Motivic Periods, Coleman Functions, and the Unit Equation

An Ongoing Project

D. Corwin[1]   I. Dan-Cohen[2]

[1]MIT

[2]Ben Gurion University of the Negev

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Motivation: The Unit Equation

Let Z be an integer ring with a finite set of primes inverted ($= \mathcal{O}_k[1/S]$) and $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Theorem

There are finitely many $x, y \in Z^\times$ such that $x + y = 1$
Equivalently, $|X(Z)| < \infty$.

Originally proven by Siegel using Diophantine approximation around 1929.

Problem

Find $X(Z)$ for various Z, or even find an algorithm.

In 2004, Minhyong Kim gave a proof in the case $k = \mathbb{Q}$ using fundamental groups and p-adic analytic Coleman functions.

Refined Problem (Chabauty-Kim Theory)

Find p-adic analytic (Coleman) functions on $X(\mathbb{Z}_p)$ that vanish on $X(Z)$.

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Periods

Let (Y, D, ω, γ) be a *period datum*, i.e., a smooth algebraic variety Y of dimension d over $\mathbb{Q}$, a normal crossings divisor D in Y, an element $\omega \in \Omega^d(Y)$, and an element $\gamma \in H_d(Y(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$.

Definition

A *period* is a complex number equal to an integral $I(Y, D, \omega, \gamma) := \int_\gamma \omega$ for some period datum (Y, D, ω, γ).

Examples

Algebraic numbers, $2\pi i$, $\log(r)$, $\zeta(k)$, $\mathrm{Li}_k(r)$, π, $\cdots$ (for $r \in \mathbb{Q}_{>0}$ and $k \in \mathbb{Z}_{>0}$)

- Algebraic numbers: If α is a root of $p(x) \in \mathbb{Q}[x]$, let $(Y, D, \omega, \gamma) = (\mathbb{A}^1, \{p(z) = 0\} \cup \{0\}, dz, [0, \alpha])$.
- $2\pi i$: $(\mathbb{G}_m, \emptyset, \frac{dz}{z}, \gamma)$ with γ a counterclockwise loop around 0.
- $\log(r)$ for $r \in \mathbb{Q}_{>0}$: $(\mathbb{G}_m, \{1, r\}, \frac{dz}{z}, [1, r])$.

Polylogarithms as Periods

- As we will use them extensively, we now explicitly describe the period datum that gives rise to a polylogarithm.
- Let $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Q})$ and $\gamma\colon (0,1] \to \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{C})$ a path from 0 to z.
- For $k \geq 0$, we set $z_i = \gamma(t_i)$ for $i = 1, \cdots, k$.
- One may write

$$\mathrm{Li}_k^{\gamma}(z) = \int_{0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1} \frac{dz_1}{1 - z_1} \frac{dz_2}{z_2} \cdots \frac{dz_k}{z_k}$$

- One may obtain this integral from a period datum as follows:
- One first considers the variety $Y = X^k$ and the divisor $D = \{z_1 = 0\} \cup \{z_1 = z_2\} \cup \{z_2 = z_3\} \cup \cdots \cup \{z_{k-1} = z_k\} \cup \{z_k = z\}$.
- To deal with the improperness of the integral, one must use a blowup procedure to resolve the singularity in the integrand.
- The latter follows from Deligne's theory of tangential basepoints.

Relations Between Periods

One may deduce relations between periods using the following rules:

