosen to ensure that it lies in the interval $[\frac{T_i}{\sqrt{2}}, \sqrt{2} T_i]$. Furthermore, it is clear that $(T_1^o, T_2^o, \ldots, T_n^o, T_L^o)$ is a feasible solution to (P_{NS}).

Theorem 2. *Given any feasible solution $(T_1, \ldots, T_n, T_L)$ to Problem (P_R) with cost $G(T)$, Algorithm B returns a power-of-two policy $(T_1^o, T_2^o, \ldots, T_n^o, T_L^o)$ with expected cost at most $\frac{G(T)}{\sqrt{2}\log(2)} \approx 1.02\, G(T)$.*

Proof: Without loss of generality, we may assume $T_L = 1$. Then

$$E(T_i^o) = \frac{\int_1^{z_i} 2^{p_i+1} dy + \int_{z_i}^2 2^{p_i} dy}{\sqrt{2}\log 2}$$

$$= \frac{2^{p_i}[2(z_i-1)+(2-z_i)]}{\sqrt{2}\log 2} = \frac{T_i}{\log 2\sqrt{2}}.$$

Similarly,

$$E(1/T_i^o) = \frac{\sqrt{2}\int_1^{z_i} 2^{-p_i-1}(1/y^2)dy + \sqrt{2}\int_{z_i}^2 2^{-p_i}(1/y^2)dy}{\log 2}$$

$$= \frac{\sqrt{2}2^{-p_i}(1/2 - \frac{1}{2z_i} - 1/2 + \frac{1}{z_i})}{\log 2} = \frac{1}{T_i \log 2\sqrt{2}},$$

and the theorem follows. □

De-randomization. Suppose $z_1 \leq z_2 \leq \ldots \leq z_n$. For y in $[z_i, z_{i+1})$, suppose the algorithm returns a policy with cost $A/y + By$, then for all other y' in the same interval, the algorithm returns a policy with cost $A/y' + By'$. By choosing a y' in the interval that maximizes this term, and doing the same for each interval partitioned by the z_i's, we obtained an $O(n \log n)$ deterministic algorithm, which is exactly Roundy's rounding procedure.

The argument used above can easily be adapted to analyze more general costs in the objective function. For instance, we have the following:

Theorem 3. *Under Algorithm B,*

$$E\{(T_i^o)^2\}/T_i^2 = T_i^2 E\{(\frac{1}{T_i^o})^2\} = \frac{3}{4\log(2)} \approx 1.082 \ ;$$

$$\frac{1}{\sqrt{2}\log(2)} \leq E(\frac{1}{T_i^o T_j^o}) T_i T_j \leq \frac{3}{4\log(2)} \ ;$$

$$\frac{1}{\sqrt{2}\log(2)} \leq E(T_i^o T_j^o) \frac{1}{T_i T_j} \leq \frac{3}{4\log(2)} \ ;$$

$$E(\frac{T_i^o}{T_j^o}) \leq 1.06 \frac{T_i}{T_j}.$$

The above inequalities imply new bounds (91.8%) if there are $T_i^2, \frac{1}{T_i^2}, T_i T_j$ or $\frac{T_i}{T_j}$ terms in the objective function.

2.3 Unification of the 94% and 98% bounds

The 94% and 98% performance bounds assume that the base period is fixed and optimally selected respectively. The 94% bound is attained by a power-of-two policy, where every order interval is a fixed multiple of a preselected base period. The 98% approximation algorithm, however, cannot ensure that the base planning period belongs to a preselected set. In this section, we propose a technique to bridge the gap between the performance of these two algorithms, by giving progressively more flexibility to the choice of base periods. We assume that the allowed base periods are in the set $\mathcal{S} = \{2^{\frac{j}{p}} : j \text{ integer}\}$.

Consider the following randomized rounding algorithm.

