Problem 1

(10pts)

Let $M \subset \mathbb{R}^{n+1}$ be an $n$-dimensional $C^1$ embedded submanifold with a $C^0$ unit normal $n : M \to S^n$. (Thus, $M$ is orientable.) Prove that, for every $p \in M$,

$$T_p M = T_{n(p)} S^n$$

as subspaces of $\mathbb{R}^{n+1}$.

Note: Thus, if $M$ is $C^2$, then $dn|_p : T_p M \to T_{n(p)} S^n = T_p M$, as indicated in lecture.

Problem 2

(40(+40)pts: 15+25(+40))

For each of the following sets $M \subset \mathbb{R}^{n+1}$, do the following things:

- Prove that $M$ is an $n$-dimensional $C^2$ embedded submanifold of $\mathbb{R}^{n+1}$. (In fact, they are all smooth; i.e., $C^k$ for all $k$.)

- Prove that $M$ is orientable by writing down a continuous unit normal $n : M \to S^n$ to $M$. (Of course, $n$ will be smooth.)

- For your particular choice of $n$, compute the shape operator $dn|_p : T_p M \to T_p M$ at a general point $p \in M$; i.e., pick a convenient basis for $T_p M$ and write down how the linear map $dn|_p$ acts on this basis (i.e., write down the matrix for the linear map relative to your basis).

For full credit, do not just pick your favorite $p \in M$, but handle all $p \in M$.

(a) $M = \{ x \in \mathbb{R}^{n+1} : x \cdot e = c \}$ for fixed $e \in \mathbb{R}^{n+1} \setminus \{0\}$, $c \in \mathbb{R}$. (“Hyperplane.”)

(b) $M = S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \subset \mathbb{R}^{n+1}$. (“Unit $n$-sphere.”)

(c) (Extra credit) $M = S^1 \times \mathbb{R} = \{ (x, y) \in \mathbb{R}^2 \times \mathbb{R} : x \in S^1, y \in \mathbb{R} \} \subset \mathbb{R}^3$. (“Product cylinder.”)
Problem 3

(25pts)

In this problem, you will prove that every embedded submanifold looks like a graph if you restrict to a small enough neighborhood.

Suppose that $M \subset \mathbb{R}^{n+m}$ is an $n$-dimensional smoothly embedded submanifold. Without loss of generality, assume that $0 \in M$. Prove that there exists a permutation $\sigma \in S_{n+m}$ of $\{1, \ldots, n+m\}$, an open set $U \subset \mathbb{R}^{n+m}$ containing $0$, an open set $W \subset \mathbb{R}^n$ containing $0 \in \mathbb{R}^n$, and a smooth $G : W \to \mathbb{R}^m$ such that $G(0) = 0$ and

\[ M \cap U = \{(x_1, \ldots, x_{n+m}) \in \mathbb{R}^{n+m} : (x_{\sigma(n+1)}, \ldots, x_{\sigma(n+m)}) = G(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \text{ and } (x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \in W \} \cap U. \]

(Hint: Inverse function theorem.)

Problem 4

(130pts: 10+15+15+20+30+40)

In this problem, we compute the unit normal, the second fundamental form, the shape operator, and the mean curvature of graphical $n$-dimensional submanifolds of $\mathbb{R}^{n+1}$. This is largely (if not entirely) a linear algebra exercise. For notational brevity, throughout this problem we write $\partial_i$ for $\partial/\partial x_i$.

Let $U \subset \mathbb{R}^n$ be a nonempty open set, and let $f : U \to \mathbb{R}$ be smooth. We call $M := \{(x, f(x)) : x \in U \} \subset \mathbb{R}^{n+1}$ the “graph of $f$.” Note that $M$ is the image of the so-called “graph map” $F : U \to \mathbb{R}^{n+1}$ given by $F(x) := (x, f(x))$.

(a) Prove, directly, that $F$ is an injective immersion and that the inverse of $F : U \to F(U)$ is continuous. (Thus, $M$ is an $n$-dimensional embedded submanifold that is globally parametrized by a single map, $F$.)