- Linearity: $I(Y, D, \omega, \gamma)$ is linear in ω and γ.
- Algebraic Change of Variables: If $f\colon (Y_1, D_1) \to (Y_2, D_2)$ is a morphism of pairs over $\mathbb{Q}$, $f^*(\omega_2) = \omega_1$, and $f_*(\gamma_1) = \gamma_2$, then
$$I(Y_1, D_1, \omega_1, \gamma_1) = I(Y_2, D_2, \omega_2, \gamma_2).$$
- Stokes' Theorem: Let (Y, D) be a pair as above, $\tilde{D}$ the normalization of D, and $\tilde{D}_1$ the divisor with normal crossings in $\tilde{D}$ coming from double points in D. If $\beta \in \Omega^{d-1}(\tilde{D})$ and $\gamma \in H_d(Y(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$, then
$$I(Y, D, d\beta, \gamma) = I(\tilde{D}, \tilde{D}_1, \beta, \delta\gamma),$$
where $\delta\colon H_d(Y(\mathbb{C}), D(\mathbb{C}); \mathbb{Q}) \to H_{d-1}(\tilde{D}(\mathbb{C}), \tilde{D}_1(\mathbb{C}); \mathbb{Q})$ is the boundary map.

For example, one can deduce $6\zeta(2) = \pi^2$ in this way (see the Kontsevich-Zagier article mentioned in the last slide).

Motivic Periods

Defintion

The ring $\mathcal{P}^{\mathfrak{m},+}$ of *effective motivic periods* is the formal $\mathbb{Q}$-algebra generated by period data, modulo the relations in the previous slide.

There is then a natural map $I\colon \mathcal{P}^{\mathfrak{m},+} \to \mathbb{C}$ given by integration.

Conjecture (Kontsevich-Zagier)

The map $I\colon \mathcal{P}^{\mathfrak{m},+} \to \mathbb{C}$ is injective.

Examples

We denote the corresponding "motivic special values" by $(2\pi i)^{\mathfrak{m}}, \log^{\mathfrak{m}}(r)$, $\zeta^{\mathfrak{m}}(k)$, $\mathrm{Li}_k^{\mathfrak{m}}(r)$, $\cdots$

However, we would like something more adapted to Coleman integration.

De Rham Periods

Coleman integrals use de Rham cohomology (specifically, the Frobenius and Hodge filtration) but not Betti cohomology. We therefore need:

Definition

The ring $\mathcal{P}^{\mathfrak{dr},+}$ of *effective de Rham periods* is a variant of $\mathcal{P}^{\mathfrak{m},+}$ in which γ represents a (relative) de Rham homology class.

Examples

We similarly write $\log^{\mathfrak{dr}}(r)$, $\zeta^{\mathfrak{dr}}(k)$, $\mathrm{Li}_k^{\mathfrak{dr}}(r), \cdots$ where for γ we take the "canonical de Rham path" between any two points.

Fact

There is a subring $\mathcal{P}^{\mathfrak{dr},+}_{\mathcal{MT}}(Z) \subseteq \mathcal{P}^{\mathfrak{dr},+}$ of effective mixed Tate de Rham periods over Z that contains all periods coming from unirational pairs (Y, D) with good reduction over Z.

Mixed Tate Periods

- From now on, we assume that $\mathrm{Frac}(Z) = \mathbb{Q}$.
- Furthermore, the Coleman version of $\zeta(2)$, written $\zeta^p(2)$, is zero.

Our Motivic Periods

We will therefore work with

$$\mathcal{O}(\pi_1^{\mathrm{un}}(Z)) := \mathcal{P}_{\mathcal{MT}}^{\mathfrak{dr},+}(Z)/\zeta^{\mathfrak{dr}}(2)$$

- We let $\zeta^{\mathfrak{u}}(r)$, $\log^{\mathfrak{u}}(r)$, $\zeta^{\mathfrak{u}}(k)$, $\mathrm{Li}_k^{\mathfrak{u}}(r), \cdots$ be the images in $\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$ of the corresponding elements of $\mathcal{P}_{\mathcal{MT}}^{\mathfrak{dr},+}(Z)$.
- Coleman integration gives a map $\mathrm{per}_{\mathfrak{p}} \colon \mathcal{O}(\pi_1^{\mathrm{un}}(Z)) \to \mathbb{Q}_p$ for $p \in \mathrm{S}pec(Z)$.
- An inclusion $Z \subseteq Z'$ induces an inclusion $\mathcal{O}(\pi_1^{\mathrm{un}}(Z)) \subseteq \mathcal{O}(\pi_1^{\mathrm{un}}(Z'))$.
- In particular, $\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{Q}))$ is the union of $\mathcal{O}(\pi_1^{\mathrm{un}}((\mathbb{Z}[1/S]))$ over all S.