Algorithm C:
Let $T_i = 2^{p_i} z_i$, where $1 \leq z_i < 2$. Let Y be a random number generated in the interval $[2^{-\frac{1}{2p}}, 2^{\frac{1}{2p}})$, with distribution function $F(y) = \frac{2^{1/p}y^2 - 1}{(2^{1/p}-1)(1+y^2)}$. Construct a power-of-two policy as follows:

Select the base period $T_L = 2^{j/p}$ with probability $\frac{1}{p}$.

$$T_i^o = \begin{cases} 2^{p_i+1} 2^{\frac{i}{p}} \; if \; z_i > \sqrt{2}\, 2^{\frac{i}{p}} Y \\ 2^{p_i-1} 2^{\frac{i}{p}} \; if \; z_i < \frac{2^{\frac{i}{p}} Y}{\sqrt{2}} \\ 2^{p_i} 2^{\frac{i}{p}} \qquad \text{otherwise.} \end{cases}$$

Theorem 4. *Given any feasible lot-sizing policy $(T_1, \ldots, T_n)$ in (P_R) with cost $G(T)$, Algorithm C returns a power-of-two policy T^o with expected cost at most*

$$\left(\frac{2^{\frac{1}{p}} + 1}{2\sqrt{2}p(2^{\frac{1}{p}} - 1)} \right) G(T).$$

Note that for $p = 1$ and $p = \infty$, we obtain the 94% and 98% bounds respectively. For $p = 2$, the bound already improves to 97%. This observation implies that for the fixed base period model, the 94% bound might be improved considerably by considering only two distinct base periods, both integral multiples of T_L. In the next section, we use this observation to derive an improved approximation algorithm.

3 An improved approximation algorithm for the fixed base period model

In this section, we propose an improved approximation algorithm for the general problem (P_{NS}) under the fixed base period model. The improvement over the 94% bound comes from having a tighter representation of the objective function over the discrete points $\{T_L, 2T_L, 3T_L\}$ in the interval $[T_L, 3T_L]$. We consider an improved relaxation of the original problem:

$$(P_R') \quad \min \sum_{i=1}^{n} f_i'(T_i)$$

$$\text{subject to } T_i \geq T_j \text{ if } (i,j) \in A, \quad T_i \geq 0,$$

where $f'(\cdot)$, which is depicted in Figure 1, represents the piecewise linearization of $f_i(T_i) = K_i/T_i + H_i T_i$ over the points $\{T_L, 2T_L, 3T_L\}$, i.e.,

$$f_i'(T_i) = \begin{cases} (T_i - T_L) f_i(2T_L) + (2T_L - T_i) f_i(T_L) & \text{if } T_L \leq T_i \leq 2T_L, \\ (T_i - 2T_L) f_i(3T_L) + (3T_L - T_i) f_i(2T_L) & \text{if } 2T_L \leq T_i \leq 3T_L. \end{cases}$$

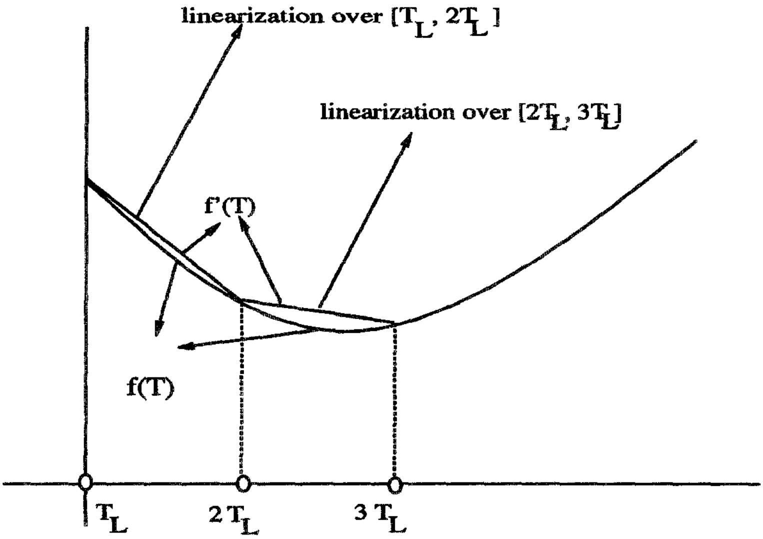

Fig. 1. Piecewise linearization of the objective function over the points $T_L, 2T_L, 3T_L$.

