15. Lecture 15

The goal is to cover:

- Mean curvature flow introduction.
- Huisken’s monotonicity formula.

Your primary source can be [14] from the website. You should feel free to assume that we have global motion by MCF, i.e., wherever you see "((M_t) moves by MCF in U × I)," just assume U = R^{n+1}, I = [0, T). This will make things simpler. Be careful of sign conventions for the unit norm and second fundamental form, which are different than the ones we’re using in the course (do Carmo’s).

- Define MCF. In our notation, 
  \[ F : \Sigma \times [0, T) \to \mathbb{R}^{n+1} \]
  \[ \partial_t F = H \nu \]
  where \( \nu \) denotes an inward pointing normal.

- Just like with CSF, one has short-time existence (for \( C^{2,\alpha} \) submanifolds; for \( C^2 \) submanifolds one argues as in a previous problem set), preservation of embeddedness, and the avoidance principle (we called it the “separation” principle, I believe).

- In CSF you showed that at the first singular time \( T \), \( \lim_{t \to T} \max_{(S^1,t)} |k| \to \infty \). In MCF, at the first singular time, \( \lim_{t \to T} \max_{(\Sigma,t)} |A| \to \infty \).

- Huisken proved the analog of the Gage–Hamilton theorem: if the initial surface is convex (i.e., positive definite second fundamental form: \( A|_p(X,Y) > 0 \) for all \( p \in \Sigma, X, Y \in T_p \Sigma \)), then the flow remains convex and flows to a round point. However, there is no analog of \( T = A(0)/2\pi \).

- In the non-convex setting, singularities can and do in fact often happen in MCF before the entire flow becomes extinct. This will be exhibited in Shengtong, Varkey, and Yannick’s project.

- So, let’s switch gears to prove Huisken’s monotonicity for smooth MCFs.

- State it: it’s Theorem 3.5.1 for \( \phi = 1 \). Careful about sign conventions in [14]: there, \( H \) is a vector and corresponds to what we would be denoting in our class as \( H \nu \) where \( \nu \) is the inward pointing normal.

- Lemma: prove that \( \text{div}_\Sigma \nu = -H \). (This follows from the definition of \( \text{div}_\Sigma \), and \( H = \sum g^{ij} A_{ij} \) This has nothing to do with MCF, by the way.

- Lemma: Also prove the following result, which allows us to compute \( \Delta_\Sigma f \) in case \( f \) is a function defined on an open subset \( U \subset \mathbb{R}^{n+1} \) rather than just on an embedded submanifold \( \Sigma^n \subset \mathbb{R}^{n+1} \): if \( f : U \to \mathbb{R} \) is smooth, then on \( \Sigma \cap U \) we have:
  \[ \Delta_\Sigma f = \text{div}_\Sigma \nabla f + (\partial_\nu f) H. \]

This also has nothing to do with MCF, but it’s the analog of (2.3.1.b) in [14]. Careful about sign conventions in [14]: there, \( H \) is a vector and corresponds to what we would be denoting in our class as \( H \nu \) where \( \nu \) is the inward pointing normal. (Worse yet, [14] uses outward pointing normals.)

- Prove (2.3.1.a) in [14]. This MCF-specific.

- Recall: under MCF, \( \partial_t g_{ij} = -2H A_{ij} \) and \( \partial_t \text{Area}(F(\Sigma,t)) = -\int_{\Sigma} H^2. \)

These are consequences of computations performed in a previous lecture.
The latter evolution, together with the first variation formula, tells us that MCF is the gradient flow for the area.

- State and prove whichever other parts of Corollary 3.1.4 will be useful for your proof of Huisken’s monotonicity formula in a little bit.
- Write down the backward heat kernel and prove Proposition 3.4.1.
- Prove Huisken’s monotonicity: Theorem 3.5.1 for $\phi = 1$.
- So, what happens if you have a smooth MCF for which the monotone quantity is constant? What can you say about how the flow moves? It will be a self-similarly shrinking flow. These self-shrinkers describe singularity models for MCF. (Heuristically: you zoom into a singularity and dilate space and time, and the monotone quantity for the original MCF becomes constant.) Their study is a very active field of research. They are classified for CSF, convex-MCF, mean-convex MCF, but are open otherwise.