Why Motivic Periods

The reason working with motivic periods rather than ordinary periods is useful is that they have a nice algebraic structure.

Fact (Deligne, Goncharov, Voevodsky, Borel, ...)

$\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$ has the structure of a graded Hopf algebra, and as such is abstractly isomorphic to an explicit free shuffle algebra.Assuming $\mathrm{Frac}(Z) = \mathbb{Q}$, it is the free shuffle algebra

$$\mathbb{Q}\langle\{\{\tau_p\}_{p\in S}, \{\sigma_{2n+1}\}_{n\geq 1}\}\rangle,$$

where each τ_p has degree 1, and σ_{2n+1} has degree $2n+1$.

As a graded vector space, it's the free non-commutative algebra in these generators. However, it's equipped with a commutative product denoted by Ш, which we describe more precisely in the next slide.

Free Shuffle Algebras

Let I be an index set and $d\colon I \to \mathbb{Z}$ a function, and let $\mathbb{Q}\langle\{x_i\}_{i\in I}\rangle$ be the graded vector space underlying the free non-commutative graded algebra over $\mathbb{Q}$ in the variables x_i, where x_i has degree $d(i)$.

We define a graded Hopf algebra structure on $\mathbb{Q}\langle\{x_i\}_{i\in I}\rangle$ as follows:

- The group $\text{Ш}(r,s)$ is the set of permutations σ of $\{1,2,\cdots,r+s\}$ such that $\sigma(1)<\sigma(2)<\cdots<\sigma(r)$ and $\sigma(r+1)<\sigma(r+2)<\cdots<\sigma(r+s)$.
- The product of two words w_1, w_2 in the x_i's is given as follows:

$$w_1 \text{Ш} w_2 := \sum_{\sigma\in\text{Ш}(\ell(w_1),\ell(w_2))} \sigma(w_1 w_2),$$

  where ℓ denotes length, and $w_1 w_2$ denotes concatenation.
- The coproduct Δ is given by

$$\Delta w := \sum_{w_1 w_2 = w} w_1 \otimes w_2$$

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Definition

Let $\pi_1^{\mathrm{PL}}(X) := Spec(\mathbb{Q}[\log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \mathrm{Li}_2^{\mathfrak{u}}, \cdots])$.

- As before, let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let $z \in X(\mathbb{Q})$. For each integer k, using the representation of the kth polylogarithm as an iterated integral, one can define a motivic period $\mathrm{Li}_k^{\mathfrak{u}}(z) \in \mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{Q}))$.
- It follows that each $z \in X(\mathbb{Q})$ defines a homomorphism $\kappa(z)\colon \mathcal{O}(\pi_1^{\mathrm{PL}}(X)) \to \mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{Q}))$ sending $\mathrm{Li}_k^{\mathfrak{u}}$ to $\mathrm{Li}_k^{\mathfrak{u}}(z)$.

Fact

$z \in X(Z)$ iff $\mathrm{Image}(\kappa(z)) \subseteq \mathcal{O}(\pi_1^{\mathrm{un}}(Z))$

Hopf Algebra Structure

There is furthermore a graded Hopf algebra structure on $\mathcal{O}(\pi_1^{\mathrm{PL}}(X))$, in which $\mathrm{Li}_k^{\mathfrak{u}}$ has degree k, $\log^{\mathfrak{u}}$ has degree 1, and the reduced coproduct Δ' is given by:

$$\Delta' \mathrm{Li}_k^{\mathfrak{u}} = \sum_{i=1}^{k-1} \mathrm{Li}_{k-i}^{\mathfrak{u}} \otimes \frac{(\log^{\mathfrak{u}})^i}{i!}.$$

Fact

For $z \in X(\mathbb{Q})$, the homomorphism $\kappa(z)$ is a homomorphism of graded Hopf algebras.

