

PROBABILISTIC ANALYSIS OF THE HELD AND KARP LOWER BOUND FOR THE EUCLIDEAN TRAVELING SALESMAN PROBLEM*

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We analyze probabilistically the classical Held-Karp lower bound derived from the 1-tree relaxation for the Euclidean traveling salesman problem (ETSP). We prove that, if n points are identically and independently distributed according to a distribution with bounded support and absolutely continuous part $f(x) dx$ over the d -cube, the Held-Karp lower bound on these n points is almost surely asymptotic to

$$\beta_{HK}(d)n^{(d-1)/d} \int f(x)^{(d-1)/d} dx,$$

where $\beta_{HK}(d)$ is a constant independent of n . The result suggests a probabilistic explanation of the observation that the lower bound is very close to the length of the optimal tour in practice, since the ETSP is almost surely asymptotic to

$$\beta_{TSP}(d)n^{(d-1)/d} \int f(x)^{(d-1)/d} dx.$$

The techniques we use exploit the polyhedral description of the Held-Karp lower bound and the theory of subadditive Euclidean functionals.

1. Introduction. During the last two decades combinatorial optimization has been a fast growing area in the field of mathematical programming. Some important contributions were Lagrangian relaxation, polyhedral theory and probabilistic analysis.

The landmarks in the development of Lagrangian relaxation (see Geoffrion [6] or Fisher [5]) for combinatorial optimization problems are the two papers for the traveling salesman problem (TSP) by Held and Karp [10], [11]. In the first paper, Held and Karp [10] propose a Lagrangian relaxation based on the notion of 1-tree for the TSP. Using a complete characterization of the 1-tree polytope, which follows from a result of Edmonds [4] for matroids, they show that this Lagrangian relaxation gives the same bound as the linear programming relaxation of a classical formulation of the TSP. In the second paper, Held and Karp [11] introduce a method, which is now known under the name of subgradient optimization (Held, Wolfe and Crowder [12]), to solve the Lagrangian dual. The 1-tree relaxation has been extensively and successfully used to devise branch and bound procedures to solve the TSP (see Held and Karp [11], Helbig Hansen and Krarup [9], Smith and Thompson [22], Volgenant and Jonker [25] or, for a survey, Balas and Toth [1]). These computational studies have shown that, on the average, the Held-Karp lower bound is extremely close to the length of the optimal tour. According to most of the above authors (see also Christofides [3] and Johnson [13]) the relative gap is often less or much less than 1%.

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On a theoretical ground, a result due to Wolsey [26] and later rediscovered by Shmoys and Williamson [21] states that the Held-Karp lower bound is never less than $2/3$ of the length of the optimal tour when the triangle inequality is satisfied. However, this worst-case analysis does not capture the efficiency of the bound in practice. The probabilistic analysis developed in this paper is aimed at shedding new light on the behavior of the Held-Karp lower bound.

The area of probabilistic analysis has its origin in the pioneering paper by Beardwood, Halton and Hammersley [2]. The authors characterize very sharply the asymptotic behavior of the TSP if the points are uniformly and independently distributed in the Euclidean plane or, more generally, in R^d . The potential importance of this early work is demonstrated in Karp [14]. Steele [23] analyzes probabilistically a general class of combinatorial optimization problems by developing the notion of subadditive Euclidean functionals. In Karp and Steele [15] the original proof of Beardwood et al. [2] is simplified using the Efron-Stein inequality. Steele [24] presents an even simpler proof using martingale inequalities. Martingale inequalities were first applied to the probabilistic analysis of combinatorial optimization problems by Rhee and Talagrand [19].

In this paper, we combine the combinatorial interpretation of the Held-Karp lower bound with the probabilistic techniques of Steele [23]. We first prove that, if n points are uniformly and independently distributed over the d -dimensional unit cube, the Held-Karp lower bound on these n points divided by $n^{(d-1)/d}$ is almost surely asymptotic to a constant $\beta_{HK}(d)$. Furthermore, we extend this result to the case in which the n points are identically and independently distributed according to a distribution with bounded support and absolutely continuous part $f(x) dx$ over the d -cube. In this case we prove that the Held-Karp lower bound on these n points is almost surely asymptotic to

$$\beta_{HK}(d)n^{(d-1)/d} \int f(x)^{(d-1)/d} dx.$$

When $d = 2$, we prove the complete convergence of the Held-Karp lower bound divided by \sqrt{n} . We exploit the fact that the bound can be viewed as the cost of the best convex combination of 1-trees such that each vertex has degree 2 on the average. Our analysis is also based on a linear programming formulation for the bound the validity of which can only be proved when the costs satisfy the triangle inequality (Goemans and Bertsimas [8]). Relying on computational studies for the TSP and the matching problem in the Euclidean plane, we estimate that the asymptotic gap $(\beta_{TSP} - \beta_{HK})/\beta_{TSP}$ is less than 3%. To our best knowledge this is the first time that a linear relaxation of a combinatorial optimization problem is analyzed probabilistically using subadditivity techniques.

The rest of the paper is structured as follows. §2 reviews briefly the main results of the Held and Karp [10] paper and also offers a new formulation. In §3 we first prove that the Held-Karp lower bound is monotone and subadditive and then prove the main theorem. In §4 we establish the almost sure convergence of the Held-Karp lower bound in the case of nonuniformly distributed points. In §5 we use a martingale inequality to derive some sharp bounds for the Held-Karp lower bound and we establish its complete convergence when $d = 2$. §6 contains some closing remarks.

2. The Held-Karp lower bound. In this section, we summarize the main results of Held and Karp [10] and also include a new formulation, which is valid when the costs satisfy the triangle inequality. Given a complete undirected graph with vertex set V and costs c_{ij} defined for $i, j \in V$ ($i \neq j$), Held and Karp [10] present a lower bound on the length of the optimal tour to the symmetric ($c_{ij} = c_{ji}$) traveling salesman problem. This bound can be described in several equivalent ways.

