18.994 - Problem Set 2

Due: February 18, 2020

Reading: Notation from analysis

Let $\Omega \subset \mathbb{R}$ be an interval, possibly $\Omega = \mathbb{R}$. For continuous $f : \Omega \to \mathbb{R}$, and $\alpha \in (0,1]$, we denote

$$
\|f\|_{C^\alpha(\Omega;\mathbb{R})} := \sup_{x \in \Omega} |f(x)|,
$$

and

$$
[f]_{C^\alpha(\Omega;\mathbb{R})} := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in \Omega, x \neq y \right\}.
$$

If $k \in \mathbb{N}$ and $f : \Omega \to \mathbb{R}$ is $k$-times continuously differentiable, we denote the $\ell$-th derivative of $f$, where $\ell = 0, 1, \ldots, k$, by $f^{(\ell)} : \Omega \to \mathbb{R}$. One often writes $f'$, $f''$, etc. for $f^{(1)}$, $f^{(2)}$, etc. We denote

$$
\|f\|_{C^k(\Omega;\mathbb{R})} := \sum_{\ell=0}^k \|f^{(\ell)}\|_{C^\alpha(\Omega;\mathbb{R})},
$$

and

$$
\|f\|_{C^{k,\alpha}(\Omega;\mathbb{R})} := \|f\|_{C^k(\Omega;\mathbb{R})} + [f^{(k)}]_{C^\alpha(\Omega;\mathbb{R})}.
$$

Slightly abusing notation, it is sometimes convenient to denote $\|f\|_{C^{k,\alpha}(\Omega;\mathbb{R})} := \|f\|_{C^{k,\alpha}(\Omega;\mathbb{R})}$. Finally, if $f, f_j : \Omega \to \mathbb{R}$, $j = 1, 2, \ldots$, are $k$-times continuously differentiable functions, then we say that $f_j \to f$ in $C^k$ (resp. $C^{k,\alpha}$) on $\Omega$ when $\|f_j - f\|_{C^k(\Omega;\mathbb{R})}$ (resp. $\|f_j - f\|_{C^{k,\alpha}(\Omega;\mathbb{R})}$) $\to 0$ as $j \to \infty$.

(N.b. These norms can be $\infty$ even when $f$ is smooth, i.e., infinitely differentiable.)

Let’s extend this to $\mathbb{R}^n$-valued functions, too. Any $f : \Omega \to \mathbb{R}^n$ can be written out in component form as $f = (f^1, \ldots, f^n)$, where $f^i : \Omega \to \mathbb{R}$ for $i = 1, \ldots, n$. Suppose $f$ is $k$-times continuously differentiable (i.e., its components $f^i$ are $k$-times continuously differentiable). We denote the $\ell$-th derivative of $f$ by $f^{(\ell)} : \Omega \to \mathbb{R}^n$, where $f^{(\ell)} := ((f^1)^{(\ell)}, \ldots, (f^n)^{(\ell)})$. We often write $f'$, $f''$, etc. for $f^{(1)}$, $f^{(2)}$, etc. Finally, we set

$$
\|f\|_{C^k(\Omega;\mathbb{R}^n)} := \sum_{i=1}^n \|f^i\|_{C^k(\Omega;\mathbb{R})},
$$

and, abusing notation again, $\|f\|_{C^{k,\alpha}(\Omega;\mathbb{R}^n)} := \|f\|_{C^k(\Omega;\mathbb{R}^n)}$. If $f, f_j : \Omega \to \mathbb{R}^n$, $j = 1, 2, \ldots$, are $k$-times continuously differentiable functions, then we say that $f_j \to f$ in $C^k$ (resp. $C^{k,\alpha}$) on $\Omega$ when $\|f_j - f\|_{C^k(\Omega;\mathbb{R}^n)}$ (resp. $\|f_j - f\|_{C^{k,\alpha}(\Omega;\mathbb{R}^n)}$) $\to 0$ as $j \to \infty$. 

1
Problem 1

(25pts: 5+5+10+5)

In this problem, we will set up our $C^k$ and $C^{k, \alpha}$ spaces on the unit circle.

It feels geometrically unsatisfactory to parametrize closed curves by $\mathbb{R}^2$-valued functions on intervals (i.e., $[0, 2\pi R_0]$), as we did, e.g., in Problem 3 of Problem Set 1. On the other hand, it tends to be analytically convenient to do so, since parametrizing with intervals means we can write down explicit formulas without any fear of confusion. With a bit of thought, one can have the best of both worlds by identifying spaces of functions on the unit circle with suitable spaces of \textit{periodic} functions on $\mathbb{R}$. The point is that the prior are convenient geometrically (e.g., when studying closed curves), while the latter are convenient analytically (e.g., to invoke tools from single variable calculus, or to write down formulas, as you inevitably did in Problem 3 of Problem Set 1).

