The topics to be covered are:

- integration on hypersurfaces,
- the first variation formula.

The main reference that we’d like to be following is (for better or for worse!) my undergraduate thesis on MCF; this is [14] on our website. While trying to understand the material, keep in mind that our problem sets (including problem set 7) are designed with the intention of making the material more accessible to you. So, you’ll want to have completed problem set 6 prior to/while trying to parse through this material.

- Define the notion of the $n$-dimensional area of the parametrized image of an injective immersion $F : B^n \to \mathbb{R}^{n+m}$ (say, bounded in $C^1$).

$$\text{Area}(F(B)) = \int_B (\det DF(x)^T DF(x))^{1/2} \, dx.$$  

This is sometimes suggestively called the ”area formula.”

- Prove that this is well-defined. In other words, if $G : B' \to \mathbb{R}^n$ is another injective immersion parametrizing $F(B)$, then $\text{Area}(F(B)) = \text{Area}(G(B'))$. This is nothing but a consequence of the change of variables formula from multivariable calculus!

- Next we want to define the notion of area of a compact embedded $n$-dimensional submanifold $M \subset \mathbb{R}^{n+m}$.

- To do so, we need the notion of a partition of unity subordinate to a cover $\{U_i\}$. State (without proof) the standard definition/theorem regarding the existence of partitions of unity.

- Define the area of a compact embedded $M \subset \mathbb{R}^{n+m}$. Cover $M$ with open $U_i \subset \mathbb{R}^{n+m}$ s.t. every $M \cap U_i$ is the image of an injective immersion $F_i : B_i \to M \cap U_i$ with a continuous inverse, as in the definition of embedded submanifolds. Consider a partition of unity $\{\phi_i\}$ subordinate to this cover. Then,

$$\text{Area}(M) = \sum_i \int_{B_i} \phi_i(F_i(x))(\det DF_i(x)^T DF_i(x))^{1/2} \, dx$$

(For more about partitions of unity, see Chapter 10 of baby Rudin.)

- More generally, one defines arbitrary integrals over $M$ by using such a partition of unity. Namely, if $h$ is some scalar-valued function defined on $M$, then one defines:

$$\int_S h = \sum_i \int_{B_i} h(F_i(x))\phi_i(F_i(x))(\det DF_i(x)^T DF_i(x))^{1/2} \, dx$$

- Switch gears to the first variation formula. It’s one of the most important tools in geometric analysis. First we’ll state it; then we’ll prove it.

- Suppose $M$ is a compact $n$-dimensional embedded submanifold in $\mathbb{R}^{n+1}$. (This is called a compact embedded hypersurface.) Suppose that we have a smooth family of such $M$’s being moved according to a function $G : M \times (-\delta, \delta) \to \mathbb{R}^{n+1}$; here, $G(\cdot, t) : M \to \mathbb{R}^{n+1}$, for $|t| < \delta$, are smooth injective immersions. Assume $G(\cdot, 0) : M \to \mathbb{R}^n$ is the identity map and
that the time derivative $\partial_t \mathbf{G}(\cdot, 0)$ at $t = 0$ is a vector normal to $M$:

$$\partial_t \mathbf{G}(\cdot, 0) = f \nu$$

for some scalar field $f : M \to \mathbb{R}$, and where $\nu : M \to \mathbb{R}^n$ is the inward pointing unit normal. (This is well-defined for compact embedded $n$-dimensional $M \subset \mathbb{R}^n$, because they are orientable.) Then:

$$\left[ \frac{d}{dt} \text{Area}(\mathbf{G}(M, t)) \right]_{t=0} = - \int_{\Sigma} f H,$$

where $H$ is the mean curvature of $M$.

- Do you see how this is a generalization of the evolution formula for the arclength element we proved in Lecture 2? It might help you (the presenter) to recall how that simpler proof went.
- First, prove the first variation formula in the case where $M$ consists of just one chart, i.e.:

$$\left[ \frac{d}{dt} \text{Area}(\mathbf{F}(B, t)) \right]_{t=0} = - \int_B f H(\det DF(x)^T DF(x))^{1/2}.$$

- To do so, it’s best to understand the "metric" and the "second fundamental form" a little better, let’s say, for a fixed parametrization $\mathbf{F} : U \to \mathbb{R}^{n+1}$.
- The metric was underhandedly introduced in PS 6 Problem 2 Part (b); it’s the matrix $DF(x)^T DF(x)$, whose components are often denoted $g_{ij}(x)$, $i, j = 1, \ldots, n$. It’s also often the case that we need to use the inverse of this matrix, whose components we denote $g^{ij}(x)$. In the notation of the metric, note that

$$\text{Area}(\mathbf{F}(B)) = \int_B (\det DF(x)^T DF(x))^{1/2} dx.$$

- The second fundamental form was explicitly introduced in Problem 2 Part (e). In particular, if $t = 0$ variation is $f \nu$, then for every $i, j$:

$$\left[ \frac{d}{dt} g_{ij} \right]_{t=0} = -2 f A_{ij}$$

See [14, Proposition 2.3.2 (a)], where this was done for a particular choice of $f$, and where $A_{ij}$ is denoted $-h_{ij}$.
- In the homework you will have showed that

$$H(x) = \sum_{i,j=1}^n g^{ij}(x) A_{ij}(x).$$

- In particular, using the above and the formula for differentiating determinants, one can deduce the first variation formula. See [14, Proposition 2.3.2 (c)]
- Finally, conclude the first variation formula if there are many $\mathbf{F}_i$’s parametrizing $M$. 