- In particular,

$$\Delta' \mathrm{Li}_k^{\mathfrak{u}}(z) = \sum_{i=1}^{k-1} \mathrm{Li}_{k-i}^{\mathfrak{u}}(z) \otimes \frac{(\log^{\mathfrak{u}}(z))^i}{i!}.$$

Motivic Kim's Cutter

For a prime p, this gives us a diagram:

$$\begin{tikzcd} X(Z) \arrow[r] \arrow[d, "\kappa"'] & X(\mathbb{Z}_p) \arrow[d] \\ \mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathcal{O}(\pi_1^{\mathrm{un}}(Z))) \arrow[r, "\mathrm{per}_{\mathfrak{p}}"] & \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathbb{Q}_p) \end{tikzcd}$$

- We recall the integration map $\mathrm{per}_{\mathfrak{p}} \colon \mathcal{O}(\pi_1^{\mathrm{un}}(Z)) \to \mathbb{Q}_p$ for $p \in Spec(Z)$.
- This induces $\mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathcal{O}(\pi_1^{\mathrm{un}}(Z))) \xrightarrow{\mathrm{per}_{\mathfrak{p}}} \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathbb{Q}_p)$.
- In addition, an arbitrary $z \in X(\mathbb{Z}_p)$ induces a homomorphism $\mathcal{O}(\pi_1^{\mathrm{PL}}(X)) \to \mathbb{Q}_p$ sending $\mathrm{Li}_k^{\mathfrak{u}}$ to $\mathrm{Li}_k^p(z)$.

Motivic Kim's Cutter, cont.

$$\begin{tikzcd} X(Z) \arrow[r] \arrow[d, "\kappa"'] & X(\mathbb{Z}_p) \arrow[d] \\ \mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathcal{O}(\pi_1^{\mathrm{un}}(Z))) \arrow[r, "\mathrm{per}_{\mathfrak{p}}"] & \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathbb{Q}_p) \end{tikzcd}$$

- The above diagram is known as *Kim's Cutter*.
- We may upgrade the bottom horizontal morphism to a map of schemes, as follows:
- We define a scheme $Z_{\mathrm{PL}}^{1,\mathbb{G}_m}$ over $\mathbb{Q}$ by $Z_{\mathrm{PL}}^{1,\mathbb{G}_m}(R) = \mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)), \mathcal{O}(\pi_1^{\mathrm{un}}(Z))) \otimes R)$ for a $\mathbb{Q}$-algebra R.
- The bottom arrow may then be viewed as a map of $\mathbb{Q}_p$-schemes

$$Z_{\mathrm{PL}}^{1,\mathbb{G}_m} \otimes \mathbb{Q}_p \to \pi_1^{\mathrm{PL}}(X) \otimes \mathbb{Q}_p$$

Motivic Kim's Cutter, cont.

$$\begin{tikzcd} X(Z) \arrow[r] \arrow[d, "\kappa"'] & X(\mathbb{Z}_p) \arrow[d] \\ Z_{\mathrm{PL}}^{1,\mathbb{G}_m} \otimes \mathbb{Q}_p \arrow[r, "\mathrm{per}_{\mathfrak{p}}"] & \pi_1^{\mathrm{PL}}(X) \otimes \mathbb{Q}_p \end{tikzcd}$$

- Dimension counts show that the bottom horizontal arrow is non-dominant, which is what proves Siegel's theorem.
- Therefore, there is a nonzero ideal $\mathcal{I}_{\mathrm{PL}}^{Z} \subseteq \mathcal{O}(\pi_1^{\mathrm{PL}}(X)) \otimes \mathbb{Q}_p$ vanishing on the image of the bottom arrow, known as the *(polylogarithmic) Chabauty-Kim ideal*.
- The right-hand vertical map is Coleman analytic, so elements of $\mathcal{I}_{\mathrm{PL}}^{Z}$ pull back to Coleman functions on $X(\mathbb{Z}_p)$ that vanish on $X(Z)$.

