The goal is threefold:

- Introduce differential operators on submanifolds.
- Discuss the maximum principle.
- Introduce the norm of the second fundamental form on submanifolds.

In all that follows, let $\Sigma^n \subset \mathbb{R}^{n+1}$ be an $n$-dimensional embedded submanifold of $\mathbb{R}^{n+1}$. For the most part we’ll work locally near some point $p \in \Sigma$, where $\Sigma \cap B_r(p)$ can be viewed as the image of an injective immersion $F : U \to \Sigma$, $p = F(0)$, and $U$ is a bounded open subset of $\mathbb{R}^n$.

- First, we want to define the tangential gradient of functions along $\Sigma$.
- Recall that if $f$ is some ambient function on some open $U \subset \mathbb{R}^{n+1}$, then $\nabla f$ is just a vector field on $U$. We, unsurprisingly, define its tangential gradient along $\Sigma$ to be literally the projection of this gradient onto $\Sigma$. Specifically, if $\nu(p)$ denotes a normal vector to $p \in \Sigma$, then:

$$\nabla_{\Sigma} f(p) = \nabla f(p) - \langle \nabla f(p), \nu(p) \rangle \nu(p).$$

Being a tangent vector, $\nabla_{\Sigma} f(p)$ can be rewritten as a linear combination of the basis vectors for $T_p \Sigma$: $\partial_1 F(0), \ldots, \partial_n F(0)$. Namely:

$$\nabla_{\Sigma} f(p) = \sum_{i,j=1}^n g^{ij}(0) \partial_i (f \circ F)(0) \partial_j F(0).$$

Here, $g^{ij}$ are the elements of a matrix $(g^{ij})$ defined as the inverse matrix of the matrix $(g_{ij})$ whose elements are given by

$$g_{ij} = \partial_i F \cdot \partial_j F.$$

- The formula above for $\nabla_{\Sigma} f$ is parametrization-independent: if $G$ also parametrizes $\Sigma$ near $p$ (i.e., $G = F \circ \varphi$ for a $C^1$ bijection $\varphi$ with $C^1$ inverse, with the correct domain/range in $\mathbb{R}^n$), then $\nabla_{\Sigma} f(p)$ is the same whether you computed using $F$ or $G$.

- One can use the formula above to define $\nabla_{\Sigma} f$ for functions $f : \Sigma \to \mathbb{R}$ for arbitrary embedded submanifolds of $\mathbb{R}^{n+1}$, even if these functions do NOT come up as restrictions of ambient functions. Notice that even if $\Sigma$ is less than $n$-dimensional, this definition still makes sense (as long as we replace $n$ with the dimension of $\Sigma$).

- One defines the tangential divergence $\text{div}_{\Sigma} Z$ along $\Sigma$ for vector fields $Z$ defined on open subsets $U \subset \mathbb{R}^{n+1}$ along the same way:

$$\text{div}_{\Sigma} Z(p) = \sum_{i,j=1}^n g^{ij}(0) \partial_i (Z \circ F)(0) \cdot \partial_j F(0).$$

Like before, $\text{div}_{\Sigma} Z$ is parametrization independent. Along the same lines as above one defines $\text{div}_{\Sigma} Z$ for vector fields that are only defined along $\Sigma$. This is also the definition uses if $\Sigma$ is less than $n$-dimensional.

- State the divergence theorem. You needn’t prove it: If $\Sigma^k \subset \mathbb{R}^{n+1}$ is a compact $k$-dimensional embedded submanifold, and $Z$ is a $C^1$ vector field on $\Sigma$ which is everywhere tangent to $\Sigma$ and $f$ is a $C^1$ function on $\Sigma$, then

$$\int_{\Sigma} \nabla_{\Sigma} f \cdot Z = - \int_{\Sigma} f \text{div}_{\Sigma} Z.$$
• We can now define the tangential Laplacian for functions $f: \Sigma \to \mathbb{R}$:

$$\Delta_\Sigma f = \text{div}_\Sigma \nabla_\Sigma f.$$  

• State the analog of the strong maximum principle. If $\Sigma^k \subset \mathbb{R}^{n+1}$ is a compact embedded submanifold, we have the differential operator

$$Lu = \Delta_\Sigma u + b \cdot \nabla_\Sigma u + cu$$

for $c \leq 0$, and $u: \Sigma \times [0,T]$ satisfies $\frac{\partial}{\partial t} u - Lu \leq 0$ (etc.) then ...

• One can now state the ODE-PDE comparison theorem on compact $\Sigma$. This is Theorem 2.1.1 in Mantegazza. (Skip the references to $u_{\text{max}}$! We have gone to great lengths in the course to not have to worry about $u_{\text{max}}$. Simply compare to the solution $h$ of the ODE he writes down later, which is just as good and in fact clearer.)

• Finally, let’s switch gears back to defining things. Let’s assume $\Sigma$ is $n$-dimensional again, rather than $k \leq n$ dimensional.

• We’d like to define the ”norm” of the second fundamental form at $p$. Up until now, we’ve viewed the second fundamental form as an $n \times n$ matrix with entries

$$A_{ij}(0) := -\partial_i \nu(0) \cdot \partial_j \mathbf{F}(0), \ i, j = 1, \ldots, n,$$

where $\nu: \mathbb{B} \to \mathbb{R}^{n+1}$ is a normal to $\Sigma$.

• Could its (squared) norm be defined to be something like

$$\sum_{i,j=1}^n |A_{ij}(0)|^2$$

No! This quantity isn’t “geometric” – i.e., it’s not independent of parametrization. Do you see why?

• The *correct* (squared) norm is defined to be

$$|A(0)|^2 = \sum_{i,j,k,\ell} g^{ik}(0) g^{j\ell}(0) A_{ij}(0) A_{k\ell}(0).$$

This quantity *is* independent under reparametrization! Note: the direct equivalent of this for curves in $\mathbb{R}^2$ is the square of the length of the curvature vector, $|\kappa|^2$ (equivalently, the square of the curvature scalar, $k^2$).