Jump Number of Two-Directional Orthogonal Ray Graphs

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Abstract. We model *maximum cross-free matchings* and *minimum biclique covers* of *two-directional orthogonal ray graphs* (2-dorgs) as maximum independent sets and minimum hitting sets of an associated family of rectangles in the plane, respectively. We then compute the corresponding maximum independent set using linear programming and uncrossing techniques. This procedure motivates an efficient combinatorial algorithm to find a cross-free matching and a biclique cover of the same cardinality, proving the corresponding min-max relation.
We connect this min-max relation with the work of Györi [19], Lubiw [23], and Frank and Jordán [16] on seemingly unrelated problems. Our result can be seen as a non-trivial application of Frank and Jordán's Theorem. As a direct consequence, we obtain the first polynomial algorithm for the *jump number problem* on 2-dorgs. For the subclass of convex graphs, our approach is a vast improvement over previous algorithms. Additionally, we prove that the *weighted* maximum cross-free matching problem is NP-complete for 2-dorgs and give polynomial algorithms for some subclasses.

1 Introduction

The *jump number problem* is to find a linear extension L of a poset P minimizing the number of *jumps*, that is, the number of pairs of consecutive elements in L that are incomparable in P. For bipartite posets, Chaty and Chein [8] have shown that this is equivalent to find a maximum alternating-cycle-free matching in the underlying comparability graph, which is NP-hard as shown by Pulleyblank [30].

Two related problems are the *minimum biclique cover* and the *maximum cross-free matching* problems. Given a bipartite graph $G = (A \cup B, E)$, a *biclique* is the edge set of a complete bipartite subgraph. A *biclique cover* is a family of bicliques whose union is E. Two edges e and f *cross* if there is a biclique containing both. A *cross-free matching* is a collection of pairwise non-crossing edges. Note that the maximum size of a cross-free matching is at most the minimum size of a biclique cover. An interesting question is to find classes of graphs where the two quantities coincide.

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For *chordal bipartite graphs*, alternating-cycle-free matchings and cross-free matchings coincide, making the jump number problem equivalent to the maximum cross-free matching problem. Müller [25] has shown that the jump number problem is NP-hard in this class. There are polynomial time algorithms for the jump number problem in important subclasses, such as bipartite permutation graphs [33, 3, 12], biconvex graphs [3] and convex graphs [11]. For this last class, the fastest known algorithm runs in $O(|A \cup B|^9)$ time [11]. To our knowledge, finding efficient algorithms for the jump number problem in any natural superclass of convex bipartite graphs is open in the literature.

The minimum biclique cover problem arise in many areas (e.g. biology [27], chemistry [9] and communication complexity [22]). Orlin [28] has shown that finding a minimum biclique cover of a bipartite graph is NP-hard. Müller [26] extended this result to chordal bipartite graphs. There are polynomial time algorithms for bipartite graphs that are C_4-free [26], distance hereditary graphs [26], permutation graphs [26], and domino-free graphs [1]. To our knowledge, these are the only bipartite classes for which this problem has been explicitly shown to be polynomially solvable.

Our Results. We study both the maximum cross-free matching and the minimum biclique cover problems on the class of *two-directional orthogonal ray graphs* (2-dorgs), which is a superclass of convex bipartite graphs that has recently been studied by many authors [29, 32]. Our main result is that the size of a maximum cross-free matching is equal to the size of a minimum biclique cover in 2-dorgs, and both objects are polynomially computable. A key tool for this result is a new *geometrical reformulation* of the previous problems as the *maximum independent set* and the *minimum hitting set* of an associated collection of rectangles in the plane respectively. Our reformulation is reminiscent of the one already observed by Ceroi [5], between the jump number of a two-dimensional poset and the maximum *weighted* independent set of a collection of rectangles in the plane. Using this geometric representation, we give a linear program formulation for the maximum cross-free matching. Even though the associated polytope is not integral, we show how to find an optimal integral vertex by using an uncrossing procedure. We can also find this vertex by solving a related linear program having only integral optimal extreme points. We are further able to devise an efficient combinatorial algorithm that computes simultaneously a maximum cross-free matching and a minimum biclique cover of the same cardinality in a 2-dorg. We also study a weighted version of the maximum cross-free matching problem, which is equivalent to the *weighted jump number* studied by Ceroi [6] and show that this problem is NP-hard for 2-dorgs.

We explore the relation between our main result and the following, apparently unrelated, pairs of combinatorial problems where a min-max result exists: the *minimum rectangle cover* and the *maximum antirectangle* of an orthogonal biconvex board, studied by Chaiken et al. [7]; the *minimum base of a family of intervals* and the *maximum independent set of point-interval pairs*, studied by Györi [19] and Lubiw [23]; and finally, the *minimum edge-cover* and the *maximum half-disjoint family of set-pairs*, studied by Frank and Jordán [16]. Our result can be seen both as a generalization of Györi's result and as a non-trivial application of Frank and Jordán's Theorem.

Additionally, for special subclasses of 2-dorgs, we give new results: (1) We give new efficient algorithms to solve the maximum weight cross-free matching in bipartite permutation graphs. (2) For convex graphs, we show how to use algorithmic versions of Györi's result [18, 21] to compute a maximum cross-free matching and minimum biclique cover in $O(n^2)$ time and how to use a result by Lubiw [23] to compute a maximum weight cross-free matching in $O(n^3)$ time.

2 Preliminaries

Notation. The *rectangles* we consider in this paper are closed sets in the plane (possibly not full dimensional). Their sides are parallel to the x and y axes and their vertices are points in $\boldsymbol{Z}^2$. For a set $S \in \boldsymbol{Z}^2$, we denote by S_x the projection of S onto the x-axis. If $S = \{p\}$ is a singleton, we write S_x simply as p_x. For sets $S, S' \in \boldsymbol{Z}^2$, we write $S_x < S'_x$ if the projection S_x is to the left of the projection S'_x, that is, if $p_x < p'_x$ for all $p \in S$, $p' \in S'$. We extend this convention to $S_x > S'_x, S_x \leq S'_x, S_x \geq S'_x$ and to the projections onto the y-axis as well.

