Problem 1
(20 pts: 10 + 10)

Let \( f, f_j : [0, 1] \to \mathbb{R}^n \), \( j = 1, 2, \ldots \), be continuously differentiable functions with \( f_j \) converging to \( f \) uniformly as \( j \to \infty \).

(a) If \( |f'_j| \leq 1 \) on \([0, 1]\), for every \( j = 1, 2, \ldots \), then prove that \( |f'| \leq 1 \) on \([0, 1]\), too.

(b) Construct \( f_j, f \) as above, with \( |f'_j| \equiv 1 \) pointwise on \([0, 1]\), for all \( j = 1, 2, \ldots \), but \( f' \equiv 0 \).

The geometric significance of this problem is that, unless one is careful about how they take limits, a sequence of curves might experience length-drop in the limit, even if the convergence is uniform. The Arzelà–Ascoli theorem, which we will discuss next week, is one of the main tools used to address this issue.

Problem 2
(10 pts: 5 + 5)

Let \( I, J \subset \mathbb{R} \) be intervals and \( \gamma : I \to \mathbb{R}^n, n \geq 2 \), be an immersed curve. Prove that the quantity

\[
\tau_\gamma(t) := \frac{\gamma'(t)}{|\gamma'(t)|}
\]

is an orientation-\textit{dependent} geometric invariant, i.e., for every reparametrization \( \varphi : J \to I \),

\[
\tau_{\gamma \circ \varphi} = \pm \tau_\gamma \circ \varphi,
\]

with \( \pm \) depending on whether \( \varphi \) is orientation-preserving or orientation-reversing. Next, assuming \( \gamma \) is twice differentiable, prove that the quantity

\[
\kappa_\gamma(t) := \frac{1}{|\gamma'(t)|} \frac{d}{dt} \frac{\gamma'(t)}{|\gamma'(t)|}
\]

is an orientation-\textit{independent} geometric invariant, i.e., for every reparametrization \( \varphi : J \to I \),

\[
\kappa_{\gamma \circ \varphi} = \kappa_\gamma \circ \varphi.
\]

These quantities, \( \tau_\gamma \) and \( \kappa_\gamma \), are the unit tangent and curvature vectors of \( \gamma : I \to \mathbb{R}^n \).
Problem 3
(25 pts: 5 + 20)

Let $C_0$ denote a circle of radius $R_0 > 0$ centered at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$.

(a) Write down a 1:1, arclength parametrization $\gamma_0 : [0, 2\pi R_0) \rightarrow \mathbb{R}^2$ of $C_0$.

(b) Construct a solution $\gamma : [0, 2\pi R_0) \times [0, T) \rightarrow \mathbb{R}^2$ to the curve shortening flow

\[ \frac{\partial}{\partial t} \gamma(\cdot, t) = \kappa(\cdot, t), \]

\[ \gamma(\cdot, 0) = \gamma_0. \]

In other words, write down an ansatz for $\gamma$, verify that each $\gamma(\cdot, t)$ is an immersion, and that $\gamma(\cdot, t)$ satisfies the system above. How big can $T$ be?

Note: $\gamma(\cdot, t)$ is no longer parametrized by arclength once $t > 0$.

Problem 4
(20 pts: 5 + 5 + 10)

Suppose $\gamma : (a, b) \rightarrow \mathbb{R}^2$ is a curve that can be written out in “graph” form:

\[ \gamma(x) = \begin{bmatrix} x \\ u(x) \end{bmatrix}, \]

for some continuously differentiable $u : (a, b) \rightarrow \mathbb{R}$. Note that such a $\gamma$ is automatically an immersed curve, though not necessarily parametrized by arclength. Compute:

(a) The unit tangent vector $\tau(x)$ for $\gamma$.

(b) The upward pointing unit normal vector $n(x)$ for $\gamma$.

(c) The curvature vector $\kappa(x)$ for $\gamma$, assuming $u$ is at least twice differentiable.

Problem 5
(40pts: 10 + 20 + 10)

Let $\gamma : (a, b) \times [0, T) \rightarrow \mathbb{R}^2$ be a continuously differentiable family of immersed curves.

(a) If $\varphi : (c, d) \times [0, T) \rightarrow (a, b)$ is continuously differentiable and $\frac{\partial}{\partial x} \varphi \neq 0$ (this is the partial derivative in the first slot), then prove that the reparametrized family $\tilde{\gamma}(\cdot, t) := \gamma(\varphi(\cdot, t), t)$ is still a continuously differentiable family of immersed curves, and that

\[ \left[ \frac{\partial}{\partial t} \tilde{\gamma}(\cdot, t) \right]^\perp = \left[ (\frac{\partial}{\partial t} \gamma)(\varphi(\cdot, t), t) \right]^\perp, \]

where $[\cdot]^\perp$ denotes the projection onto the normal direction.
(b) Suppose now that we have a solution $\gamma(\cdot, t)$ of the curve shortening flow:

$$\frac{\partial}{\partial t} \gamma(\cdot, t) = \kappa(\cdot, t).$$

(Do not worry about its regularity.) For any fixed $t$, small portions of $\gamma(\cdot, t)$ can certainly be expressed in graph form (see Problem 4), as long as we’re working near points with unit tangent vector $\tau(\cdot, t) \neq \pm (0, 1)$. (You needn’t prove this.) Unfortunately, it is simply not necessarily true that we can write

$$\gamma(x, t) = \begin{bmatrix} x \\ u(x, t) \end{bmatrix}$$

if $t$ is allowed to vary. (Look at your solution to Problem 3.) It turns out we can fix this by reparametrizing $\gamma$ in a time-dependent way, as in (a). So assume (without proof, for now) that there exists $\varphi$ as in (a) so that $\tilde{\gamma}(\cdot, t) := \gamma(\varphi(\cdot, t), t)$ can be written locally in graph form:

$$\tilde{\gamma}(x, t) = \begin{bmatrix} x \\ u(x, t) \end{bmatrix},$$

for some $u(x, t), x \in (a, b), t \in [0, T)$. Prove that the function $u(x, t)$ satisfies:

$$u_t = (1 + u_x^2)^{1/2} \frac{\partial}{\partial x} \frac{u_x}{(1 + u_x^2)^{1/2}}.$$  \hspace{1cm} (†)

(c) Classify all solutions $u$ of (†) that, additionally, satisfy $u(x, t) = u(x, 0) + ct$ for some constant $c > 0$.

Note: Curves corresponding to such $u$ are called translators for the curve shortening flow, since they correspond to solutions of the curve shortening flow that evolve by translation.