First, it can be expressed as the optimal objective function value HK of the linear relaxation of the following standard formulation of the TSP:

$$(1) \quad \text{Min} \sum_{i \in V} \sum_{j \in V, j > i} c_{ij} y_{ij}$$

subject to

$$(2) \quad \sum_{j \in V, j > i} y_{ij} + \sum_{j \in V, j < i} y_{ji} = 2 \quad \forall i \in V,$$

$$(3) \quad \sum_{i \in S} \sum_{j \in S, j > i} y_{ij} \leq |S| - 1 \quad \forall \emptyset \neq S \subset V,$$

$$(4) \quad 0 \leq y_{ij} \leq 1 \quad \forall i, j \in V, j > i,$$

$$(5) \quad y_{ij} \text{ integer} \quad \forall i, j \in V, j > i.$$

In this program, y_{ij} indicates whether cities i and j are adjacent in the optimal tour; c_{ij} represents the cost of traveling from city i to city j or, by symmetry, from city j to city i . The subtour elimination constraints (3) can equivalently be expressed by the constraints

$$(6) \quad \sum_{i \in S} \sum_{j \notin S, i < j} y_{ij} + \sum_{i \in S} \sum_{j \notin S, j < i} y_{ji} \geq 2 \quad \forall \emptyset \neq S \subset V.$$

In that case, the constraints $y_{ij} \leq 1$ are redundant since they can be obtained by combining (2) and (6). Although the above formulation has an exponential number of constraints, one can compute the Held-Karp lower bound in polynomial time either using the Ellipsoid algorithm, since the separation problem corresponding to the polytope (2)–(4) can be reduced to a maximum flow problem, or using Karmarkar's algorithm, since a polynomial size formulation for the LP (1)–(4) can be obtained through the max-flow min-cut theorem.

In [8] we propose another formulation for the Held-Karp lower bound. We show that, under the triangle inequality, constraints (2) can be relaxed, i.e. the Held-Karp lower bound can be written as

$$(7) \quad \text{Min} \sum_{i \in V} \sum_{j \in V, j > i} c_{ij} y_{ij}$$

subject to

$$(8) \quad \sum_{i \in S} \sum_{j \notin S, i < j} y_{ij} + \sum_{i \in S} \sum_{j \notin S, j < i} y_{ji} \geq 2 \quad \forall \emptyset \neq S \subset V,$$

$$(9) \quad y_{ij} \geq 0 \quad \forall i, j \in V, j > i.$$

The proof of this result, given in [8], is based on a modification of a result due to Lovasz [16] on connectivity properties of Eulerian graphs. In this formulation, y_{ij} can be interpreted as a capacity on edge (i, j) and the Held-Karp lower bound can be viewed as the minimum cost of a fractional network in which 2 units of flow can be sent from any vertex to any other vertex. We use the formulation (7)–(9) in §4 to perform the probabilistic analysis in the case in which the points are nonuniformly distributed.

We now give two alternative definitions of a 1-tree which constitute the core of the other formulations.

DEFINITION 1. $T = (V, E)$ is a 1-tree (rooted at vertex 1) if T consists of a spanning tree on $V \setminus \{1\}$, together with two edges incident to vertex 1.

From now on we shall always assume, unless otherwise stated, that the root node is identical for any 1-tree, say vertex 1.

DEFINITION 2. $T = (V, E)$ is a 1-tree if

- (1) T is connected.
- (2) $|V| = |E|$.
- (3) T has a cycle containing vertex 1.
- (4) The degree in T of vertex 1 is 2.

Held and Karp [10] highlight the relation between the linear program (1)–(4) and the class of 1-trees. More precisely, they show that the feasible solutions to (2)–(4) can be equivalently characterized as convex combinations of 1-trees such that each vertex has degree 2 on the average. Hence, we may rewrite (1)–(4) in the following way:

$$(10) \quad HK = \text{Min} \sum_{r=1}^k \lambda_r c(T_r)$$

subject to

$$(11) \quad \sum_{r=1}^k \lambda_r = 1,$$

$$(12) \quad \sum_{r=1}^k \lambda_r d_j(T_r) = 2 \quad \forall j \in V \setminus \{1\},$$

$$(13) \quad \lambda_r \geq 0, \quad r = 1, \dots, k,$$

where

- $\{T_r\}_{r=1, \dots, k}$ constitutes the class of 1-trees defined on the vertex set V ,
- $c(G) = \sum_{e=(i,j) \in E} c_{ij}$ is the total cost of the subgraph $G = (V, E)$, and
- $d_j(T)$ denotes the degree in T of vertex j .

Finally, the most common approach to find the Held-Karp lower bound is to take the Lagrangian dual of (10)–(13) with respect to (12). We then obtain

$$(14) \quad HK = \max_{\mu} L(\mu)$$

subject to

$$(15) \quad L(\mu) = \min_{r=1, \dots, k} c_{\mu}(T_r) - 2 \sum_{j \in V} \mu_j$$

where $c_{\mu}(T_r)$ is the cost of the 1-tree T_r with respect to the costs $c_{ij} + \mu_i + \mu_j$.

3. The main theorem. Let the n points $X^{(n)} = (X_1, X_2, \dots, X_n)$ be uniformly and independently distributed in the d -cube $[0, 1]^d$. Let $HK(X^{(n)})$ denote the Held-Karp lower bound on $X^{(n)}$ as defined by any of the five equivalent formulations

(1)–(4), (7)–(9) (possibly along with (2)), (10)–(13) and (14)–(15) of §2. We are interested in the behavior, as n tends to infinity, of $HK(X^{(n)})$. Steele [23] proves that the asymptotic behavior of a particular class of Euclidean functionals L defined on finite subsets of R^d to R can be characterized very sharply as follows:

THEOREM 1 (Steele [23]). *Let L be a monotone [$L(A \cup \{x\}) \geq L(A) \forall x \in R^d, \forall A \subset R^d$], Euclidean [$L(ax_1, ax_2, \dots, ax_n) = aL(x_1, x_2, \dots, x_n)$, $L(x_1 + x, x_2 + x, \dots, x_n + x) = L(x_1, x_2, \dots, x_n)$] functional of finite variance [$\text{Var}[L(X^{(n)})] < \infty$] which satisfies the subadditivity hypothesis:*

If $\{Q_i; 1 \leq i \leq m^d\}$ is a partition of the d -cube $[0, 1]^d$ into m^d identical subcubes with edges parallel to the axes and $tQ_i = \{tx; x \in Q_i\}$, then there exists a constant $C > 0$ such that $\forall m \in N \setminus \{0\}, \forall t > 0$, we have that

$$L(\{x_1, \dots, x_n\} \cap [0, t]^d) \leq \sum_{i=1}^{m^d} L(\{x_1, \dots, x_n\} \cap tQ_i) + Ctm^{d-1}.$$

Then there exists a constant $\beta_L(d)$ such that

$$\lim_{n \rightarrow \infty} \frac{L(X^{(n)})}{n^{(d-1)/d}} = \beta_L(d)$$

almost surely.