Two special sets of points $A, B \subseteq \boldsymbol{Z}^2$, not necessarily disjoint, represent the vertices of the bipartite graphs we consider in this paper. We use a and b to denote points of A and B respectively. For $a_x \leq b_x$ and $a_y \leq b_y$, we use $\Gamma(a, b)$ to the denote the rectangle with bottom-left corner a and top-right corner b. Let $\mathcal{R}(A, B)$ (or simply $\mathcal{R}$) be the set of rectangles with bottom-left corner in A and top-right corner in B. This is, $\mathcal{R} = \{\Gamma(a, b) \colon a \in A, b \in B, a_x \leq b_x, a_y \leq b_y\}$. The subset of inclusion-wise minimal rectangles of $\mathcal{R}$ is denoted by $\mathcal{R}_\downarrow$. Abusing notation, we denote by $G = (A \cup B, \mathcal{R})$ the bipartite graph with bipartition A and B, where there is an edge between $a \in A$ and $b \in B$ if and only if $\Gamma(a, b) \in \mathcal{R}$. Finally, for a given rectangle $R \in \mathcal{R}$, we denote by $A(R)$ (resp. $B(R)$) the bottom-left (resp. top-right) corner of the rectangle R.

We claim that the class of graphs arising from the previous construction is equivalent to the class of *two-directional orthogonal ray graphs* (2-dorgs) recently considered by Shrestha et al. [32]. A 2-dorg is a bipartite graph on $A \cup B$ where each vertex v is associated to a point $(v_x, v_y) \in \boldsymbol{Z}^2$, so that $a \in A$ and $b \in B$ are connected if and only the rays $[a_x, \infty) \times \{a_y\}$ and $\{b_x\} \times (-\infty, b_y]$ intersect each other. Since this condition is equivalent to $\Gamma(a, b) \in \mathcal{R}(A, B)$, the claim follows.

It is an easy exercise to prove that every 2-dorg admits a geometric representation where no two points of $A \cup B$ are in the same horizontal or vertical line, and furthermore, $A \cup B$ are points of the grid $[n]^2 = \{1, \ldots, n\} \times \{1, \ldots, n\}$, where $n = |A \cup B|$. When these conditions hold, we say that $G = (A \cup B, \mathcal{R})$ is in *rook representation*. For the rest of the paper, we consider a 2-dorg as being *both the graph and its geometric representation.*

Graph Definitions. We recall the following definitions of a nested family of bipartite graph classes. We keep the notation $G = (A \cup B, \mathcal{R})$ since we can show they are 2-dorgs.

A *bipartite permutation graph* is the comparability graph of a two dimensional poset of height 2, where A is the set of minimal elements and B is the complement of this set. A *two dimensional poset* is a collection of points in $\boldsymbol{Z}^2$ with the relation $p \leq_{\boldsymbol{Z}^2} q$ if $p_x \leq q_x$ and $p_y \leq q_y$.

In a *convex graph*, there is a labeling for $A = \{a_1, \ldots, a_k\}$ so that the neighborhood of each $b \in B$ is a set of consecutive elements of A. In a *biconvex graph*, there is also a labeling for $B = \{b_1, \ldots, b_l\}$ so that the neighborhood of each $a \in A$ is consecutive in B.

In an *interval bigraph*, each vertex $v \in A \cup B$ is associated to a real closed interval I_v (w.l.o.g. with integral extremes) so that $a \in A$ and $b \in B$ are adjacent if and only if $I_a \cap I_b \neq \emptyset$.

It is known that bipartite permutation $\subset$ biconvex $\subset$ convex $\subset$ interval bigraph $\subset$ 2-dorg and that all inclusions are strict [4, 32]. We give a simple proof of the last inclusion using our geometrical interpretation of 2-dorgs. Let G be an interval bigraph with parts A and B. For $a \in A$ with interval $I_a = [s, t]$ and $b \in B$ with interval $I_b = [s', t']$, we identify a with the point $(s, -t) \in \boldsymbol{Z}^2$ and b with the point $(t', -s') \in \boldsymbol{Z}^2$. By definition, ab is an edge of G if and only if $[s, t] \cap [s', t'] \neq \emptyset$, or equivalently if $s \leq t'$ and $-t \leq -s'$. This shows that $\Gamma(a, b)$ is in $\mathcal{R}(A, B)$. Intuitively, the previous assignment maps A (resp. B) to points weakly below (resp. weakly above) the diagonal line $y = -x$ in such a way that their horizontal and vertical projections onto this line define the corresponding intervals. We illustrate this construction in Fig. 1.

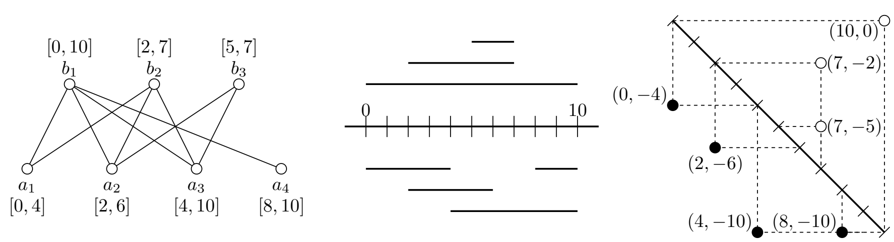


Fig. 1: An interval bigraph, its interval representation and a geometric representation as 2-dorg.

Geometric Interpretation. Given a bipartite graph $G = (V, E)$, a *biclique* is the edge set of a complete bipartite subgraph of G. A *biclique cover* is a family of bicliques whose union is E. Two distinct edges $e = ab$ and $f = a'b'$ *cross* if there is a biclique containing both e and f. This holds if and only if ab' and $a'b$ are also edges of the graph. In particular, two edges incident to the same vertex cross. A *cross-free matching* is a collection of edges pairwise non-crossing. We denote the maximum size of a cross-free matching by $\alpha^*(G)$ and the minimum size of a biclique cover by $\kappa^*(G)$. For the last quantity, we can restrict to biclique covers using maximal bicliques.