We introduce the following notation. Let p, q be nonnegative numbers, such that $p + q = 1$. Let

$$a(p) = 2\sqrt{\frac{p + 3q/2}{p + 2q/3}},\ b(p) = 2\sqrt{\frac{2p + 3q/2}{p/2 + 2q/3}},$$

and

$$F(p, z) = \frac{\frac{1}{4}(p + \frac{2}{3}q)z^2 - \frac{3}{2}q - p}{p(1 + \frac{1}{8}z^2)},$$

$$F'(p, z) = \frac{\frac{1}{9}(q + \frac{3}{4}p)z^2 - \frac{4}{3}p - q}{q(1 + \frac{1}{18}z^2)}.$$

Note that $F(p, a(p)) = F'(p, b(p)) = 0$, and $F(p, b(p)) = F'(p, 2a(p)) = 1$. Furthermore, F and F' are nondecreasing in z and are valid distribution functions. Suppose further that T^* is an optimal solution to (P_R'). Note that for $T_i^* \leq 3T_L$, we may assume that $T_i^* \in \{T_L, 2T_L, 3T_L\}$. This follows from the following lemma:

Lemma 5. *There exists an optimal solution T^* with the property that $T_i^* \in \{T_L, 2T_L, 3T_L\}$ if $T_i^* \leq 3T_L$.*

Consider the following rounding algorithm:

Algorithm D: Let $p = 0.7, q = 0.3$. Let $a = a(0.7)$ and $b = b(0.3)$. Note that $a < b < 2a$. Select Policy 1 below with probability p, and Policy 2 with probability q.
Policy 1: Let $T_i^* = 2^{p_i} T_L z_i$, where z_i is in the interval $[1, 2)$. Let Y be a random number generated in the interval $[a(p), b(p)]$ with distribution function $F(p, y)$. Let

$$T_i^1 = \begin{cases} 2^{p_i} T_L & \text{if } 2z_i < Y, \\ 2^{p_i+1} T_L & \text{if } 2z_i \geq Y. \end{cases}$$

Policy 2: Let $T_i^* = 3 \cdot 2^{p_i} T_L z_i'$, where z_i' is in the interval $[1, 2)$. Let Y' be a random number generated in the interval $[b(p), 2a(p)]$ with distribution function $F'(p, y)$. For $T_i \geq 3T_L$, let

$$T_i^2 = \begin{cases} 3 \cdot 2^{p_i} T_L & \text{if } 3z_i' < Y', \\ 3 \cdot 2^{p_i+1} T_L & \text{if } 3z_i' \geq Y'. \end{cases}$$

For all items i with $T_i = 2T_L$, we round them (simultaneously) to $3T_L$ with probability $\frac{9}{14}$ and to T_L with probability $\frac{5}{14}$. Note that in this way, for $T_i = 2T_L$,

$$\frac{E(T_i^2)}{2T_L} = \frac{8}{7} = 2T_L E(\frac{1}{T_i^2}).$$

Finally, if $T_i^* = T_L$, $T_i^2 = T_L$.

Let T denote the vector of ordering intervals under the selected policy.

Theorem 6. *The expected cost of the policy T produced by algorithm D is at most 1.043 times the value of the continuous relaxation (P_R').*

Proof: Without loss of generality, we assume $T_L = 1$. If $T_i^* = 2$, then

$$\frac{E(T_i)}{2T_L} = 2T_L E(\frac{1}{T_i}) = p + \frac{8}{7} q = 1.0428...$$

Thus we only need to consider the case when T_i^* greater than 3. Suppose T_i^* lies in (1) $[2^{k_i} a, 2^{k_i} b]$ or (2) $(2^{k_i} b, 2^{k_i+1} a]$. In case (1), Policy 2 always rounds T_i to $3 \cdot 2^{k_i}$, whereas in case (2), Policy 1 always rounds T_i to 2^{k_i+2}.
Case (1) : T_i^* lies in $[2^{p_i} a, 2^{p_i} b]$, i.e., $T_i^* = 2^{p_i} w_i$, where $w_i \in [a, b]$. Then

$$E(T_i) = pE(T_i^1) + qE(T_i^2) = T_i^* (\frac{3q}{w_i} + p \frac{2(1 + F(p, w_i))}{w_i}),$$

and

$$E(\frac{1}{T_i}) = pE(\frac{1}{T_i^1}) + qE(\frac{1}{T_i^2}) = \frac{1}{T_i^*}(q\frac{w_i}{3} + p(1 - \frac{F(p,w_i)}{2})w_i/2).$$