We emphasize that the critical property in Theorem 1 is the subadditivity hypothesis. It can easily be seen that HK is a Euclidean functional. Moreover, $HK(X^{(n)})$ has finite variance since it has bounded support, namely

$$0 \leq HK(X^{(n)}) \leq TSP(X^{(n)}) \leq c\sqrt{n}$$

for some constant c . Proposition 2 proves that the subadditivity hypothesis holds for the functional HK . The monotonicity of HK is proved in Proposition 3. For these propositions the formulation of the Held-Karp lower bound we use is (10)–(13). A more concise proof can be obtained using the new formulation (7)–(9) instead of (10)–(13). For convenience and clarity we denote by $P(A)$ the linear program (10)–(13) corresponding to the set A of cities.

PROPOSITION 2. *HK is subadditive, i.e. $\exists C > 0$, such that $\forall m \in N \setminus \{0\}, \forall t > 0$*

$$HK(\{x_1, \dots, x_n\} \cap [0, t]^d) \leq \sum_{i=1}^{m^d} HK(\{x_1, \dots, x_n\} \cap tQ_i) + Ctm^{d-1}$$

for any finite subset $\{x_1, x_2, \dots, x_n\}$ of R^d .

PROOF. Using the fact that HK is a Euclidean functional, we may restrict ourselves to the case $t = 1$. Let $V = \{x_1, \dots, x_n\} \cap [0, 1]^d$ and $V_i = \{x_1, \dots, x_n\} \cap Q_i$ for $i = 1, \dots, m^d$. Let $p = m^d$. We arbitrarily choose a root vertex 1_i in every V_i . Let $\{T_{i1}, T_{i2}, \dots, T_{ik_i}\}$ be the class of 1-trees defined on V_i (with respect to the root 1_i). We

consider the optimal solution $\{\lambda_{ir}\}_{r=1, \dots, k_i}$ to $P(V_i)$, i.e. $\{\lambda_{ir}\}_{r=1, \dots, k_i}$ satisfies

$$(16) \quad \sum_{r=1}^{k_i} \lambda_{ir} = 1,$$

$$(17) \quad \sum_{r=1}^{k_i} \lambda_{ir} d_j(T_{ir}) = 2 \quad \forall j \in V_i \setminus \{1_i\},$$

$$(18) \quad \lambda_{ir} \geq 0, \quad r = 1, \dots, k_i,$$

$$(19) \quad HK(V_i) = \sum_{r=1}^{k_i} \lambda_{ir} c(T_{ir}).$$

From these optimal solutions we shall construct a feasible solution to $P(V)$ whose cost is less than or equal to

$$(20) \quad \sum_{i=1}^p HK(V_i) + Cm^{d-1}$$

where $C = 2\sqrt{d+3}$. For this purpose, we consider every possible combination of selecting one 1-tree in each subcube Q_i . There are $(\prod_{i=1}^p k_i)$ such combinations. Let us focus on one of them, say $\{T_{ir}\}_{i=1, \dots, p}$. Let Λ be the indices (r_1, r_2, \dots, r_p) of the corresponding 1-trees. From these p 1-trees we shall construct a 1-tree T_Λ rooted at 1_1 , spanning V and satisfying the following conditions:

$$(21) \quad d_j(T_\Lambda) = d_j(T_{ir_i}) \quad \text{if } j \in Q_i,$$

$$(22) \quad c(T_\Lambda) \leq \sum_{i=1}^p c(T_{ir_i}) + Cm^{d-1}.$$

We claim that, by assigning a weight of $\lambda_\Lambda = \prod_{i=1}^p \lambda_{ir_i}$ to each 1-tree T_Λ , we get a feasible solution to $P(V)$ whose cost is less than (20). Indeed,

(1) Using (16) recursively, we have

$$(23) \quad \begin{aligned} \sum_{\Lambda} \lambda_{\Lambda} &= \sum_{\Lambda=(r_1, \dots, r_p)} \prod_{i=1}^p \lambda_{ir_i} \\ &= \sum_{r_1=1}^{k_1} \lambda_{1r_1} \sum_{r_2=1}^{k_2} \lambda_{2r_2} \cdots \sum_{r_p=1}^{k_p} \lambda_{pr_p} \\ &= 1. \end{aligned}$$

(2) Consider any vertex $j \in V$. Assume that $j \in Q_i$. We have that

$$\begin{aligned}
 (24) \quad \sum_{\Lambda} \lambda_{\Lambda} d_j(T_{\Lambda}) &= \sum_{\Lambda} \lambda_{\Lambda} d_j(T_{r_i}) \\
 &= \sum_{r_i=1}^{k_i} \lambda_{r_i} d_j(T_{r_i}) \prod_{j \in \{1, \dots, p\} \setminus \{i\}} \sum_{r_j=1}^{k_j} \lambda_{r_j} \\
 &= \sum_{r_i=1}^{k_i} \lambda_{r_i} d_j(T_{r_i}) \\
 &= 2
 \end{aligned}$$

using (21), (16) and (17) respectively.

(3) $\lambda_{\Lambda} \geq 0$ follows from (18).

1, 2 and 3 imply that the solution is feasible in $P(V)$. The cost of this solution is given by

$$\begin{aligned}
 (25) \quad \sum_{\Lambda} \lambda_{\Lambda} c(T_{\Lambda}) &\leq \sum_{\Lambda=(r_1, \dots, r_p)} \lambda_{\Lambda} \left(\sum_{i=1}^p c(T_{r_i}) \right) + \sum_{\Lambda} C m^{d-1} \lambda_{\Lambda} \\
 &= \sum_{i=1}^p \sum_{r_i=1}^{k_i} \lambda_{r_i} c(T_{r_i}) + C m^{d-1} \\
 &= \sum_{i=1}^p HK(V_i) + C m^{d-1}
 \end{aligned}$$

using (22), (16), (23) and (19) respectively. The last point left in this proof is the construction of the 1-tree T_{Λ} satisfying (21) and (22). We proceed in 2 steps:

(1) (Figure 1) In each 1-tree T_{r_i} ($i = 1, \dots, p$) we delete one of the 2 edges incident to the root 1_i , say $(1_i, 2_i)$. Note that typically 2_i depends on r_i .

(2) (Figure 2) Assume that the numbering of the subcubes is such that the subcubes Q_i and Q_{i+1} ($i = 1, \dots, p-1$) are adjacent. Such a numbering exists for every d . A possible numbering for the case $d = 2$ is represented in Figure 3. We now add the edges $(2_i, 1_{i+1})$ ($i = 1, \dots, p-1$) and the edge $(2_p, 1_1)$. If there are points on cell borders, they are simply assigned to specific cells, which can be done when the cube is first partitioned.