Theorem 1. *Let $G = (A \cup B, \mathcal{R})$ be a 2-dorg. Two edges R and R' cross if and only if R and R' intersect as rectangles.*

Proof. Let $R = \Gamma(a, b)$ and $R' = \Gamma(a', b')$ be distinct edges in $\mathcal{R}$. Both edges cross if and only if $\Gamma(a, b')$ and $\Gamma(a', b)$ are also in $\mathcal{R}$. This is equivalent to $\max(a_x, a'_x) \leq \min(b_x, b'_x)$ and $\max(a_y, a'_y) \leq \min(b_y, b'_y)$. From here, if R and R' cross, then the point $p = (\max(a_x, a'_x), \max(a_y, a'_y))$ is in the intersection of R and R' as rectangles. Conversely, if there is a point $p \in R \cap R'$, then $\max(a_x, a'_x) \leq p_x \leq \min(b_x, b'_x)$ and $\max(a_y, a'_y) \leq p_y \leq \min(b_y, b'_y)$, and so, R and R' cross.

It is natural to study the following problems. Given a collection $\mathcal{C}$ of rectangles, an *independent set* is a family of pairwise disjoint rectangles in $\mathcal{C}$ and a *hitting set* is a set of points such that every rectangle in $\mathcal{C}$ contains at least one point of this set. We denote by $\text{mis}(\mathcal{C})$ and $\text{mhs}(\mathcal{C})$ the sizes of a *maximum independent set of rectangles* in $\mathcal{C}$ and a *minimum hitting set* for $\mathcal{C}$ respectively.

Consider the intersection graph $\mathcal{I}(\mathcal{C}) = (\mathcal{C}, \{RS : R \cap S \neq \emptyset\})$. Naturally, independent sets of $\mathcal{C}$ correspond to stable sets in $\mathcal{I}(\mathcal{C})$. On the other hand, since the family $\mathcal{C}$ has the *Helly property*,[3] we can assign to every clique in $\mathcal{I}(\mathcal{C})$ a unique *witness point*, defined as the leftmost and lowest point contained in all rectangles of the clique. Since different maximal cliques have different witness points, it is easy to prove that $\mathcal{C}$ admits a minimum hitting set consisting only of witness points of maximal cliques. In particular, $\text{mhs}(\mathcal{C})$ is equal to the minimum size of a *clique-cover* of $\mathcal{I}(\mathcal{C})$.

For both problems defined above we can restrict ourselves to the family $\mathcal{C}_\downarrow$ of inclusion-wise minimal rectangles in $\mathcal{C}$, since any maximum independent set in $\mathcal{C}_\downarrow$ is also maximum in $\mathcal{C}$ and any minimum hitting set for $\mathcal{C}_\downarrow$ is also minimum for $\mathcal{C}$. Using Theorem 1 we conclude the following.

Theorem 2. *For a 2-dorg $G = (A \cup B, \mathcal{R})$, the cross-free matchings of G correspond to the independent sets of $\mathcal{R}$ and the biclique covers of G using maximal bicliques correspond to the hitting sets for $\mathcal{R}$ using witness points of maximal cliques (and to clique-covers in $\mathcal{I}(\mathcal{R})$ using maximal cliques). In particular, $\alpha^*(G) = \text{mis}(\mathcal{R}) = \text{mis}(\mathcal{R}_\downarrow)$ and $\kappa^*(G) = \text{mhs}(\mathcal{R}) = \text{mhs}(\mathcal{R}_\downarrow)$.*

3 A Linear Programming Approach

Consider the integer program formulation for the maximum independent set of a collection of rectangles $\mathcal{C}$ with vertices on the grid $[n]^2$:

$$\text{mis}(\mathcal{C}) = \max \left\{ \sum_{R \in \mathcal{C}} x_R \colon \sum_{R\colon\, q \in R} x_R \leq 1, q \in [n]^2; x \in \{0,1\}^{\mathcal{C}} \right\} .$$

[3] If a collection of rectangles intersect, then all of them share a rectangular region in the plane.

Let $\mathrm{P}(\mathcal{C})$ be the polytope $\{x \in \boldsymbol{R}^{\mathcal{C}} \colon \sum_{R\colon q\in R} x_R \leq 1, q \in [n]^2; x \geq 0\}$ and the relaxation of $\mathrm{mis}(\mathcal{C})$ given by $\mathrm{LP}(\mathcal{C}) = \max\{\sum_{R\in\mathcal{C}} x_R \colon x \in \mathrm{P}(\mathcal{C})\}$.

Let $G = (A \cup B, \mathcal{R})$ be a 2-dorg with all its vertices in the grid $[n]^2$. If $\mathrm{LP}(\mathcal{R})$ (or $\mathrm{LP}(\mathcal{R}_\downarrow)$) has an integral solution then, by Theorem 2, this solution induces a maximum cross-free matching of G. This happens, for example, when $\mathrm{P}(\mathcal{R})$ or $\mathrm{P}(\mathcal{R}_\downarrow)$ is an integral polytope.

Theorem 3. *Given a family of rectangles $\mathcal{C}$ with vertices in $[n]^2$, the polytope $\mathrm{P}(\mathcal{C})$ is integral if and only if the intersection graph $\mathcal{I}(\mathcal{C})$ is* perfect.

Theorem 3 follows from the fact that $\mathrm{P}(\mathcal{C})$ is the *clique-constrained stable set polytope* of the intersection graph $\mathcal{I}(\mathcal{C})$, usually denoted by $\mathrm{QSTAB}(\mathcal{I}(\mathcal{C}))$. A classical result on perfect graphs (See, e.g. [31]) establishes that a graph H is perfect if and only if $\mathrm{QSTAB}(H)$ is integral.