We have chosen $F(p,\cdot)$ such that

$$q\frac{3}{w_i} + p\frac{2(1+F(p,w_i))}{w_i} = q\frac{w_i}{3} + p(1 - \frac{F(p,w_i)}{2})w_i/2.$$

With this choice of $F(p,\cdot)$, and $p = 0.7, q = 0.3$, we can optimize the bound over the range of w_i to obtain

$$\frac{E(T_i)}{T_i^*} = T_i^* E(\frac{1}{T_i}) \leq 1.043.$$

Case (2) : T_i^* lies in $(2^{p_i}b, 2^{p_i+1}a]$, i.e. $T_i^* = 2^{p_i}w_i$ where $w_i \in (b, 2a]$. Then

$$E(pT_i^1 + qT_i^2) = T_i^*(p\frac{4}{3w_i} + q\frac{3(1+F'(p,w_i))}{w_i}),$$

and

$$E(p\frac{1}{T_i^1} + q\frac{1}{T_i^2}) = \frac{1}{T_i^*}(p\frac{3w_i}{4} + q(1 - \frac{F'(p,w_i)}{2})w_i/3).$$

We have chosen $F'(p,\cdot)$ such that

$$p\frac{4}{3w_i} + q\frac{3(1+F'(p,w_i))}{w_i} = p\frac{3w_i}{4} + q(1 - \frac{F'(p,w_i)}{2})w_i/3.$$

With this choice of $F'(p,\cdot)$, again we have

$$\frac{E(pT_i^1 + qT_i^2)}{T_i^*} = T_i^* E(p\frac{1}{T_i^1} + q\frac{1}{T_i^2}) \leq 1.043.$$

Hence the result follows. □

We next show that if $T_i^* \geq \sqrt{6}T_L$ for all i we can improve the approximation guarantee. This result will be useful in the next section. We consider the following modified rounding algorithm:

> **Algorithm E**: Let $p = 0.5, q = 0.5$. Select Policy 1 with probability p and Policy 2 otherwise:
> **Policy 1**: The same as in Algorithm D.
> **Policy 2**: For $T_i^* \geq 3T_L$, the same as in Algorithm D. For T_i^* in $[\sqrt{6}T_L, 3T_L]$, we round T_i to $3T_L$.

The following result follows from a similar analysis to Theorem 6.

Theorem 7. *If $T_i^* \geq \sqrt{6}T_L$ for all i, then the expected cost of the policy T obtained from Algorithm E is at most 1.031 times the optimal value of the continuous relaxation (P_R).*

4 An improved approximation algorithm for the (OWMR) problem

In this section we improve the guarantee of 1.043 to 1.031 for the problem of a single warehouse supplying and distributing items to a group of n retailers. For distribution systems Roundy [18] has showed that the optimal nested policy can be arbitrarily bad compared to the optimal stationary policy. Under the assumption that the retailers place their order only when their inventory level is zero, he showed that there is an optimal stationary policy which satisfies the integer ratio property, i.e., the ratio of the ordering interval T_i for retailer i and the ordering interval T_0 of the warehouse is either an integer or 1 over an integer. He has also constructed similar 94% and 98% approximation algorithms for problem $(OWMR)$, with fixed and variable base period respectively.

The problem can be modelled as follows (see [18]):

$$(P_{OWMR}) \quad \min\ C(T) = \sum_{i=0}^{n}(K_i/T_i) + \sum_{i=1}^{n}(g_i \max(T_0, T_i) + H_i T_i)$$

$$\text{subject to } \frac{T_i}{T_0} \in \{k_i, \frac{1}{k_i} : k_i \text{ integer}\},$$

$$\frac{T_i}{T_L} \text{ integer for all } i = 0, \ldots, n,$$

where $g_i = \frac{1}{2}h_0 d_i$, and $H_i = \frac{1}{2}(h_i - h_0)d_i$. We consider the following relaxation:

$$(P_{ROWMR}) \quad \min\ \sum_{i=0}^{n}(K_i/T_i) + \sum_{i=1}^{n}(g_i \max(T_0, T_i) + H_i T_i)$$

$$\text{subject to } T_i \geq T_L \quad \text{for all } i = 0, \ldots, n.$$

The constraint $T_i \geq T_L$ is a relaxation of the condition that each T_i is an integral multiple of T_L. Let T_i^*, $i = 0, 1, \ldots, n$ be a solution of the relaxation (P_{ROWMR}).