Consider the mapping $\sigma : \mathbb{R} \to \mathbb{R}^2$,

$$\sigma(\theta) := (\cos \theta, \sin \theta),$$

whose image (the unit circle in $\mathbb{R}^2$) we denote $S^1$. For $n \in \mathbb{N} \setminus \{0\}$, we denote:

$$C^k_{\text{per}}([0, 2\pi]; \mathbb{R}^n) := \{k\text{-times continuously differentiable } f : [0, 2\pi] \to \mathbb{R}^n \text{ such that } f^{(\ell)}(0) = f^{(\ell)}(2\pi) \forall \ell = 0, 1, \ldots, k\},$$

$$C^k(S^1; \mathbb{R}^n) := \{f : S^1 \to \mathbb{R}^n \text{ such that } f \circ \sigma \text{ is } k\text{-times continuously differentiable}\}.$$

We will write $\|f\|_{C^k(S^1; \mathbb{R}^n)} := \|f \circ \sigma\|_{C^k(\mathbb{R}; \mathbb{R}^n)}$ whenever $f \in C^k(S^1; \mathbb{R}^n)$. Note (without proof):

- For $f \in C^k(S^1; \mathbb{R}^n)$, $\ell = 0, 1, \ldots, k$, there is uniquely defined “$\ell$-th derivative” $f^{(\ell)} : S^1 \to \mathbb{R}^n$ so that $f^{(\ell)} \circ \sigma = (f \circ \sigma)^{(\ell)}$.

Now:

(a) If $f \in C^k_{\text{per}}([0, 2\pi]; \mathbb{R}^n)$, then verify that $\|f\|_{C^k_{\text{per}}([0, 2\pi]; \mathbb{R}^n)} < \infty$.
(b) Construct a (“natural”) \textit{linear bijection} $\iota_k : C^k(S^1; \mathbb{R}^n) \to C^k_{\text{per}}([0, 2\pi]; \mathbb{R}^n)$ such that

$$\|\iota_k(f)\|_{C^k_{\text{per}}([0, 2\pi]; \mathbb{R}^n)} = \|f\|_{C^k(S^1; \mathbb{R}^n)}.$$

(In functional analysis, such a $\iota_k$ is called an \textit{isometry}.) This obviously implies $\|f\|_{C^k(S^1; \mathbb{R}^n)} < \infty$ for all $f \in C^k(S^1; \mathbb{R}^n)$.

Now also fix $\alpha \in (0, 1]$. We denote:

$$C^k_{\text{per}}([0, 2\pi]; \mathbb{R}^n) := \{k\text{-times continuously differentiable } f : [0, 2\pi] \to \mathbb{R}^n \text{ such that } f^{(\ell)}(0) = f^{(\ell)}(2\pi) \forall \ell = 0, 1, \ldots, k, \text{ and } \|f\|_{C^{k, \alpha}([0, 2\pi]; \mathbb{R}^n)} < \infty\},$$

$$C^{k, \alpha}(S^1; \mathbb{R}^n) := \{f : S^1 \to \mathbb{R}^n \text{ such that } f \circ \sigma \text{ is } k\text{-times continuously differentiable and } \|f \circ \sigma\|_{C^{k, \alpha}(\mathbb{R}; \mathbb{R}^n)} < \infty\}.$$

We will write $\|f\|_{C^{k, \alpha}(S^1; \mathbb{R}^n)} := \|f \circ \sigma\|_{C^{k, \alpha}(\mathbb{R}; \mathbb{R}^n)}$ whenever $f \in C^{k, \alpha}(S^1; \mathbb{R}^n)$. Note (without proof):
• By (a), (b), \( C_{\text{per}}^k([0, 2\pi]; \mathbb{R}^n) = C_{\text{per}}^{k,0}([0, 2\pi]; \mathbb{R}^n) \) and \( C^k(S^1; \mathbb{R}^n) = C^{k,0}(S^1; \mathbb{R}^n) \), where the left sides are the spaces defined at beginning of this problem, and the right sides are the spaces defined just now, with \( \alpha = 0 \). (Recall the notation convention we’re using when \( \alpha = 0 \).