The construction is now complete. We first claim that the resulting subgraph $T_{\Lambda} = (V, E_{\Lambda})$ is a 1-tree rooted at vertex 1_1 . This follows from Definition 2. Indeed T_{Λ} is clearly connected, the number of edges of T_{Λ} is

$$|E_{\Lambda}| = \sum_{i=1}^p (|E_{r_i}| - 1) + p = \sum_{i=1}^p |E_{r_i}| = \sum_{i=1}^p |V_i| = |V|,$$

T_{Λ} has a cycle containing vertex 1_1 and the degree in T_{Λ} of vertex 1_1 is 2. Secondly, from the construction, it is evident that we have not changed the degree of any vertex. Therefore (21) holds. Finally, we have added $(p-1)$ edges of cost at most $\sqrt{d+3}/m$

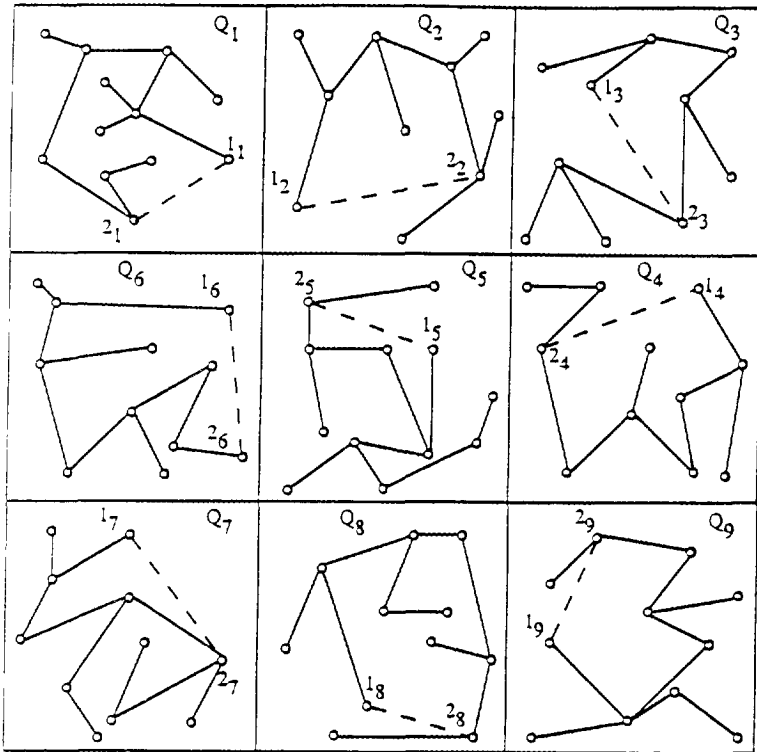


FIGURE 1 Step 1 in the construction of T_n .

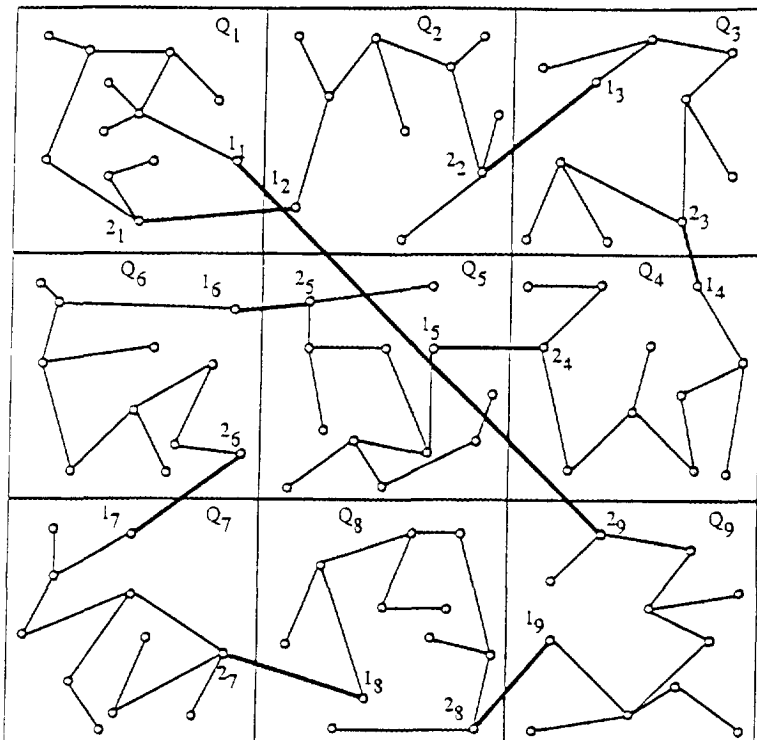


FIGURE 2 Step 2 in the Construction of T_n .

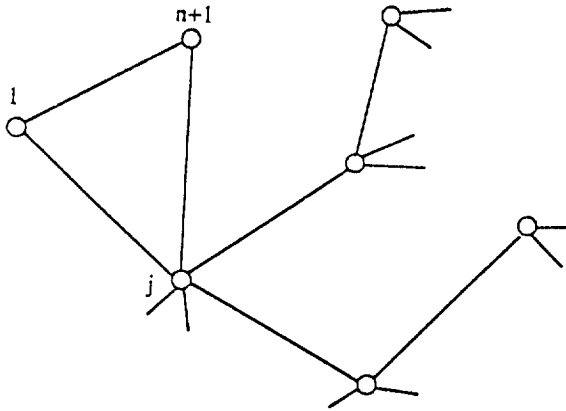


FIGURE 4. 1-Tree in \mathcal{S}_Δ .

$HK(V)$. For this purpose, we first need to show that the optimal solution to $P(V)$ can be decomposed in such a way that \mathcal{S} does not contain some particular 1-trees. Let $\mathcal{S}_\Delta = \{T \in \mathcal{S}, d_{n+1}(T) = 2, \exists j \in V: (1, j), (1, n + 1), (j, n + 1) \in T\}$. A possible candidate for \mathcal{S}_Δ is represented in Figure 4.

Claim 1. Without loss of generality, \mathcal{S}_Δ can be assumed to be empty.