In this section, we improve on the approximation bound for the fixed base period model, by using six stronger relaxations. These relaxations correspond to the requirement that either $T_0^* \geq 6T_L$ or $T_0^* = kT_L$ for k in $\{1, 2, 3, 4, 5\}$.

We first consider the relaxation

$$(P_6) \quad Z_6 = \min\{\sum_{i=1}^{n}(f_i'(T_i) + g_i \max(T_0, T_i)) + K_0/T_0 : T_0 \geq 6T_L, T_i \geq T_L\},$$

where $f_i'(T_i) = f_i(T_i) = K_i/T_i + H_i T_i$ if $T_i \geq 3T_L$, and $f_i'(T_i)$ is the piecewise linearization of $f_i(T_i)$ over the points $\{T_L, 2T_L, 3T_L\}$. Note that this relaxation provides a lower bound to the optimal value of (P_{OWMR}). Z_6 can be computed in $O(n)$ time by using a linear time median finding algorithm, as suggested in

Queyranne [15] or Lu Lu and Posner [12]. Let T_i^* be the optimal solution of relaxation (P_6). Following Lemma 5 we may assume that $T_i^* \in \{T_L, 2T_L, 3T_L\}$ if $T_i^* \leq 3T_L$. We apply Algorithm E that leaves those T_i^* with values T_L or $2T_L$ unchanged.

Lemma 8. *Algorithm E applied to an optimal solution to relaxation (P_6) produces an integer ratio policy with cost within 1.031 of Z_6.*

Proof: Policies 1 and 2 of Algorithm E round those T_i^* with values greater than or equal to $3T_L$ to a power-of-two policy of the type $2^{p_i}T_L$ or $2^{p_i}(3T_L)$. Those T_i^* with values T_L or $2T_L$ are left unchanged. The expected gap between T_i^* and the rounded value T_i again satisfies

$$\frac{E(T_i)}{T_i^*} \leq 1.031, \ \frac{E(1/T_i)}{1/T_i^*} \leq 1.031.$$

Note that in addition, because of the dependence in the rounding process,

$$E[\max(T_i, T_0)] = \max(E[T_i], E[T_0])] \leq 1.031 \max(T_i^*, T_0^*).$$

Note that since $T_0^* \geq 6T_L$, T_0^* is rounded to a multiple of $4T_L$ (under Policy 1) or multiples of $6T_L$ (under Policy 2). Therefore, the policy constructed need not satisfy the condition $T_0 \geq 6T_L$, since Policy 1 might round T_0^* down to $4T_L$. However, the policy obtained is an integer ratio policy. □

We next consider the case that $T_0^* = kT_L$, $k \in \{1, 2, 3, 4, 5\}$. Let $f_i^k(T_i)$ denote a partial piecewise linearization of $f_i(T_i)$ in the interval $[T_L, 3kT_L]$, over the points $T_L, kT_L, 2kT_L, 3kT_L$. Particularly for $k = 4$, in addition to $T_L, 4T_L, 8T_L, 12T_L$ we include the point $2T_L$ in the linearization. For $k \in \{1, 2, 3, 4, 5\}$ we consider the following five relaxations, in which we fix the value of T_0 to be kT_L and consider the linearization $f_i^k(T_i)$ instead of $f_i(T_i)$:

$$(P_k) \ \ Z_k = \min\{C^k(T) = K_0/(kT_L) + \sum_{i=1}^{n}(g_i \max(kT_L, T_i) + f_i^k(T_i)) : T_i \geq T_L\}.$$

Note that each relaxation can be solved in $O(n)$. Moreover,

Lemma 9. *There exists an optimal solution T^k to Z_k with the property that if $T_i^k \leq 3kT_L$, then*

$$T_i^k \in \{T_L, kT_L, 2kT_L, 3kT_L\} \text{ for } k = 1, 2, 3, 5$$

$$T_i^k \in \{T_L, 2T_L, 4T_L, 8T_L, 12T_L\} \text{ for } k = 4.$$

We next show that Algorithm E applied to the optimal solution of relaxation (P_k) produces an integer ratio policy within 1.031 of Z_k.