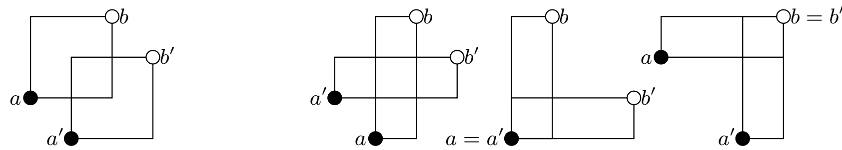
If $G = (A \cup B, \mathcal{R})$ is a 2-dorg such that $\mathcal{I}(\mathcal{R})$ (or $\mathcal{I}(\mathcal{R}_\downarrow)$) is perfect, then $\alpha^*(G) = \kappa^*(G)$ and solving the linear program $\mathrm{LP}(\mathcal{R})$ (or $\mathrm{LP}(\mathcal{R}_\downarrow)$) gives a polynomial time algorithm for finding a maximum cross-free matching. Since there are also polynomial time algorithms to find minimum clique-covers of perfect graphs, we can also obtain a minimum biclique cover of G.

For bipartite permutation graphs $G = (A \cup B, \mathcal{R})$, the intersection graph $\mathcal{I}(\mathcal{R}) \cong (\mathcal{R}, \{ef : e \text{ and } f \text{ cross}\})$ is known to be perfect since it is both weakly chordal [26] and co-comparability [3]. By the previous discussion we can find a maximum cross-free matching and minimum biclique cover for these graphs in polynomial time. Using the structure of bipartite permutation graphs, linear-time algorithms have been developed for both problems [33, 3, 12]. For biconvex graphs $G = (A \cup B, \mathcal{R})$, the graph $\mathcal{I}(\mathcal{R})$ is not necessarily perfect, but we can show that there is a geometric representation $(A' \cup B', \mathcal{R}')$ for which $\mathcal{I}(\mathcal{R}'_\downarrow)$ is perfect and that this property does not extend to convex graphs. We defer the proof of these facts for the full version of this paper.

Even if the intersection graph of the rectangles of a 2-dorg is not perfect, we can still find a maximum cross-free matching. Let $G = (A \cup B, \mathcal{R})$ be a 2-dorg in *rook representation*. Since the rectangles in $\mathcal{R}_\downarrow$ are inclusion-wise minimal, no rectangle $R = \Gamma(a, b)$ in $\mathcal{R}_\downarrow$ contains a third point in $(A \cup B) \setminus \{a, b\}$. Therefore, there are only four ways in which a pair of rectangles can intersect. They are depicted in Fig. 2. We say that two intersecting rectangles have *corner-intersection* if all the vertices involved are distinct and each rectangle contain a corner of the other. If this does not happen, we say that the intersection is *corner-free*. A *corner-free-intersection* (c.f.i.) family is a collection of inclusion-wise minimal rectangles having no corner-intersections.

Let z^* be the optimal value of $\mathrm{LP}(\mathcal{R}_\downarrow)$ and for every rectangle $R \in \mathcal{R}_\downarrow$, let $\mu(R) > 0$ be its area.[4] Let $\bar{x}$ be an *optimal extreme point* of the following linear

[4] For our discussion, the *area* or a rectangle $R = \Gamma(a, b)$ is defined as $(b_x - a_x)(b_y - a_y)$. However, our techniques also works if we define the *area* of a rectangle as the number of grid points it contains.



(a) corner-intersection (b) corner-free intersections

Fig. 2: The ways two rectangles in $\mathcal{R}_\downarrow$ can intersect each other.

program

$$\mathrm{LP}(z^*, \mathcal{R}_\downarrow) = \min\Big\{ \sum_{R\in\mathcal{R}_\downarrow} \mu(R)x_R \colon \sum_{R\in\mathcal{R}_\downarrow} x_R = z^* \text{ and } x \in \mathrm{P}(\mathcal{R}_\downarrow)\Big\} \ .$$

Note that $\bar{x}$ is a solution to $\mathrm{LP}(\mathcal{R}_\downarrow)$ minimizing its total *weighted area* and it is also an optimal extreme point of $\max\{\sum_{R\in\mathcal{R}_\downarrow}(1-\varepsilon\mu(R))x_r \colon x \in \mathrm{P}(\mathcal{R}_\downarrow)\}$ for a small value of ε.

Theorem 4. *The point $\bar{x}$ is an integral point.*

Proof. Suppose that $\bar{x}$ is not integral. Let $\overline{\mathcal{R}} = \{R \in \mathcal{R}_\downarrow \colon \bar{x}_R > 0\}$ be the set of rectangles in the support of $\bar{x}$. We claim that $\overline{\mathcal{R}}$ is a c.f.i. family. Assume for sake of contradiction that $R = \Gamma(a,b)$ and $R' = \Gamma(a',b')$ are rectangles in $\overline{\mathcal{R}}$ having corner-intersection as in Fig. 2a. We apply the following *uncrossing procedure*: let $\varepsilon = \min(\bar{x}_R, \bar{x}_{R'}) > 0$ and consider the rectangles $S = \Gamma(a,b')$ and $S' = \Gamma(a',b)$ in $\mathcal{R}_\downarrow$. Consider the vector $\tilde{x}$ obtained from $\bar{x}$ by decreasing x_R and $x_{R'}$ by ε and increasing by the same amount x_S and $x_{S'}$. It is easy to see that $\tilde{x}$ is a feasible solution of $\mathrm{LP}(z^*, \mathcal{R}_\downarrow)$ with strictly smaller weighted area than $\bar{x}$, contradicting its optimality.

Now we prove that the intersection graph $\mathcal{I}(\overline{\mathcal{R}})$ is a comparability graph, and therefore it is perfect. Consider the following partial order in $\overline{\mathcal{R}}$:

$$R \preceq R' \text{ if and only if } R_y \subseteq R'_y \text{ and } R_x \supseteq R'_x \ , \tag{1}$$

Since $\overline{\mathcal{R}}$ is a c.f.i. family, it follows that every pair of intersecting rectangles in $\overline{\mathcal{R}}$ are comparable by $\preceq$. The converse trivially holds. Therefore, $\mathcal{I}(\overline{\mathcal{R}})$ is the comparability graph of $(\overline{\mathcal{R}}, \preceq)$.