Indeed, let $T \in \mathcal{S}_\Delta$ such that $(1, j), (1, n + 1), (j, n + 1) \in T$ and $d_{n+1}(T) = 2$. As $n + 1 \geq 4$, the degree of vertex j in T is at least 3. Therefore, since the degree of each vertex is 2 on the average, there exists a 1-tree $T' \in \mathcal{S}$ such that $d_j(T') = 1$. Our goal is to construct a feasible solution without changing the value of the objective function by converting T and T' to \bar{T} and \bar{T}' , where $\bar{T}, \bar{T}' \notin \mathcal{S}_\Delta$. Let i_1 and i_2 be the two vertices adjacent to vertex 1 in T' . Without loss of generality, we may assume that $i_1 \neq n + 1$. Moreover, since T' is a 1-tree with $d_j(T') = 1$, we have that $i_1 \neq j$ and $i_2 \neq j$. Otherwise, removing vertex 1 would disconnect the graph on $V \setminus \{1\}$. If we replace $(1, j)$ in T by $(1, i_1)$ and $(1, i_1)$ in T' by $(1, j)$, we get two 1-trees \bar{T} and \bar{T}' which are not in \mathcal{S}_Δ . This basically follows from the fact that $(1, j) \notin \bar{T}$ while $(1, n + 1), (j, n + 1) \in \bar{T}$ and that if $(1, n + 1)$ and $(j, n + 1)$ were both in \bar{T}' and $d_{n+1}(\bar{T}') = 2$ then T' would not be a 1-tree since $T' \setminus \{(1, i_1), (1, n + 1)\}$ would be disconnected. But $\mathcal{S} \setminus \{T, T'\} \cup \{\bar{T}, \bar{T}'\}$ represents the same optimal solution as previously since T and T' have the same weight λ and they have simply “traded” edges. Hence, by applying this procedure repeatedly, we see that we may assume, without loss of generality, that $\mathcal{S}_\Delta = \emptyset$.

Let $\mathcal{S}_i = \{T \in \mathcal{S}: d_{n+1}(T) = i\}$, $i = 1, 2$. We duplicate every 1-tree T in $\mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$ $(d_{n+1}(T) - 2)$ times and we associate to each copy a weight of $\lambda / (d_{n+1}(T) - 2)$ in order to keep the solution unchanged. Call \mathcal{S}_3 the resulting set. Note that the weight associated to the 1-trees in \mathcal{S}_1 or \mathcal{S}_2 is still λ while the weight associated to a 1-tree T in \mathcal{S}_3 is $\lambda / (d_{n+1}(T) - 2)$.

Claim 2. $|\mathcal{S}_1| = |\mathcal{S}_3|$.

Since vertex $n + 1$ has degree 2 on the average, we have

$$(26) \quad \sum_{T \in \mathcal{S}_1} \lambda + \sum_{T \in \mathcal{S}_2} 2\lambda + \sum_{T \in \mathcal{S}_3} d_{n+1}(T) \frac{\lambda}{d_{n+1}(T) - 2} = 2.$$

Now the claim follows by subtracting the equality

$$\sum_{T \in \mathcal{S}_1} \lambda + \sum_{T \in \mathcal{S}_2} \lambda + \sum_{T \in \mathcal{S}_3} \frac{\lambda}{d_{n+1}(T) - 2} = 1$$

twice from (26).

This means that we can regroup \mathcal{S}_1 and \mathcal{S}_3 into a set \mathcal{S}_{13} of pairs (T_1, T_3) of 1-trees of \mathcal{S}_1 and \mathcal{S}_3 ($|\mathcal{S}_{13}| = |\mathcal{S}_1| = |\mathcal{S}_3|$). From \mathcal{S}_2 and \mathcal{S}_{13} we shall construct a feasible solution to $P(V')$ whose total cost is less than or equal to $HK(V)$. More precisely, we associate to each 1-tree $T \in \mathcal{S}_2$ (to each pair $(T_1, T_3) \in \mathcal{S}_{13}$, respectively) a 1-tree T' (a pair (T'_1, T'_3) of 1-trees, respectively) defined on V' such that

$$(27) \quad \lambda c(T') \leq \lambda c(T),$$

$$(28) \quad \left(\lambda c(T'_1) + \frac{\lambda}{d_{n+1}(T_3) - 2} c(T'_3) \leq \lambda c(T_1) + \frac{\lambda}{d_{n+1}(T_3) - 2} c(T_3), \text{ resp.} \right),$$

$$\lambda d_j(T') = \lambda d_j(T) \quad \forall j \in V',$$

$$\left(\lambda d_j(T'_1) + \frac{\lambda}{d_{n+1}(T_3) - 2} d_j(T'_3) \right. \\ \left. = \lambda d_j(T_1) + \frac{\lambda}{d_{n+1}(T_3) - 2} d_j(T_3), \text{ resp.} \right) \quad \forall j \in V'$$

hold. Combining (27) and (28) we clearly see that, by keeping the old weights, we get a feasible solution to $P(V')$ whose cost is less than or equal to the cost of the optimal solution to $P(V)$ which is $HK(V)$.

The construction of T' and (T'_1, T'_3) is as follows:

(1) $T \in \mathcal{S}_2$. Let $(i, n+1)$ and $(j, n+1)$ be the two edges incident to vertex $n+1$ in T . Let

$$T' = T \setminus \{(i, n+1), (j, n+1)\} \cup \{i, j\}.$$

The fact that T' is a 1-tree on V' follows from Definition 2 and the fact that we can assume without loss of generality that $\mathcal{S}_\Delta = \emptyset$ (Claim 1). Clearly (28) is satisfied and the triangle inequality implies that (27) holds.

(2) $(T_1, T_3) \in \mathcal{S}_{13}$. Let i be the unique vertex adjacent to $(n+1)$ in T_1 ($i \neq 1$). Let $\nu = d_{n+1}(T_3) \geq 3$. Let j_1, \dots, j_ν be the vertices adjacent to $n+1$ in T_3 . We may assume without loss of generality that i is in the same connected component as j_1 when we remove the vertices 1 and $n+1$ in T_3 . Moreover, if vertex 1 is adjacent to vertex $n+1$ in T_3 , we let j_2 be vertex 1 if and only if $(1, j_1) \notin T_3$. The transformation is the following (see Figure 5):

$$T'_1 \leftarrow T_1 \setminus \{(i, n+1)\},$$

$$T'_3 \leftarrow T_3 \setminus \{(j_1, n+1), \dots, (j_\nu, n+1)\} \cup \{(j_1, j_2), (j_3, i), \dots, (j_\nu, i)\}.$$