• \( C_{\text{per}}^{k,\alpha}([0, 2\pi]; \mathbb{R}^n) \subseteq C_{\text{per}}^{\ell,\beta}([0, 2\pi]; \mathbb{R}^n) \) and \( C^{k,\alpha}(S^1; \mathbb{R}^n) \subseteq C^{\ell,\beta}(S^1; \mathbb{R}^n) \) whenever \( k, \ell \in \mathbb{N}, \alpha, \beta \in [0, 1], \) and \( k + \alpha > \ell + \beta \).

• \( C_{\text{per}}^{k+1,0}([0, 2\pi]; \mathbb{R}^n) \subsetneq C_{\text{per}}^{k+1,0}([0, 2\pi]; \mathbb{R}^n) \) and \( C^{k+1,0}(S^1; \mathbb{R}^n) \subsetneq C^{k+1}(S^1; \mathbb{R}^n) \).

Next:
(c) Construct a (“natural”) linear bijection \( t_{k,\alpha} : C^{k,\alpha}(S^1; \mathbb{R}^n) \rightarrow C_{\text{per}}^{k,\alpha}([0, 2\pi]; \mathbb{R}^n) \) such that
\[
\| t_{k,\alpha}(f) \|_{C_{\text{per}}^{k,\alpha}([0, 2\pi]; \mathbb{R}^n)} \leq \| f \|_{C^{k,\alpha}(S^1; \mathbb{R}^n)} \leq c_{k,\alpha} \| t_{k,\alpha}(f) \|_{C_{\text{per}}^{k,\alpha}([0, 2\pi]; \mathbb{R}^n)},
\]
for some \( c_{k,\alpha} \geq 1 \) independent of \( f \). (In functional analysis, such a \( t_{k,\alpha} \) is called an isomorphism.)

(d) When can the uniform constant \( c_{k,\alpha} \geq 1 \) be taken to equal 1?

Note (again, without proof):
• Using \( t_k, t_{k,\alpha} \), we can extend our \( C^k \) and \( C^{k,\alpha} \) convergence notions to functions on \( S^1 \).

• Your \( t_k \)'s and \( t_{k,\alpha} \)'s should be “natural” in the sense that the inclusion diagrams coming from \( C^k \subset C^\ell \) \( (k > \ell) \) and \( C^{k,\alpha} \subset C^{\ell,\beta} \) \( (k + \alpha > \ell + \beta) \) had better commute.

Problem 2

\( 20 \text{pts: } 15 + 5 \)

We deduce the implications of Arzelà–Ascoli on our \( C^{k,\alpha}(S^1; \mathbb{R}^n) \) spaces. Suppose that \( \{f_j\}_{j=1,2,...} \subseteq C^{k,\alpha}(S^1; \mathbb{R}^n) \) is a sequence with
\[
\sup_{j=1,2,...} \| f_j \|_{C^{k,\alpha}(S^1; \mathbb{R}^n)} \leq M < \infty.
\]

(a) If \( \alpha \in (0, 1] \), then prove that there exists a subsequence \( \{f_{j'}\}_{j'=1,2,...} \) and a \( f_\infty \in C^{k,\alpha}(S^1; \mathbb{R}^n) \) such that
\[
\| f_\infty \|_{C^{k,\alpha}(S^1; \mathbb{R}^n)} \leq M,
\]
and, for any \( \ell \in \mathbb{N}, \beta \in [0, 1] \) such that \( \ell + \beta < k + \alpha \),
\[
\lim_{j' \to \infty} \| f_{j'} - f_\infty \|_{C^{\ell,\beta}(S^1; \mathbb{R}^n)} = 0.
\]

Explain, also, how this limit needn’t hold when \( \ell + \beta = k + \alpha \), no matter what \( k \in \mathbb{N}, \alpha \in [0, 1] \) we take, i.e., no matter how much regularity we force on our functions.

(b) What conclusions can you draw when \( \alpha = 0 \) and \( k \in \mathbb{N} \)? (i.e., what type of subsequential convergence can one expect if only assuming \( C^k \) bounds, but not Hölder bounds?)
Problem 3
(50pts: 25+25)

A $C^k$ immersed curve $\gamma : I \to \mathbb{R}^n$, $n \geq 2$, is said to be closed when $I = \mathbb{R}$ and $\gamma$ is periodic. We may take the shortest period of $\gamma$ to be $2\pi$. By virtue of Problem 1, $\gamma$ corresponds to both:

- an element $\gamma|_{[0,2\pi]} \in C^k_{\text{per}}([0,2\pi];\mathbb{R}^n)$, and
- an element $i_k^{-1}(\gamma|_{[0,2\pi]}) \in C^k(S^1;\mathbb{R}^n)$.