Using Theorem 3 we show that $\mathrm{P}(\overline{\mathcal{R}})$ is an integral polytope. Consider the set $F = \mathrm{P}(\overline{\mathcal{R}}) \cap \big\{\sum_{R\in\overline{\mathcal{R}}} x_R = z^*, \sum_{R\in\overline{\mathcal{R}}} \mu(R)x_R = \sum_{R\in\overline{\mathcal{R}}} \mu(R)\bar{x}_R\big\}$. This set is a face of $\mathrm{P}(\overline{\mathcal{R}})$ containing only optimum solutions of $\mathrm{LP}(z^*, \mathcal{R}_\downarrow)$. To conclude the proof of the theorem we show that $\bar{x}$ is a vertex of F, and therefore, a vertex of $\mathrm{P}(\overline{\mathcal{R}})$. If $\bar{x}$ is not a vertex of F, then we can find two different points in F such that $\bar{x}$ is a convex combination of those points. This means that $\bar{x}$ is a convex combination of different optimum solutions of $\mathrm{LP}(z^*, \mathcal{R}_\downarrow)$, contradicting the choice of $\bar{x}$.

Using Theorem 4 we can find a maximum cross-free matching of a 2-dorg G in rook representation as follows: solve the linear program $\mathrm{LP}(\mathcal{R}_\downarrow)$ and find an optimal extreme point of $\mathrm{LP}(z^*, \mathcal{R}_\downarrow)$. Moreover, since 2-dorgs are *chordal-bipartite graphs* [32], for which the maximum cross-free matching and the jump number problems are equivalent, we conclude the following.

Theorem 5. *For a 2-dorg, the maximum cross-free matching (and equivalently, the jump number) can be computed in polynomial time.*

For any family of rectangles $\mathcal{C}$ with vertices in $[n]^2$, consider the linear program $\mathrm{DP}(\mathcal{C}) = \min\{\sum_{q\in[n]^2} y_q \colon \sum_{q\in R} y_q \geq 1, R \in \mathcal{C}; y \geq 0\}$. The integer feasible solutions of $\mathrm{DP}(\mathcal{C})$ correspond exactly to hitting sets for $\mathcal{C}$ that are contained in the grid $[n]^2$. Since there are minimum hitting sets for $\mathcal{C}$ of this type, we conclude that $\mathrm{DP}(\mathcal{C})$ is a linear program relaxation for $\mathrm{mhs}(\mathcal{C})$.

The programs $\mathrm{LP}(\mathcal{C})$ and $\mathrm{DP}(\mathcal{C})$ are dual to each other. Hence, they have a common optimum value $z^*(\mathcal{C})$ and $\mathrm{mis}(\mathcal{C}) \leq z^*(\mathcal{C}) \leq \mathrm{mhs}(\mathcal{C})$. In particular, for the family $\mathcal{R}_\downarrow$ of inclusion-wise minimal rectangles coming from a 2-dorg G in rook representation, we have

$$\alpha^*(G) \leq z^*(\mathcal{R}_\downarrow) \leq \kappa^*(G) \ . \tag{2}$$

In this section we have shown not only that the first inequality in (2) is an equality, but also that an integer optimal solution for $\mathrm{LP}(\mathcal{R}_\downarrow)$ can be found by optimizing an easy to describe linear function over the optimal face of this linear program. It would be interesting to show a similar result for $\mathrm{DP}(\mathcal{R}_\downarrow)$, since as a consequence of the combinatorial algorithm we give in Sect. 4, this polytope also admits an integer optimal solution.

4 A Combinatorial Algorithm

In this section we give a combinatorial algorithm that computes simultaneously a maximum cross-free matching and a minimum biclique cover of a 2-dorg. Our procedure is based on the algorithmic proof of Györi's min-max result of intervals [19] given by Frank [15]. In order to keep the discussion self-contained, we do not rely on previous results. The relation of our approach to other previous work will be explored in Sect. 5.

In Sect. 3 we have shown that a maximum cross-free matching of a 2-dorg $G = (A \cup B, \mathcal{R})$ can be obtained from a maximum independent set of the c.f.i. family $\overline{\mathcal{R}}$. In what follows, we show that this result also holds if we replace $\overline{\mathcal{R}}$ by certain *maximal greedy* c.f.i. subfamily of $\mathcal{R}_\downarrow$.

Given a 2-dorg $G = (A \cup B, \mathcal{R})$ in rook representation, we say that a rectangle $R \in \mathcal{R}_\downarrow$ appears before a rectangle $S \in \mathcal{R}_\downarrow$ in *right-top order* if we either have $A(R)_x < A(S)_x$, or if $A(R)_x = A(S)_x$ and $B(R)_y < B(S)_y$. This defines a total order on $\mathcal{R}_\downarrow$. Construct a family $\mathcal{K}$ by processing the rectangles in $\mathcal{R}_\downarrow$ in right-top order and adding only those that keep $\mathcal{K}$ corner-free.

Since $\mathcal{I}(\mathcal{K})$ is the comparability graph of $(\mathcal{K}, \preceq)$, where $\preceq$ is as in (1), the size of a maximum independent set $\mathcal{R}_0$ of $\mathcal{K}$ is equal to the size of a minimum hitting set H_0 for $\mathcal{K}$. We can find optimal solutions of these problems by computing a maximum antichain and a minimum chain-cover of the poset $(\mathcal{K}, \preceq)$, using any polynomial time algorithm for Dilworth's chain-partitioning problem (See, e.g. [13]). We modify H_0 to obtain a set of points H^* of the same size hitting $\mathcal{R}$.

An *admissible flip* of a hitting set H for $\mathcal{K}$ is an ordered pair of points p and q in H with $p_x < q_x$ and $p_y < q_y$, such that the set $H \setminus \{p, q\} \cup \{(p_x, q_y), (q_x, p_y)\}$ obtained by *flipping p and q* is still a hitting set for $\mathcal{K}$. Construct H^* from H_0 by flipping admissible flips while this is possible. Note that flipping two points reduces the potential $\psi(H) = \sum_{p \in H} p_x p_y$ by at least one unit. Since ψ is positive and $\psi(H_0) \leq |H_0| n^2 \leq n^3$, the previous construction can be done efficiently.