Lemma 10. *For $k = 1, \ldots, 5$ Algorithm E applied to an optimal solution of relaxation (P_k) that satisfies Lemma 9 produces an integer ratio policy with cost within 1.031 of Z_k.*

Combining Lemmas 8 and 10 we obtain

Theorem 11. *For the one-warehouse-multi-retailer problem with fixed base period, there is an $O(n)$ time 96.9% approximation algorithm.*

5 Extensions

Since our prior analysis did not utilize any structure of the optimal solution, our proof techniques cover several extensions of the basic models almost effortlessly. Our techniques produce randomized rounding algorithms for the following problems considered in the literature:

1. Capacitated lot-sizing problems, in which we add constraints $2^{l_i}T_L \leq T_i \leq 2^{u_i}T_L$ for each i. Since the Algorithms A and B preserve these properties, Theorems 1 and 2 apply also for this capacitated version of the problem, giving rise to 94% and 98% power-of-two policies respectively. The same result was also derived in Federgruen and Zheng [6] by extending Roundy's approach to the capacitated version.

2. Submodular ordering costs introduced in Federgruen et al. [5] and Zheng [21]:

$$(P_{SUB}) \quad Z = \min_T \max_k \sum_{j \in N} \left(\frac{k_i}{T_i} + H_i T_i\right)$$
$$T_i \leq T_j \text{ if } (i,j) \in A,$$
$$T_i \geq T_L \text{ for each } i.$$
$$k \in \mathcal{P},$$

   where
$$\mathcal{P} = \{k : \sum_{j \in S} k_j \leq K(S), \sum_{j \in N} k_j = K(N), k_j \geq 0\},$$

   and $K(S)$ submodular. Algorithms A and B can be used to round the fractional optimal solution in (P_{SUB}) to 94% and 98% optimal power-of-two solutions. Furthermore, if $T_i^* \geq \sqrt{6}T_L$ for all i, then the fixed base period bound can be improved further to 96.9%, using Theorem 7.

3. Resource constrained lot-sizing problems considered in Roundy [20], in which we add to (P_{NS}) constraints of the type

$$\sum_j a_{ij}/T_j \leq A_i, \quad i = 1, \ldots, m.$$

He showed that there is a power-of-two policy for the variable base period case with cost at most 1.44 times the optimal solution. We can generalize this result to the lot-sizing problems with submodular joint cost function. Consider the following algorithm:

> **Algorithm F**:
> Let (k^*, T^*) be an optimal solution to (P_{SUB}) with the resource constraints added. Use $T_j = \sqrt{2}T_j^*$ in Algorithm B to obtain a power-of-two policy T^o.

First note that T_j^o lies in the interval $[T_j/\sqrt{2}, T_j\sqrt{2}]$ and hence $T_j^o \geq T_j^*$. Therefore, T_j^o satisfies the resource constraints.

Theorem 12. *Let T^* be an optimal solution to the resource constrained version of (P_{SUB}). Using Algorithm F on T^*, we obtain a power-of-two policy with cost at most 1.44 times of the optimal.*

Proof: Since scaling by $\sqrt{2}$ does not affect the ordering of T_j^*, the solution k^* is also a maximum solution to $G(T^o)$. Therefore, the result follows directly from the following observation:

$$E(T_j^o) \leq \frac{1}{\sqrt{2}\log(2)}T_j = \frac{1}{\sqrt{2}\log(2)}\sqrt{2}T_j^* \approx 1.44T_j^*$$

and

$$E(\frac{1}{T_j^o}) \leq \frac{1}{\sqrt{2}\log(2)T_j} \leq \frac{1}{T_j^*}.$$

□

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