18.994 - Assignment 8

Due: April 5, 2018

Problem 1. Embedded submanifolds can be locally written as graphs.

Suppose that \( \Sigma \subset \mathbb{R}^n \) is a \( d \)-dimensional smoothly embedded submanifold passing through \( 0 \in \mathbb{R}^n \). Prove that there exists a permutation \( \sigma \in S_n \) of \( (1, \ldots, n) \), an open set \( U \subset \mathbb{R}^n \) containing \( 0 \in \mathbb{R}^n \), an open set \( W \subset \mathbb{R}^d \) containing \( 0 \in \mathbb{R}^d \), and a smooth \( G : W \to \mathbb{R}^{n-d} \) such that \( G(0) = 0 \) and

\[
\Sigma \cap U = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_{\sigma(d+1)}, \ldots, x_{\sigma(n)}) = G(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), (x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \in W\} \cap U.
\]

(Hint: Implicit function theorem.)

Problem 2. In this problem we compute the unit normal, the second fundamental form, the shape operator, and the mean curvature of graphical \( n \)-dimensional submanifolds of \( \mathbb{R}^{n+1} \). This is largely (if not entirely) a linear algebra exercise.

Let \( B \subset \mathbb{R}^n \) be an open \( n \)-dimensional ball, e.g., of radius 1, and let \( f : B \to \mathbb{R} \) be smooth (with bounded derivatives of all orders). We call

\[
\Sigma_f := \{(x, f(x)) : x \in B\} \subset \mathbb{R}^{n+1}
\]

the “graph of \( f \).” Note that \( \Sigma_f \) is an \( n \)-dimensional embedded submanifold of \( \mathbb{R}^{n+1} \) (you don’t need to prove this). It is also convenient to consider the “graph map” \( F : B \to \mathbb{R}^{n+1} \), which is just

\[
F(x) := (x, f(x)),
\]

and parametrizes \( \Sigma_f \). Also recall that for every \( x \in B \), the tangent space of \( \Sigma_f \) at \( F(x) \) is defined to be the image of the derivative \( DF(x) : \mathbb{R}^n \to \mathbb{R}^{n+1} \):

\[
T_{F(x)} \Sigma_f := \text{span}\{\partial_i F(x) : i = 1, \ldots, n\}.
\]

Since \( F \) is an immersion, \( T_{F(x)} \Sigma_f \) is an \( n \)-dimensional subspace of \( \mathbb{R}^{n+1} \) with basis vectors \( \partial_i F(x) = e_i + \partial_i f(x)e_{n+1}, i = 1, \ldots, n \).
1. Compute, in terms of $f$, the matrix 

$$(\partial_i F(x) \cdot \partial_j F(x))_{i,j=1,...,n}.$$ 

Then show that the inverse of this matrix is 

$$\left( \delta_{ij} - \frac{(\partial_i f(x))(\partial_j f(x))}{1 + |\nabla f(x)|^2} \right)_{i,j=1,...,n}$$ 

where $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and 0 otherwise).

2. Compute, in terms of $f$, the upward pointing unit normal $\nu : B \to \mathbb{R}^{n+1}$ of $\Sigma_f$; that is, find the $\nu(x)$, $x \in B$, which is the unique upward pointing unit vector orthogonal to $T_{F(x)} \Sigma_f$.

3. Prove that the linear map $D\nu(x) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ for the derivative of $\nu$ at $x$, in fact, satisfies $\text{img } D\nu(x) \subset T_{F(x)} \Sigma_f$.

4. Compute, in terms of $f$, the $n \times n$ matrix $(A_{ij}(x))_{i,j=1,...,n}$ given by 

$$A_{ij}(x) := \partial_i \nu(x) \cdot \partial_j F(x).$$ 

This $n \times n$ matrix is called the second fundamental form of $\Sigma_f$ at $F(x)$, with respect to the basis $(\partial_i F(x))_{i=1,...,n}$. Note: $(A_{ij}(x))_{i,j=1,...,n}$ is not the matrix representing the linear transformation $D\nu(x) : \mathbb{R}^n \to T_{F(x)} \Sigma_f$ with respect to the basis $(\partial_i F(x))_{i=1,...,n}$, unless the basis is orthonormal.