Lemma 1. *H^* is a hitting set for $\mathcal{R}_\downarrow$, and therefore, a hitting set for $\mathcal{R}$.*

Proof. Suppose this is not the case. Let $R = \Gamma(a, b)$ be the *last* rectangle of $\mathcal{R}_\downarrow \setminus \mathcal{K}$ not hit by H^*, in right-top order. Let also $R' = \Gamma(a', b')$ be the *first* rectangle of $\mathcal{K}$ having corner-intersection with R, in right-top order. We have $a'_x < a_x < b'_x < b_x$ and $b'_y > b_y > a'_y > a_y$ (See Fig. 3). In particular, the rectangles $S = \Gamma(a', b)$ and $T = \Gamma(a, b')$ are in $\mathcal{R}$. They are also inclusion-wise minimal, otherwise there would be a point $v \in A \cup B \setminus \{a, a', b, b'\}$ in $S \cup T \subseteq R \cup R'$, contradicting the minimality of R or R'. Since T appears after R in right-top order, it is hit by a point $q \in H^*$.

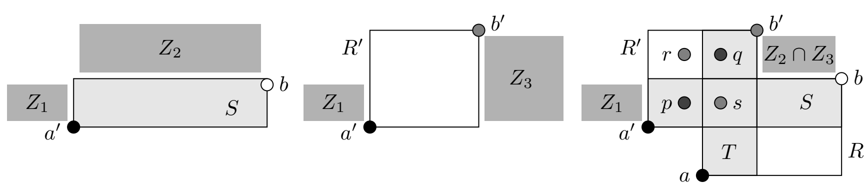


Fig. 3: Positions of rectangles and zones defined in the proof of Lemma 1.

We claim that the rectangle S is in $\mathcal{K}$. If this is not the case, then there is a rectangle $R'' = \Gamma(a'', b'') \in \mathcal{K}$ appearing before S and having corner-intersection with S. In particular corner a'' lies in $Z_1 = (-\infty, a'_x - 1] \times [a'_y + 1, b_y - 1]$, and corner b'' lies in the zone $Z_2 = [a'_x + 1, b_x - 1] \times [b_y + 1, \infty)$ as shown on the left of Fig. 3. Note that the top-left corner (a'_x, b_y) of rectangle S is in both R' and R''. Since both are in $\mathcal{K}$, the rectangles R' and R'' have corner-free intersection. Using this last fact, and that $a'' \in Z_1$ we conclude that b'' is either equal to b' or it lies in the zone $Z_3 = [b'_x + 1, \infty) \times [a'_y + 1, b_y - 1]$. See the center of Fig. 3.

Since $a'' \in Z_1$ and $b'' \in \{b'\} \cup (Z_2 \cap Z_3)$, we conclude that R and R'' have corner-intersection, contradicting the choice of R' since R'' appears before R' in right-top order. This proves the claim and, since $S \in \mathcal{K}$, it must be hit by a point $p \in H^*$.

Now we show that p and q are an admissible flip for H^*, contradicting the construction of H^*. Since $p \in S \setminus R$ and $q \in T \setminus R$ we have $p_x < a_x \leq q_x$ and

$p_y \leq b_y < q_y$. Let $r = (p_x, q_y)$, $s = (q_x, p_y)$ and suppose there is a rectangle $U \in \mathcal{K}$ hit by H^* but not by $(H^* \setminus \{p, q\}) \cup \{r, s\}$. If the rectangle U is hit by p (but not by r or s), then its upper-right corner $B(U)$ must be in the region $[p_x, q_x - 1] \times [p_y, q_y - 1]$. In particular, $B(U) \in R' \setminus \{a', b'\}$, contradicting the inclusion-wise minimality of R'. If on the other hand U is hit by q (but not by r or s), then its bottom-left corner $A(U)$ must be in $[p_x + 1, q_x] \times [p_y + 1, q_y]$. As before, this means that $A(U) \in R' \setminus \{a', b'\}$, contradicting the inclusion-wise minimality of R'. Therefore, p and q are an admissible flip, concluding the proof of the lemma.

Using Lemma 1, we can find a maximum cross-free matching and a minimum biclique cover of a 2-dorg $G = (A \cup B, \mathcal{R})$ in rook representation using the following algorithm: Compute the c.f.i. greedy family $\mathcal{K}$ as above. Use an algorithm for Dilworth's chain-partitioning problem to compute a maximum independent set $\mathcal{R}_0$ and a minimum hitting set H_0 for $\mathcal{K}$. Finally, compute the set H^* as described above. Since H^* is a hitting set for $\mathcal{R}$, and $\mathcal{R}_0$ is an independent set of $\mathcal{R}$ with $|H^*| = |H_0| = |\mathcal{R}_0|$, we conclude they are both optima; therefore, they induce a maximum cross-free matching and a minimum biclique cover for G.

Theorem 6. *The previous algorithm computes a maximum cross-free matching and a minimum biclique cover of the same size for any 2-dorg G in polynomial time. In particular, $\alpha^*(G) = \kappa^*(G)$.*

Dilworth's chain-partitioning problem on $(\mathcal{K}, \preceq)$ can be solved by finding a maximum matching on a bipartite graph with $2|\mathcal{K}|$ vertices [13]. This task can be done in $O(|\mathcal{K}|^{2.5})$ time using Hopcroft-Karp algorithm [20], or in $O(|\mathcal{K}|^{\omega})$ randomized time, where ω is the exponent for matrix multiplication, using an algorithm by Mucha and Sankowski [24]. A naive implementation of our algorithm using these results runs in $O(n^7)$. In the full version of the paper, we give faster implementations running in $\widetilde{O}(n^{2.5})$ time and $\widetilde{O}(n^{\omega})$ randomized time.

5 Discussion

In this section, we study the connection of maximum cross-free matchings and minimum biclique covers with other combinatorial problems in the literature. Chaiken et al. [7] have studied the problem of covering a biconvex board (an orthogonal polygon horizontally and vertically convex) with orthogonal rectangles included in this board. They have shown that the *minimum size of a rectangle cover* is equal to the *maximum size of an antirectangle* (a set of points in the board such that no two of them are covered by a rectangle included in the board). Györi [19] has shown that the same result holds even when the board is only vertically convex, as a corollary of a min-max relation we describe.