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The Chabauty-Kim Ideal

- General goal: Compute elements of $\mathcal{I}_{\mathrm{PL}}^{Z}$ and verify cases of Kim's conjecture, i.e., that they suffice to precisely determine the integral points.

Theorem (Dan-Cohen, Wewers, 2013)

For $Z = \mathbb{Z}[1/2]$ and $p \neq 2$, the following Coleman function is in $\mathcal{I}_{\mathrm{PL}}^{Z}$:

$$24 \log^p(2)\zeta^p(3)\mathrm{Li}_4^p(z) + \frac{8}{7}\left(\log^p(2)^4 + 24\mathrm{Li}_4^p(\frac{1}{2})\right)\log^p(z)\mathrm{Li}_3^p(z)$$
$$+\left(\frac{4}{21}\log^p(2)^4 + \frac{32}{7}\mathrm{Li}_4^p(\frac{1}{2}) + \log^p(2)\zeta^p(3)\right)\log^p(z)^3\log^p(1-z)$$

In 2015, Dan-Cohen posted a preprint showing that this could be made into an algorithm, whose halting is conditional on certain well-known conjectures.

Our Current Work

- Our current work revolves around improving the algorithm, extending to multiple polylogarithms, and verifying cases of Kim's conjecture.
- We need to use an explicit description of $Z_{\mathrm{PL}}^{1,\mathbb{G}_m}$, which is known as *the space of cocycles*.
- We can compute the former in terms of any good *abstract* basis of $\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$, i.e., that expresses it as a free shuffle algebra.
- We then need to apply $\mathrm{per}_{\mathfrak{p}}$, for which we must compute a good basis of $\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$ (up to a certain degree) as linear combinations of explicit polylogarithms of the form $\mathrm{Li}_k^{\mathfrak{u}}(z)$ for $z \in X(\mathbb{Q})$ and $k \geq 0$.
- More specifically, we are working on $Z = \mathbb{Z}[1/3]$ and $Z = \mathbb{Z}[1/6]$.

Understanding $Z^{1,\mathbb{G}_m}_{\mathrm{PL}}$

Fix an arbitrary cocycle $c \in Z^{1,\mathbb{G}_m}_{\mathrm{PL}}$. For each nonnegative integer k and each word w in $\Sigma := \{\{\tau_p\}_{p\in S}, \{\sigma_{2n+1}\}_{n\geq 1}\}$ of length k (or 1 if $k=0$), let

$$\phi_k^w(c) \in \mathbb{Q}$$

denote the associated matrix entry of c, so that in the notation above, we have

$$c(\mathrm{Li}^{\mathfrak{u}}_k) = \sum_w \phi_k^w(c) w.$$

Then for $0 \leq r \leq k$, $\tau_1, \ldots, \tau_r \in \Sigma$ of degree 1, and $\sigma \in \Sigma$ of degree $k-r$, we have

$$\phi_k^{\sigma\tau_1\cdots\tau_r}(c) = \phi_0^{\tau_1}(c)\cdots\phi_0^{\tau_r}(c)\phi^{\sigma}_{k-r}(c),$$

and all other coefficients vanish.

Computing the Image of the Bottom Arrow in Kim's Cutter

- We write $\pi_1^{\mathrm{un}}(Z) := Spec(\mathcal{O}(\pi_1^{\mathrm{un}}(Z)))$. We note that a cocycle is just a homomorphism of group schemes $\pi_1^{\mathrm{un}}(Z) \to \pi_1^{\mathrm{PL}}(X)$.
- It is conjectured that $\mathrm{per}_{\mathfrak{p}}$ is injective. This means we can focus instead on computing the image of

$$Z_{\mathrm{PL}}^{1,\mathbb{G}_m} \times \pi_1^{\mathrm{un}}(Z) \to \pi_1^{\mathrm{PL}}(X) \times \pi_1^{\mathrm{un}}(Z),$$