5. Compute, in terms of $f$, the shape operator $S(x) : T_{F(x)} \Sigma_f \to T_{F(x)} \Sigma_f$; since $S(x)$ is linear, you need just compute 

$$S(x)\partial_i F(x)$$ 

in terms of $f, \partial_1 F(x), \ldots, \partial_n F(x)$. Recall that $S(x)$ is the unique linear map such that $S(x)D\nu(x)u = D\nu(x)u$ for all $u \in \mathbb{R}^n$. (Use parts 1, 4.)

6. The mean curvature $H(x)$ of $\Sigma_f$ at $F(x)$ is the trace (sum of the eigenvalues) of the shape operator, $S_f(x) : T_{F(x)} \Sigma_f \to T_{F(x)} \Sigma_f$. Show:

$$H(x) = \text{div} \left( \frac{\nabla f(x)}{\sqrt{1 + |\nabla f(x)|^2}} \right).$$

**Problem 3.** In this problem we show that minimal graphs (i.e., $\Sigma_f$ with $H = 0$) have least area among all graphical submanifolds with the same boundary values.

Assume the setup of Problem 2, and assume, additionally, that $H(x) = 0$ for all $x \in B$. Such submanifolds $\Sigma_f$ are called minimal graphs.
1. Consider the vector field obtained by extending the upward pointing unit normal \( \nu : B \to \mathbb{R}^{n+1} \) (from Problem 2) to the cylinder \( B \times \mathbb{R} \subset \mathbb{R}^{n+1} \), so that it’s constant vertically. Prove that, for this ambient vector field \( \nu \),
\[
\text{div} \, \nu = 0 \text{ everywhere on } B \times \mathbb{R}.
\]

2. Let \( g : B \to \mathbb{R} \) be another smooth function, with \( g = f \) on \( \partial B \), and whose graph is denoted \( \Sigma_g \) (but isn’t necessarily minimal). Prove that
\[
\text{Vol}(\Sigma_f) \leq \text{Vol}(\Sigma_g).
\]
(Use the divergence theorem on the region trapped between \( \Sigma_f \) and \( \Sigma_g \), and part 1 above.)

**Problem 4. The avoidance principle for minimal hypersurfaces.**

Assume the setup of Problem 3. Let \( f' : B \to \mathbb{R} \) be another smooth function whose graph \( \Sigma_{f'} \) is also a minimal surface.

1. Using Problem 2, show that \( f' - f \) satisfies a partial differential equation of the form
\[
\sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij}(f' - f)(x) + \sum_{i=1}^{n} b_i(x) \partial_i(f' - f)(x) + c(x)(f' - f)(x) = 0 \text{ for } x \in B,
\]
where \( a_{ij}, b_i, c : B \to \mathbb{R} \) are bounded, continuous, \( (a_{ij}(x))_{i,j=1,...,n} \) is a symmetric matrix with smallest eigenvalue \( \geq \delta > 0 \) for all \( x \in B \).

2. Suppose that \( f' \geq f \) on \( B \). Show that either \( f' \equiv f \) or \( f' > f \) on \( B \).

**Problem 5. Minimal hypersurfaces are noncompact.**

Let \( \Sigma \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional smoothly embedded minimal submanifold, i.e., with mean curvature \( H \equiv 0 \). Prove that if \( \Sigma \) is closed (as a subset of \( \mathbb{R}^n \)), then:

1. Either there exists no \( p \in \Sigma \) so that \( \Sigma \) lies entirely on one side of \( T_p \Sigma \), or \( \Sigma \) is an \( n \)-dimensional plane.

2. \( \Sigma \) is necessarily noncompact.

(Hint: Use Problems 1 and 4.)