A collection of point-interval pairs (p_j, I_j), with $p_j \in I_j$ is *independent* if $p_j \notin I_k$ for $k \neq j$. A family of intervals is a *base* for another family if every interval of the latter is a union of intervals in the former. Györi establishes that the size of the *largest independent set of point-interval pairs* with intervals in a family F is equal to the size of the *minimum base of* F, where this base is not restricted to be a subset of F. The corresponding optimizers can be found using an algorithm of Franzblau and Kleitman [18] (or an implementation by Knuth [21]) in $O(k^2)$ time where k is the number of different endpoints of F. These algorithms can also be used to compute a maximum antirectangle and a minimum rectangle cover of a convex board.

Every biconvex graph G admits a biadjacency matrix where the set of *ones* form a biconvex board $\mathcal{B}(G)$. Brandstädt [3] notes that finding a maximum cross-free matching (which he called alternating-C_4-free matching) in a biconvex graph G is equivalent to finding a maximum antirectangle in $\mathcal{B}(G)$. By the previous discussion, this can be done in polynomial time. Here we observe that the maximal rectangles in $\mathcal{B}(G)$ correspond exactly to the maximal bicliques of G, showing that the minimum biclique cover of G can be obtained with the same algorithm. Surprisingly, to the authors knowledge, this observation has not been used in the literature concerning the minimum biclique cover problem, where this problem remains open for biconvex graphs.

Similar to the previous case, every convex graph $G = (A \cup B, \mathcal{R})$ admits a biadjacency matrix where the set of *ones* forms a vertically convex board $\mathcal{B}(G)$. However, antirectangles and rectangle covers of $\mathcal{B}(G)$ are no longer in correspondence with cross-free matchings and biclique covers of G. Nevertheless, we can still make a connection to Györi's result on intervals as follows. Every element of B can be seen as an *interval* of elements in A. Cross-free matchings in G correspond then to independent families of point-intervals, with intervals in B. Similarly, since every maximal biclique of G is defined by taking some interval I of elements in A (where I does not necessarily correspond to some $b \in B$), as $\{(a, b)\colon a \in I, b \supseteq I\}$, we obtain that minimal biclique covers of G (using maximal bicliques) correspond to minimum generating families of B. We remark that this connection has not been noted before in the literature. As a corollary of this discussion we have the following.

Corollary 1. *The maximum cross-free matching and minimum biclique cover of convex and biconvex graphs can be computed in $O(n^2)$ time using Knuth's [21] implementation of the algorithm of Franzblau and Kleitman [18].*

In a seminal paper, Frank and Jordán [16] extend Györi's result to *set-pairs*. We briefly describe a particular case of their result that concerns us. A collection of pairs of sets $\{(S_i, T_i)\}$ is *half-disjoint* if for every $i \neq j$, $S_i \cap S_j$ or $T_i \cap T_j$ is empty. A *directed-edge* (s, t) covers a set-pair (S, T) if $s \in S$ and $t \in T$. A family $\mathcal{S}$ of set-pairs is *crossing* if whenever (S, T) and (S', T') are in $\mathcal{S}$, so are $(S \cap T, S' \cup T')$ and $(S \cup T, S' \cap T')$. Frank and Jordán prove that for every crossing family $\mathcal{S}$, the *maximum size of a half-disjoint subfamily* is equal to the *minimum size of a collection of directed-edges covering* $\mathcal{S}$. They also give a linear programming based algorithm to compute both optimizers. Later, combinatorial algorithms for this result were also given (e.g. [2]). See Végh's Ph.D. thesis [34] for related references.

Theorems 5 and 6 can be seen as non-trivial applications of Frank and Jordán's result. Given a 2-dorg $G = (A \cup B, \mathcal{R})$, consider the family of set-pairs $\mathcal{S} = \{(R_x, R_y)\colon R \in \mathcal{R}_\downarrow\}$. It is easy to check that this family is crossing, that half-disjoint families of $\mathcal{S}$ correspond to independent sets in $\mathcal{R}_\downarrow$ and that coverings of $\mathcal{S}$ by directed-edges correspond to hitting sets for $\mathcal{R}_\downarrow$. We remark that this reduction relies heavily on the geometric interpretation of 2-dorgs we have presented in this paper, and that our proofs are self-contained and simpler than the ones used to prove the broader result of Frank and Jordán.

We want to point out that the combinatorial algorithm presented in Sect. 4 extends the algorithm given by Frank [15] as an alternative proof of Györy's result on intervals. But at the same time it comprises the same algorithmic ideas from other applications of Frank and Jordan's result, such as the algorithm of Frank and Végh [17] for connectivity augmentation. Our description is tailored to the instances considered in this paper, leading to a simpler description of the algorithm and a simpler running time analysis.

6 The Maximum Weight Cross-Free Matching Problem

We now consider the problem of finding the *maximum weight cross-free matching* of 2-dorgs $G = (A \cup B, \mathcal{R})$ with non-negative weights $\{w_R\}_{R\in\mathcal{R}}$. This problem is equivalent to the maximum weight jump number of (G, w) defined by Ceroi [6] and to the maximum weight independent set of $(\mathcal{R}, w)$.

The maximum weight cross-free matching is NP-hard for 2-dorgs, even if the weights are zero or one. To see this, we reduce from maximum independent set of rectangles (MISR), which is NP-hard even if the vertices of the rectangles are all distinct [14]. Given an instance $\mathcal{I}$ of MISR with the previous property, let A (resp. B) be the set of lower-left (resp. upper-right) corners of rectangles in $\mathcal{I}$. Note that the 2-dorg $G = (A \cup B, \mathcal{R})$ satisfies $\mathcal{I} \subseteq \mathcal{R}$ so we can find the MISR of $\mathcal{I}$ by finding the maximum weight cross-free matching in G, where we give a weight of one to each $R \in \mathcal{I}$, and a weight of zero to every other rectangle.