  where the map is cocycle evaluation times the identity.
- For any n, one may replace $\pi_1^{\mathrm{PL}}(X)$ by its finite-dimensional graded quotient $\pi_1^{\mathrm{PL}}(X)_{\geq -n} := Spec(\mathbb{Q}[\log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \cdots, \mathrm{Li}_n^{\mathfrak{u}}])$.
- We may similarly write $Z_{\mathrm{PL},n}^{1,\mathbb{G}_m}(R) = \mathrm{Hom}_{\mathrm{GrHopf}}(\mathcal{O}(\pi_1^{\mathrm{PL}}(X)_{\geq -n}), \mathcal{O}(\pi_1^{\mathrm{un}}(Z))) \otimes R)$.
- We are then reduced to computing the image of a map between finite-dimensional varieties over the function field of $\pi_1^{\mathrm{un}}(Z)$, meaning we can algorithmically compute its image.

Basis Computations for $A(\mathbb{Z}[1/6])$

- To simplify notation, we let $A = A(\mathbb{Z}[1/6])$ denote $\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{Z}[1/6]))$. We let A_n denote the nth graded piece.
- The abstract description as a free shuffle algebra shows that $\dim(A_0) = 1$, $\dim(A_1) = 2$, $\dim(A_2) = 4$, $\dim(A_3) = 9$, and $\dim(A_4) = 20$.
- In fact, A is a free polynomial algebra on infinitely many generators, so we only need to find such generators. There are two in degree 1, one in degree 2, three in degree 3, and five in degree 4.
- Basic tool: use the reduced coproduct Δ'. It's injective in degrees 2 and 4 and has a kernel of dimension one in degree 3, generated by $\zeta^{\mathfrak{u}}(3)$.
- Procedure: Inductively on n, write down motivic periods of the form $\mathrm{Li}_n^{\mathfrak{u}}(z)$ for $z \in X(Z)$, apply Δ', check dependence in lower degree.
- The non-injectivity of Δ' for $n = 3$ requires use of p-adic approximation to determine rational multiples of $\zeta^{\mathfrak{u}}(3)$.

Basis Computations for $A(\mathbb{Z}[1/3])$

- As $X(\mathbb{Z}[1/3]) = \emptyset$, we cannot use periods of the form $\mathrm{Li}^{\mathfrak{u}}_k(z)$ for $z \in X(\mathbb{Z}[1/3])$.
- Instead, we must find elements of $A(\mathbb{Z}[1/3])$ as special elements of $A(\mathbb{Z}[1/6])$.
- Using the abstract description of $A(\mathbb{Z}[1/6])$ as $\mathbb{Q}\langle\{\{\tau_p\}_{p|6}, \{\sigma_{2n+1}\}_{n\geq 1}\}\rangle$, the subring $A(\mathbb{Z}[1/3])$ corresponds to the vector space generated by those words without any τ_2's.
- More specifically, we have found that:

$$\sigma_3\tau_3 = \frac{18}{13}\mathrm{Li}^{\mathfrak{u}}_3(3) - \frac{3}{52}\mathrm{Li}^{\mathfrak{u}}_3(9)$$

- This has allowed us to find a new element of $\mathcal{I}^Z_{\mathrm{PL}}$ for $Z = \mathbb{Z}[1/3]$. We have used it to verify Kim's conjecture in this case for $p = 5, 7$.

Useful References

The following are on arXiv:

- Mixed Tate Motives and the Unit Equation, Ishai Dan-Cohen and Stefan Wewers
- Mixed Tate Motives and the Unit Equation II, Ishai Dan-Cohen
- Single-Valued Motivic Periods, Francis Brown
- Motivic Periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Francis Brown
- Notes on Motivic Periods, Francis Brown
- Integral Points on Curves and Motivic Periods, Francis Brown

Our definition of motivic periods comes from Periods, Kontsevich and Zagier (http://www.maths.ed.ac.uk/%7eaar/papers/kontzagi.pdf).

Thank You!