The fact that T'_1 is a 1-tree is obvious. We notice that none of the edges added to T_3 were already present in T_3 . We then check that T'_3 is connected, $|T'_3| = |T_3| - 1 = |V| - 1 = |V'|$, T'_3 has a cycle containing vertex 1 and $d_1(T'_3) = 2$. Hence, by Definition 2, T'_3 is a 1-tree. We have

$$\lambda c(T_1) + \frac{\lambda}{\nu - 2} c(T_3) - \lambda c(T'_1) - \frac{\lambda}{\nu - 2} c(T'_3) \\ = \lambda c_{i, n+1} + \sum_{k=1}^{\nu} \frac{\lambda}{\nu - 2} c_{j_k, n+1} - \frac{\lambda}{\nu - 2} c_{j_1 j_2} - \sum_{k=2}^{\nu} \frac{\lambda}{\nu - 2} c_{j_k i} \\ = \frac{\lambda}{\nu - 2} (c_{j_1, n+1} + c_{j_2, n+1} - c_{j_1 j_2}) + \frac{\lambda}{\nu - 2} \sum_{k=3}^{\nu} (c_{i, n+1} + c_{j_k, n+1} - c_{j_k i})$$

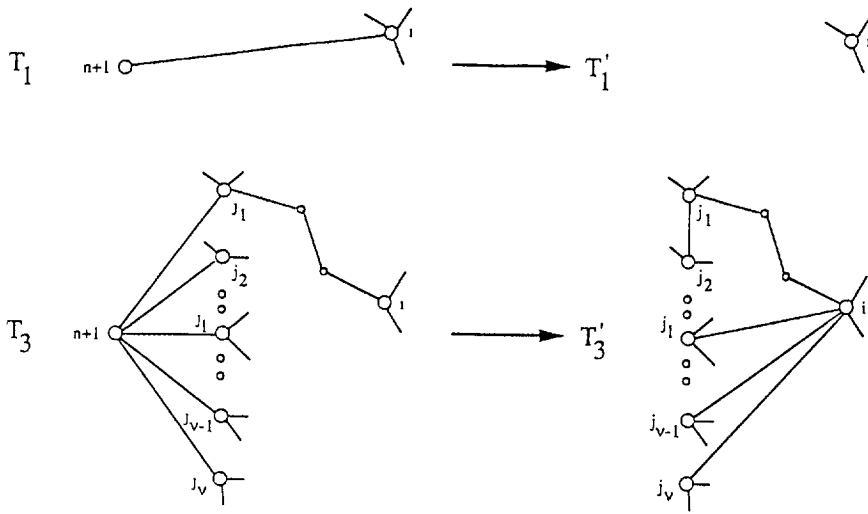


FIGURE 5. Construction of T'_1 and T'_3 .

and therefore (27) holds using the triangle inequality. Moreover, since

$$d_j(T'_1) - d_j(T_1) = \begin{cases} -1 & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_j(T'_3) - d_j(T_3) = \begin{cases} \nu - 2 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

we see that

$$\lambda d_j(T'_1) + \frac{\lambda}{\nu - 2} d_j(T'_3) - \lambda d_j(T_1) - \frac{\lambda}{\nu - 2} d_j(T_3) = 0.$$

Hence, (28) is satisfied. This completes the proof of Proposition 3. ■

Alternative proofs of the monotonicity of the Held-Karp lower bound are given in [8], [21]. We may now deduce the asymptotic behavior of HK as a corollary to Theorem 1 and Propositions 2 and 3.

THEOREM 4. *Let the n points $X^{(n)} = (X_1, \dots, X_n)$ be uniformly and independently distributed in the d -dimensional unit cube. Then there exists a constant $\beta_{HK}(d)$ such that*

$$\lim_{n \rightarrow \infty} \frac{HK(X^{(n)})}{n^{(d-1)/d}} = \beta_{HK}(d)$$

almost surely.

A number of combinatorial optimization problems, like the Euclidean traveling salesman problem, the Euclidean minimum spanning tree problem and the Euclidean minimum weight matching problem, have a similar asymptotic behavior although with a different constant β (see Beardwood, Halton and Hammersley [2] and Papadimitriou [18]). It is therefore interesting to compare $\beta_{HK}(d)$ to the value of β for closely related combinatorial optimization problems. In particular, it is clear that

$\beta_{HK}(d) \leq \beta_{TSP}(d)$. Moreover, since the value of the Held-Karp lower bound on n points is never less than the cost of the minimum spanning tree on a subset of $n - 1$ points, $\beta_{HK}(d) \geq \beta_T(d)$ where $\beta_T(d)$ is the corresponding constant for the Euclidean minimum spanning tree problem. The relationship between $\beta_{HK}(d)$ and $\beta_M(d)$, where $\beta_M(d)$ is the constant for the Euclidean minimum weight matching problem, is a little less obvious. Using a complete characterization of the perfect matching polytope, we may express the cost M of the minimum weight matching as

$$(29) \quad M = \text{Min} \sum_{i \in V} \sum_{j \in V, j > i} c_{ij} x_{ij}$$

subject to

$$(30) \quad \sum_{j \in V, j > i} x_{ij} + \sum_{j \in V, j < i} x_{ji} = 1 \quad \forall i \in V,$$

$$(31) \quad \sum_{i \in S} \sum_{j \in S, j > i} x_{ij} \leq \frac{|S| - 1}{2} \quad \forall S \subset V, |S| \text{ odd},$$

$$(32) \quad 0 \leq x_{ij} \quad \forall i, j \in V, j > i.$$

Substituting x_{ij} by $y_{ij}/2$ we get a relaxation of the linear program (1)–(4). Hence $M \leq HK/2$ (Wolsey [26]) which implies that $\beta_{HK}(d) \geq 2\beta_M(d)$. We thus obtain the following proposition:

PROPOSITION 5. $\max(2\beta_M(d), \beta_T(d)) \leq \beta_{HK}(d) \leq \beta_{TSP}(d)$.

From Proposition 5 we can establish the following analytic bounds on $\beta_{HK}(d)$. From [2] $\beta_{TSP}(d) \leq 12^{1/(2d)}\sqrt{d/6}$ and, for $d = 2$, $\beta_{TSP}(2) \leq 0.9204$. On the other hand, by considering the nearest and second nearest neighbors of each point one can find that

$$\beta_{HK}(d) \geq c_d^{-1/d} \Gamma(1/d + 1) \left(1 + \frac{1}{2d}\right),$$

where $c_d = \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of a ball of unit radius in d dimensions. In particular, $\beta_{HK}(2) \geq 0.625$. As a result, we can establish the following explicit bounds for $\beta_{HK}(d)$:

$$\frac{1}{\sqrt{\pi}} \left[\Gamma\left(\frac{d}{2} + 1\right) \right]^{1/d} \Gamma\left(\frac{1}{d} + 1\right) \left(1 + \frac{1}{2d}\right) \leq \beta_{HK}(d) \leq 12^{1/(2d)} \sqrt{\frac{d}{6}}.$$

As $d \rightarrow \infty$ the bounds become

$$\sqrt{d/2\pi e} \leq \beta_{HK}(d) \leq \sqrt{d/6},$$

and hence we establish that $\beta_{HK}(d) = \Theta(\sqrt{d})$.