It will be important that we be able to go back and forth between these options without breaking a sweat. In particular, we will tend to prefer to phrase results in terms of immersions $\gamma \in C^k(S^1;\mathbb{R}^n)$, and we will conveniently “hide” our analytic parametrizations inside our proofs.

In Lecture 2 we also defined closed $C^k$ embedded curves in $\mathbb{R}^n$ as being those closed $C^k$ immersed curves that are injective when restricted to their shortest period. In our most recent notation,

*Closed $C^k$ embedded curves are injective immersions in $C^k(S^1;\mathbb{R}^n)$.*

This definition does a wonderful job of modeling periodic curves with no points of self-intersection, and is the definition we will assume throughout our discussion of curve shortening flow.

Unfortunately, the simplicity of our definition of closed curves prevents it from generalizing well to higher dimensional objects. Its issue is that it “presupposes” that, for some reason, closed curves always ought to be parametrizable with $S^1$’s. (In high dimensions, we don’t have exhaustive lists of model background manifolds!) Let us try to find a less presumptuous definition, and then prove that it’s equivalent to the one above. We will stick to embedded objects, since these are the ones we’ll be most interested in when going to higher dimensions to study mean curvature flow.

(a) We wish to say that closed $C^k$ embedded curves are compact subsets of $\mathbb{R}^n$ that are locally comprised of injective immersions of open intervals. Let’s try this definition:

*A compact connected subset $\Gamma \subset \mathbb{R}^n$ is said to be a closed $C^k$ embedded curve if, for every $\mathbf{x} \in \Gamma$,

there exists a $r(\mathbf{x}) > 0$ such that $\Gamma \cap B_{r(\mathbf{x})}(\mathbf{x})$ is the image of an injective $C^k$ immersion $\gamma(\mathbf{x}) : (0,1) \to \mathbb{R}^n$.*

Prove that this definition is *not* good by constructing a figure-8 type curve that fits this definition but clearly possesses a point of self-intersection. ($C^1$ examples suffice.)

(b) However, prove that the following definition is *equivalent* to ours:

*A compact connected subset $\Gamma \subset \mathbb{R}^n$ is said to be a closed $C^k$ embedded curve if, for every $\mathbf{x} \in \Gamma$,

there exists a $r(\mathbf{x}) > 0$ such that $\Gamma \cap B_{r(\mathbf{x})}(\mathbf{x})$ is the image of an injective $C^k$ immersion $\gamma(\mathbf{x}) : (0,1) \to \Gamma \cap B_{r(\mathbf{x})}(\mathbf{x})$ whose inverse is continuous.*
Extra reading material: integrals over closed curves

To practice working with $S^1$-parametrizations of closed curves, let us discuss some integral identities we used in Lecture 2. Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^n$ be a “smooth” family of closed immersed curves in the sense that

$$ R \times [0, T) \ni (p, t) \mapsto \gamma(\sigma(p), t) \in \mathbb{R}^n $$

is smooth in the standard sense of multivariable calculus. Here, $\sigma : \mathbb{R} \to S^1$ is as in Problem 1.

- One thing we asserted was:

$$ \frac{d}{dt} \int_{S^1} \left| \frac{\partial \gamma}{\partial p}(p, t) \right| dp = \int_{S^1} \frac{\partial}{\partial t} \left| \frac{\partial \gamma}{\partial p}(p, t) \right| dp. $$

The integral over $S^1$ is to be understood as a single variable calculus integral over $p \in [0, 2\pi]$, and $\gamma(p, t)$ is to be understood as $\gamma(\sigma(p), t)$. Convince yourselves that this statement is trivial, and does not require any periodicity.

- Another thing we asserted was:

$$ \int_{S^1} \left( \frac{\partial^2 \gamma}{\partial p \partial t}(p, t) \right) \cdot \tau(p, t) \, dp = - \int_{S^1} \left( \frac{\partial}{\partial t} \gamma(p, t) \right) \cdot \left( \frac{\partial}{\partial p} \tau(p, t) \right) \, dp. $$

Prove this by writing out these expressions as single variable calculus integrals over $p \in [0, 2\pi]$ and integrating by parts. Which derivatives of $\gamma(p, t)$ did you need to agree at $p = 0, 2\pi$ for this to work?