Theorem 7. *The maximum weight cross-free matching problem is* NP*-hard for 2-dorgs.*

We now provide an efficient algorithm for the maximum weight cross-free matching of bipartite permutation graphs. We use the natural 2-dorg representation $G = (A \cup B, \mathcal{R})$ arising from the definition of this class. We can solve the maximum weight independent set of $\mathcal{R}$ in $O(n^2)$ time using the fact that the complement of the intersection graph $\mathcal{I}(\mathcal{R})$ is a comparability graph. To see this, let us write $R \searrow S$ if R and S are disjoint and either $R_x < S_x$ or $R_y > S_y$ holds. It is not hard to verify that $D = (\mathcal{R}, \searrow)$ is a partial order whose comparability graph is the complement of $\mathcal{I}(\mathcal{R})$, and that maximum weight cross-free matchings in G corresponds to maximum weight paths in the digraph D, using w as a weight function on the vertex set $\mathcal{R}$. Since D has $|\mathcal{R}|$ vertices and $O(|\mathcal{R}|^2)$ arcs this optimal path Q^* can be found in $O(|\mathcal{R}|^2)$ time [10].

We can find Q^* faster by exploiting the structure of D. For simplicity, assume that all the weights are different. Let $R \searrow S \searrow T$ be three consecutive rectangles in Q^*, then we can extract information about S. Consider the following cases.

If $R_x < S_x$ and $S_x < T_x$, then (i) S is the heaviest rectangle to the right of R with corner $B(S)$.
If $R_x < S_x$ and $S_y > T_y$, then (ii) S is the heaviest rectangle with corner $A(S)$.
If $R_y > S_y$ and $S_x < T_x$, then (iii) S is the heaviest rectangle with corner $B(S)$.
If $R_y > S_y$ and $S_y > T_y$, then (iv) S is the heaviest rectangle below R with corner $A(S)$.

Note that for each Property (i)–(iv), the rectangle S depends on a single parameter associated to S and on at most one parameter associated to R. More precisely, the rectangle S with Properties (i), (ii), (iii) and (iv) is completely determined by the parameters $\{B(R)_x, B(S)\}$, $\{A(S)\}$, $\{B(S)\}$ and $\{A(R)_y, A(S)\}$, respectively. This motivates the following recursion. For $R \in \mathcal{R}$, let $V(R)$ be the maximum weight of a path in D starting with R. For $a \in A$ (resp. $b \in B$), let (resp. $V_{\rightarrow}(b)$) be the maximum weight of a path using only rectangles below a (resp. to the right of b). We have

$$
\begin{aligned}
V(R) &= \max\{V_{\downarrow}(A(R)), V_{\rightarrow}(B(R))\} + w_R,\\
V_{\downarrow}(a) &= \max\{V(S)\colon S \text{ rectangle below } a \text{ satisfying (iii) or (iv)}\},\\
V_{\rightarrow}(b) &= \max\{V(S)\colon S \text{ rectangle to the right of } b \text{ satisfying (i) or (ii)}\}.
\end{aligned}
$$

Given $\{V(R)\}_{R\in\mathcal{R}}$, we can easily compute the optimal path Q^*. Note that evaluating $V_{\downarrow}(a)$ (or $V_{\rightarrow}(b)$) requires to compute the maximum of $V(S)$ over a family of $O(n)$ rectangles S. If this family can be computed in $O(n)$ time, then the recursive formula for V,$V_{\downarrow}$ and $V_{\rightarrow}$ can be completely evaluated in $O(n^2)$ time. We achieve this by precomputing all possible arising families in $O(n^2)$ time. We illustrate this only for rectangles having Property (i). Suppose $B(S) = b$. Traversing the points $b' \in B$ from right to left, we can find the heaviest such rectangle S to the right of b', for all $b' \in B$, in $O(n)$ time. Iterating this for every b, we compute all the families of rectangles arising from Property (i) in $O(n^2)$ time.

If there are repeated weights, we break ties in Properties (i) and (iii) by choosing the rectangle S of smallest width and we break ties in Properties (ii) and (iv) by choosing the rectangle S of smallest height.

Theorem 8. *The maximum weight cross-free matching of a bipartite permutation graph can be computed in $O(n^2)$ time.*

Using these ideas on the weighted 0-1 case, we can compute an optimum solution in $O(n)$ time, under certain assumption about the description of the input. We defer the description of the algorithm to the full version of the paper.

By the discussion in Sect. 5, cross-free matchings of convex graphs correspond to independent sets of a certain system of point-interval pairs. Lubiw [23] gives a polynomial time algorithm for the maximum weight of a system of point-interval pairs. It is straightforward to implement her algorithm in $O(n^3)$ time.

Corollary 2. *The maximum weight cross-free matching of a convex graph can be computed in $O(n^3)$ time.*

7 Summary of Results

We include a table with the current best running times for the studied problem on a graph $G = (A \cup B, \mathcal{R})$, with $n = |A \cup B|$. The new results, in bold, include the ones obtained via their relation to other known problems, but never considered in the literature of the problems we have studied.

	Bip. Perm.	Biconvex	Convex	2-dorg
Max. cross-free matching	$O(n)$ [3, 12]	$O(n^2)$ [3]	$O(n^9)$ [11]	-
(Jump number) **(new)**	-	-	$\boldsymbol{O(n^2)}$[a]	$\boldsymbol{\widetilde{O}(n^{2.5}), \widetilde{O}(n^{\omega})}$[c]
Min. biclique-cover	$O(n)$ [3, 12]	-	-	-
(new)	-	$\boldsymbol{O(n^2)}$[a]	$\boldsymbol{O(n^2)}$[a]	$\boldsymbol{\widetilde{O}(n^{2.5}), \widetilde{O}(n^{\omega})}$[c]
Max. wt. cross-free matching	$O(\lvert\mathcal{R}\rvert^2)$ [3]	-	-	-
(new)	$\boldsymbol{O(n^2)}$	$\boldsymbol{O(n^3)}$[b]	$\boldsymbol{O(n^3)}$[b]	**NP-hard**

[a] Using Franzblau and Kleitman's result [18] or Knuth's [21] implementation.
[b] Using Lubiw's result [23].
[c] In the full version of this paper.

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