When $d = 2$, $\beta_M(d)$, $\beta_T(d)$ and $\beta_{TSP}(d)$ were estimated to be 0.35, 0.68 and 0.72 by Papadimitriou [18], Gilbert [7] and Johnson [13], respectively. Using Proposition 5, we may therefore deduce that the asymptotic gap $(\beta_{TSP} - \beta_{HK})/\beta_{TSP}$ is perhaps less than $(0.72 - 0.70)/0.70 \approx 3\%$. This suggests a probabilistic explanation of the observation that the Held-Karp lower bound is very close to the length of the optimal tour in practice.

4. The nonuniform case. In the previous section we analyzed the asymptotic behavior of the Held-Karp lower bound in the case of uniformly distributed points. In this section we extend our analysis to the case in which the points are not uniformly distributed. Let the n points $X^{(n)} = (X_1, X_2, \dots, X_n)$ be independently and identically distributed according to a distribution with bounded support in the d -cube and absolutely continuous part $f(x) dx$. In order to extend our results we use the following result of Steele [23].

THEOREM 6 (Steele [23]). *Let L be a Euclidean functional satisfying the assumptions of Theorem 1 and also having the following properties:*

(1) L is scale bounded, i.e., there is a constant B_1 such that

$$\frac{L(x_1, \dots, x_n)}{tn^{(d-1)/d}} \leq B_1,$$

for all $n \geq 1$ and $\{x_1, \dots, x_n\} \subset [0, t]^d$.

(2) L is simply subadditive, i.e. there is a constant B_2 such that

$$L(A_1 \cup A_2) \leq L(A_1) + L(A_2) + tB_2,$$

for any finite disjoint subsets A_1, A_2 of $[0, t]^d$.

(3) L is upper linear, i.e. if $\{Q_i; 1 \leq i \leq m^d\}$ is a partition of $[0, 1]^d$ into m^d identical subcubes with edges parallel to the axes, then $\forall m \in \mathbb{N} \setminus \{0\}$, we have that

$$\sum_{i=1}^{m^d} L(\{x_1, \dots, x_n\} \cap tQ_i) \leq L(\{x_1, \dots, x_n\} \cap [0, t]^d) + o(tn^{(d-1)/d}).$$

Then there exists a constant $\beta_L(d)$ such that

$$\lim_{n \rightarrow \infty} \frac{L(X^{(n)})}{n^{(d-1)/d}} = \beta_L(d) \int f(x)^{(d-1)/d} dx$$

almost surely.

Based on the new formulation (7)–(9) we will prove that the Held-Karp lower bound satisfies the assumptions of Theorem 6. Clearly, HK is scale bounded, since

$$HK(x_1, \dots, x_n) \leq TSP(x_1, \dots, x_n) \leq B_1 tn^{(d-1)/d},$$

for any $\{x_1, \dots, x_n\} \subset [0, t]^d$. In the following two propositions we prove the simply subadditivity and the upper linearity of the Held-Karp lower bound.

PROPOSITION 7. HK is simply subadditive, i.e.

$$HK(A_1 \cup A_2) \leq HK(A_1) + HK(A_2) + O(t),$$

for any finite disjoint subsets A_1, A_2 of $[0, t]^d$.

PROOF. Since HK is Euclidean we restrict our attention to the case $t = 1$. Let $R(A)$ denote the linear program (7)–(9) corresponding to the set A of cities. Our goal

is to construct a feasible solution to $R(A_1 \cup A_2)$ whose cost is $HK(A_1) + HK(A_2) + O(1)$. For this purpose, we consider the following solution defined on $A_1 \cup A_2$. We take a solution on A_1 , optimal for $R(A_1)$, a solution on A_2 , optimal for $R(A_2)$, and we set the capacity on some arbitrary edge from A_1 to A_2 to be 2, all other edges from A_1 to A_2 having zero capacity. Since every cutset has at least two units of capacity, the constructed solution is feasible for $R(A_1 \cup A_2)$. Moreover, the cost of this solution is at most $HK(A_1) + HK(A_2) + 2\sqrt{d}$. Therefore,

$$HK(A_1 \cup A_2) \leq HK(A_1) + HK(A_2) + O(t),$$

proving the result. ■

We now prove that the Held-Karp lower bound is upper linear.

PROPOSITION 8. *HK is upper linear, i.e.*

$$\sum_{t=1}^{m^d} HK(\{x_1, \dots, x_n\} \cap tQ_t) \leq HK(\{x_1, \dots, x_n\} \cap [0, t]^d) + o(tm^{(d-1)/d}),$$

for any finite subset $\{x_1, \dots, x_n\}$ of R^d .

PROOF. As before, we prove the proposition only for the case $t = 1$ since HK is Euclidean. Let $V = \{x_1, \dots, x_n\} \cap [0, 1]^d$ and $V_i = \{x_1, \dots, x_n\} \cap Q_i$ for $i = 1, \dots, m^d$. If there are points on the boundary of any subcube, they are simply assigned to specific subcubes. Let F_{ij} ($j = 1, \dots, 2d$) denote the $d - 1$ -dimensional faces of Q_i ($i = 1, \dots, m^d$). We divide each side of F_{ij} into $n^{1/(d-1)}$ identical intervals, therefore defining a partition of face F_{ij} into n identical $d - 1$ -subcubes F_{ijk} ($k = 1, \dots, n$). Let a_{ijk} be the center point of the $d - 1$ -cube F_{ijk} . Note that the distance between any point of F_{ijk} and a_{ijk} is at most $(\sqrt{d-1}/2)n^{-1/(d-1)}m^{-1}$. Let A_{ij} denote the set $\{a_{ijk} : k \in \{1, \dots, n\}\}$, and let A_i denote $\bigcup_{j=1}^{2d} A_{ij}$.