Recall that, for every $x \in U$, the tangent space of $M$ at $F(x)$ is the image of the derivative matrix $DF(x) : \mathbb{R}^n \to \mathbb{R}^{n+1}$. Since $F$ is an immersion, $T_{F(x)}M$ is an $n$-dimensional subspace of $\mathbb{R}^{n+1}$ with basis vectors $\partial_i F(x) = e_i + (\partial_i f(x)) e_{n+1}$, $i = 1, \ldots, n$.

(b) Compute, in terms of $f$, the $n \times n$ matrix $(g_{ij}(x))_{i,j=1,\ldots,n}$ given by:

\[ g_{ij}(x) := \partial_i F(x) \cdot \partial_j F(x), \quad x \in U, \quad i, j = 1, \ldots, n. \]

Then, show that $(g_{ij}(x))_{i,j=1,\ldots,n}$ is invertible, with inverse $(g^{ij}(x))_{i,j=1,\ldots,n}$ given by

\[ g^{ij}(x) = \delta_{ij} - \frac{(\partial_i f(x))(\partial_j f(x))}{1 + |\nabla f(x)|^2}, \]

where $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and 0 otherwise).
(c) Compute, in terms of \( f \), the (pullback, under \( F \), of the) downward pointing unit normal of \( M \), i.e., compute \( n \circ F : U \to \mathbb{R}^{n+1} \). That is, for \( x \in U \), find the unique downward pointing unit vector orthogonal to \( T_{F(x)}M \), and call it \((n \circ F)(x)\).

(d) Compute, in terms of \( f \), the \( n \times n \) matrix \((A_{ij}(x))_{i,j=1,...,n}\) given by

\[
A_{ij}(x) := -\partial_i (n \circ F)(x) \cdot \partial_j F(x).
\]

This \( n \times n \) matrix is called the **second fundamental form** of \( M \) (associated to \( n \)) at \( F(x) \), with respect to the basis \((\partial_i F(x))_{i=1,...,n}\). Note: \((A_{ij}(x))_{i,j=1,...,n}\) is not the matrix representing the linear transformation \( d(n|_{F(x)}): \mathbb{R}^n \to T_{F(x)}M \) with respect to the basis \((\partial_i F(x))_{i=1,...,n}\), unless the basis is orthonormal.

(e) Compute, in terms of \( f \), the **shape operator** \( d(n|_{F(x)}): T_{F(x)}M \to T_{F(x)}M \). Specifically, since this is a linear endomorphism of \( T_{F(x)}M \), write down the associated linear transformation matrix relative to our distinguished basis \((\partial_i F(x))_{i=1,...,n}\) of \( T_{F(x)}M \); i.e., write

\[
d(n|_{F(x)})(\partial_i F(x))
\]

as a linear combination of \( \partial_1 F(x), \ldots, \partial_n F(x) \), and assemble the corresponding \( n \times n \) matrix that represents the linear map \( d(n|_{F(x)}): T_{F(x)}M \to T_{F(x)}M \). (Hint: \( b \), \( d \).)

(f) The **mean curvature** \( H(x) \) of \( M \) at \( F(x) \) is the trace (sum of the eigenvalues) of the shape operator. Show:

\[
H(x) = \sum_{i,j=1}^{n} g^{ij}(x) A_{ij}(x) = \text{div} \left( \frac{\nabla f(x)}{\sqrt{1 + |\nabla f(x)|^2}} \right).
\]

**Problem 5 – extra credit**

\( (50\text{pts}) \)

In this problem you will prove the change of parameters proposition.

Suppose \( U_i \) are nonempty open subsets of \( \mathbb{R}^n \) and that \( F_i: U_i \to \mathbb{R}^{n+m}, i = 1, 2 \), are two \( k \)-times (\( k \geq 1 \)) continuously differentiable injective immersions with \( F_1(U_1) = F_2(U_2) = W \subset \mathbb{R}^{n+m} \), whose inverses \( F_i^{-1}: W \to U_i, i = 1, 2 \), are continuous.

Prove that \( F_2^{-1} \circ F_1: U_1 \to U_2 \) is \( k \)-times continuously differentiable.

**Hint**: Problem 3.