We now consider an optimal solution y^* to the linear program (7), (2), (8) and (9) corresponding to the set V of points. We will construct a feasible solution to $R(V_i \cup A_i)$ (i.e. to the LP (7)–(9) corresponding to $V_i \cup A_i$) whose cost is equal to the contribution HK_i of y^* inside Q_i increased by $O(n^{(d-2)/(d-1)})$. By the monotonicity of the Held-Karp lower bound (Proposition 3), the cost of this solution is an upper bound on $HK(V_i)$. The upper linearity now follows by adding the contribution of each subcube Q_i :

$$\begin{aligned} \sum_{i=1}^{m^d} HK(V_i) &\leq \sum_{i=1}^{m^d} HK(V_i \cup A_i) \leq \sum_{i=1}^{m^d} (HK_i + O(n^{(d-2)/(d-1)})) \\ &= HK(V) + O(n^{(d-2)/(d-1)}) \leq HK(V) + o(n^{(d-1)/d}). \end{aligned}$$

In order to complete the proof, we need to show how to construct a feasible solution to $R(V_i \cup A_i)$ with a “reasonable” increase in cost. For each edge (x_p, x_q) crossing the boundary of Q_i ($x_p \in Q_i, x_q \notin Q_i$), say in δ_{pq} , identify the $d - 1$ -cube F_{ijk} to which δ_{pq} belongs. For notational convenience, let F^{pq} denote this cube, and a^{pq} its center. Set the capacity on the edge (a^{pq}, x_p) to be equal to the original capacity on (x_p, x_q) . The solution so constructed is not yet feasible for $R(V_i \cup A_i)$,

but its cost is still reasonable, since it is equal to

$$\begin{aligned}
 HK_i + \sum_{x_p \in Q_i, x_q \notin Q_i} (|a^{pq} - x_p| - |\delta_{pq} - x_p|) y_{pq}^* \\
 \leq HK_i + \sum_{x_p \in Q_i, x_q \notin Q_i} |a^{pq} - \delta_{pq}| y_{pq}^* \\
 \leq HK_i + \frac{\sqrt{d-1}}{2} n^{-1/(d-1)} m^{-1} \sum_{x_p \in Q_i, x_q \notin Q_i} y_{pq}^* \\
 \leq HK_i + \frac{\sqrt{d-1}}{2} n^{-1/(d-1)} m^{-1} n \\
 = HK_i + O(n^{(d-2)/(d-1)}),
 \end{aligned}$$

the first inequality following from the triangle inequality, the second from the definition of a^{pq} , and the third from constraints (2). In order to obtain a feasible solution to $R(V_i \cup A_i)$, we add one unit of capacity on some Hamiltonian tour on A_i . This solution is feasible since every cutset has at least two units of capacity. A tour of length $O(n^{(d-2)/(d-1)})$ can be obtained by patching together short tours on A_{ij} , since the length of the shortest Hamiltonian tour over n vertices in $[0, 1/m]^{d-1}$ is $O(n^{(d-2)/(d-1)})$. Hence, by allowing an increase of cost of $O(n^{(d-2)/(d-1)})$, we obtain a feasible solution to $R(V_i \cup A_i)$, which completes the proof of the proposition. ■

Proposition 8 implies that partitioning algorithms a la Karp [14] are almost surely asymptotically optimal.

We may now deduce the asymptotic behavior of HK as a corollary to Theorem 6 and Propositions 7 and 8.

THEOREM 9. *Let the n points $X^{(n)} = (X_1, \dots, X_n)$ be identically and independently distributed according to a distribution with bounded support and absolutely continuous part $f(x) dx$ over the d -cube. Then there exists a constant $\beta_{HK}(d)$ such that*

$$\lim_{n \rightarrow \infty} \frac{HK(X^{(n)})}{n^{(d-1)/d}} = \beta_{HK}(d) \int f(x)^{(d-1)/d} dx$$

almost surely.

5. Large deviation inequalities and the Held-Karp lower bound. In this section we use a recent result of Rhee and Talagrand [20] to find a sharp bound on the

$$\Pr\{|HK(X^{(n)}) - E[HK(X^{(n)})]| > t\}$$

for the case $d = 2$, i.e. in the Euclidean plane. As a consequence, we shall be able to establish the finiteness of

$$\sum_{n=1}^{\infty} \Pr\left\{\left|\frac{HK(X^{(n)})}{\sqrt{n}} - \beta_{HK}\right| > \epsilon\right\}$$

for all $\epsilon > 0$, i.e. the complete convergence of the Held-Karp lower bound when

$d = 2$. This result is stronger than the almost sure convergence of Theorem 4. We use the following result:

THEOREM 10 (Rhee and Talagrand [20]). *Let L be a Euclidean functional that satisfies*

$$(33) \quad L(A) \leq L(A \cup \{x\}) \leq L(A) + 2 \min_{y \in A} |x - y|$$

for any finite subset A and any $x \in [0, 1]^2$. Then there exists a constant K such that, for all $t > 0$,

$$\Pr\{|L(X^{(n)}) - E[L(X^{(n)})]| > t\} \leq Ke^{-t^2/4K}.$$

In the following lemma, we prove the last inequality of (33).

LEMMA 11. *The Held-Karp lower bound satisfies*

$$(34) \quad HK(A \cup \{x\}) \leq HK(A) + 2 \min_{y \in A} |x - y|,$$

for any finite subset A of $[0, 1]^2$ and any $x \in [0, 1]^2$.

PROOF. Consider any optimal solution to $R(A)$. By setting the capacity on some edge linking x to some vertex y of A to be 2, we obtain a feasible solution to $R(A \cup x)$. The cost of this solution is equal to $HK(A) + 2|x - y|$. Moreover, we can choose y so that $|x - y| = \min_z |x - z|$ and, as a result, we establish (34). ■

Proposition 3 and Lemma 11 imply that the Held-Karp lower bound satisfies the assumptions of Theorem 10 and, therefore,

$$(35) \quad \Pr\{|HK(X^{(n)}) - E[HK(X^{(n)})]| > t\} \leq Ke^{-t^2/4K}.$$

The complete convergence of the Held-Karp lower bound when $d = 2$ now follows from (35) and the fact that $E[HK(X^{(n)})]/\sqrt{n}$ tends to β_{HK} as n tends to infinity. The complete convergence for general d can also be established by using martingale inequalities (for details, see the first author's Ph.D. thesis).

6. Concluding remarks. We have analyzed probabilistically the Held-Karp lower bound for the TSP. Our result corroborates the observation that the lower bound is very close to the length of the optimal tour in practice. We would like to emphasize that we have exploited the combinatorial interpretation of the Held-Karp lower bound and the theory of subadditive Euclidean functionals. We believe that the idea of combining polyhedral characterizations with probabilistic analysis has the potential to lead to very interesting results. This work has left several open questions unanswered. For example, it would be interesting to know whether the Held-Karp lower bound is asymptotically optimal, i.e. whether $\beta_{HK}(d) = \beta_{TSP}(d)$. Moreover, obtaining the exact value for $\beta_{HK}(d)$ or $\beta_{TSP}(d)$ would be a significant step in the probabilistic analysis of combinatorial optimization problems under the Euclidean model.

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